Some largest eigenvalue problems in statistics.

Iain Johnstone, Stat + Biostat, UC Berkeley
(on leave, Stanford)

AGENDA

1. EXAMPLES

   - Principal Components Analysis

   - Canonical Correlations

2. GENERALITIES

   - One & Two Wishart problems

   - 'large' $n, p, (q)$ asymptotics

3. LARGEST EIGENVALUE(S) in null cases

   - for CCA = Jacobi ($\mathbb{R}$ & $\mathbb{C}$)

     [w. P. Forrester, in progress]

   - Liouville-Green

   - message for stat: 'large' = quite small

4. SOME OPEN QUESTIONS

REVIEW OF PCA : Goal: Dimensionality reduction

E.g: 1. Hotelling (1933)

$n = 140$ cases

$p = 4$ variables

reading < speed, power

arithmetic < speed, power

2. Craddock – Flood (1969)

$n = 1095$ Daily data, 500 mb heights

$p = 130$ Locations

→ Data Y_{ij} or $Y(t_i, x_j)$

i/t_i cases/times $i = 1, \ldots, n$

j/x_j variables/places $j = 1, \ldots, p$ $(\leq n)$

Mean centered $Y_{ij} \leftarrow Y_{ij} - \bar{Y}_{\cdot j}$ $\bar{Y}_{\cdot j} = \frac{1}{n} \sum_{i=1}^{n} Y_{ij}$

'Covariance' matrix $S = (S_{jj'}) = Y'Y$

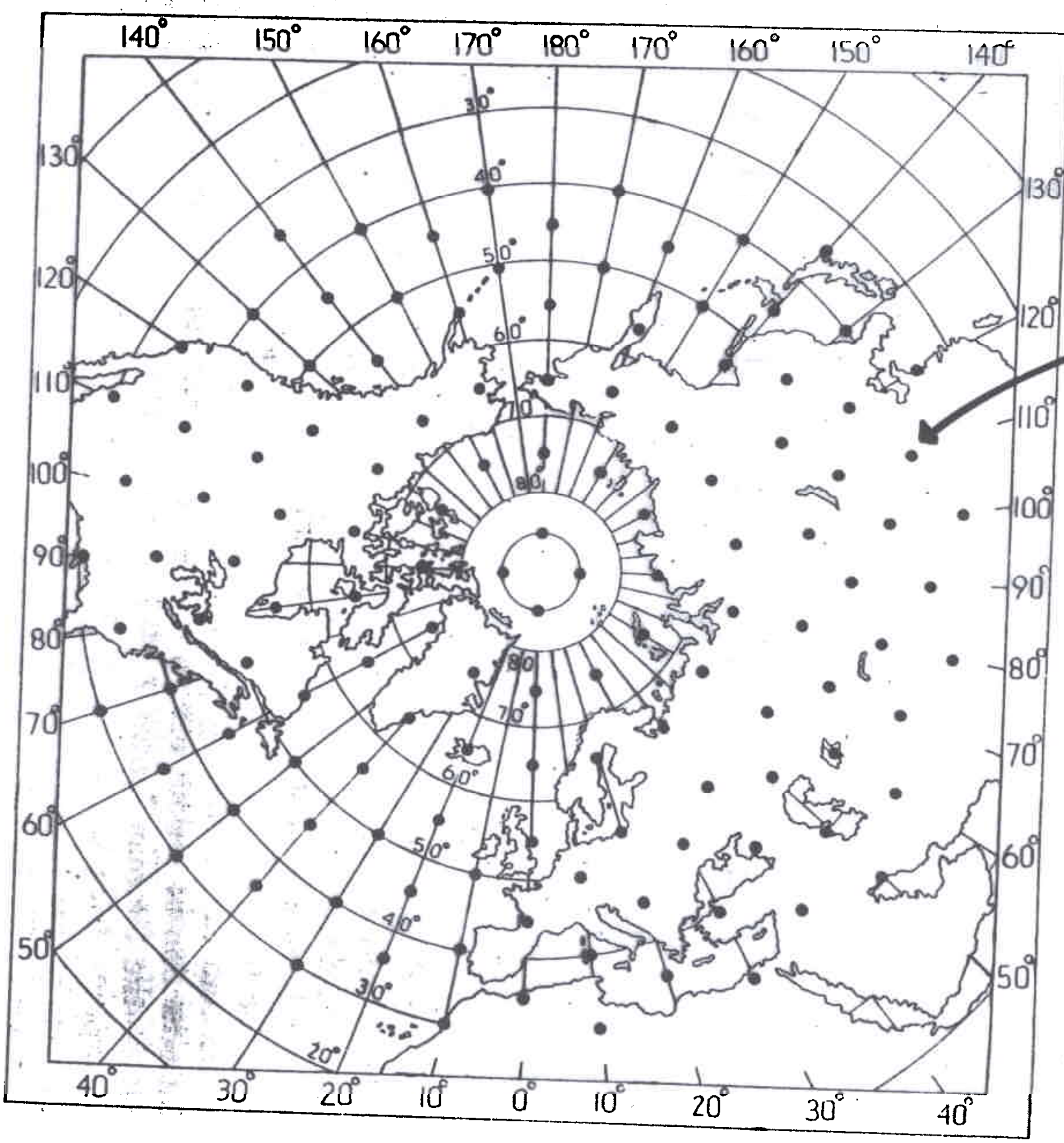
$$S_{jj'} = \sum_i Y_{ij} Y_{ij'}$$

Quart. J. R. Met. Soc. (1969), **95**, pp. 576-593

551.547.5 : 5

Eigenvectors for representing the 500 mb geopotential surface over the Northern Hemisphere

By J. M. CRADDOCK and C. R. FLOOD
Meteorological Office, Bracknell

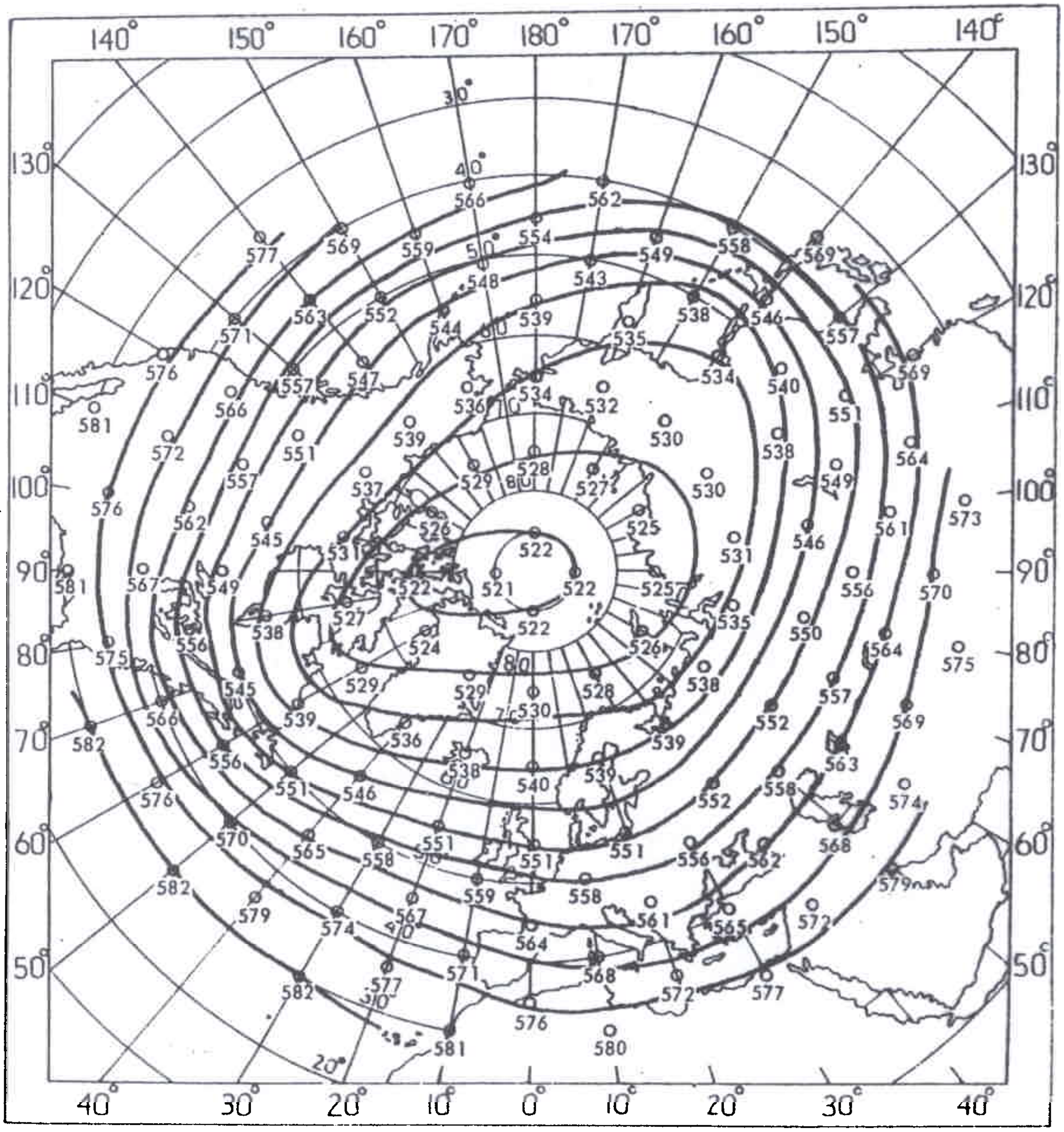


locations x_j

$p=$

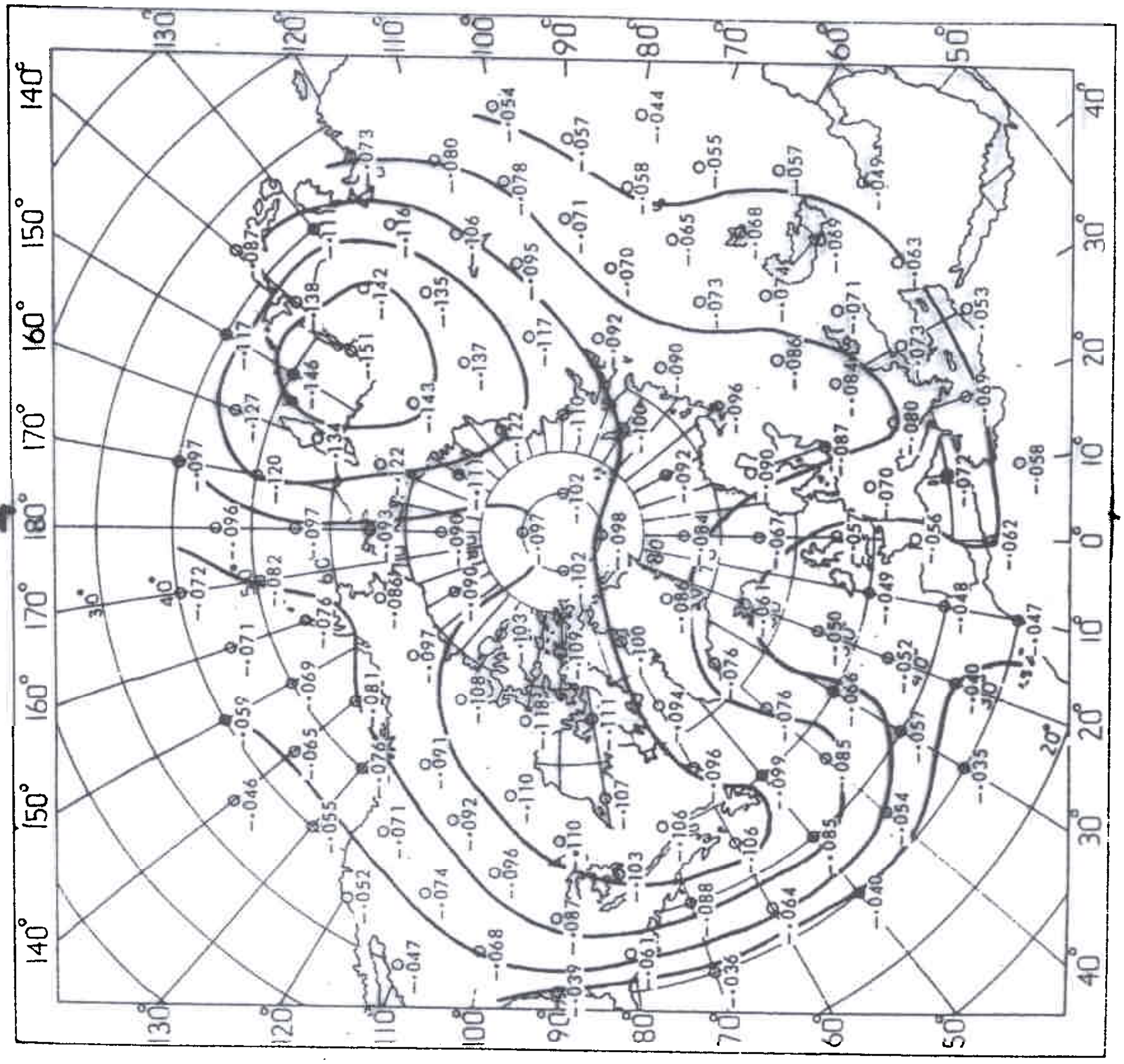
$j=1,\dots,130$

Mean 500mb field over $t_1 \ldots t_{1095}$

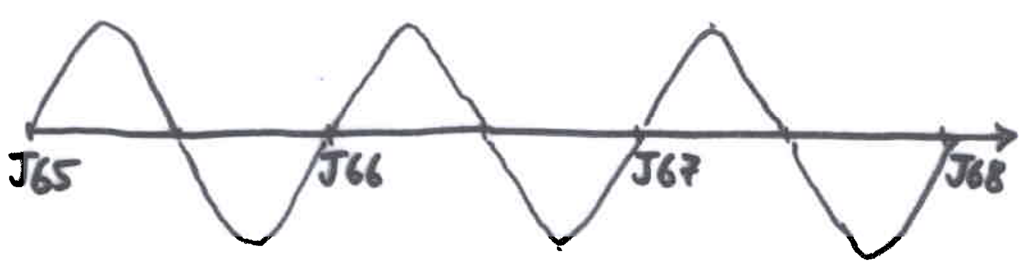


1st eigenvector $v_1(x_j)$ (e.o.f.)

$l_1 = 58.6\%$



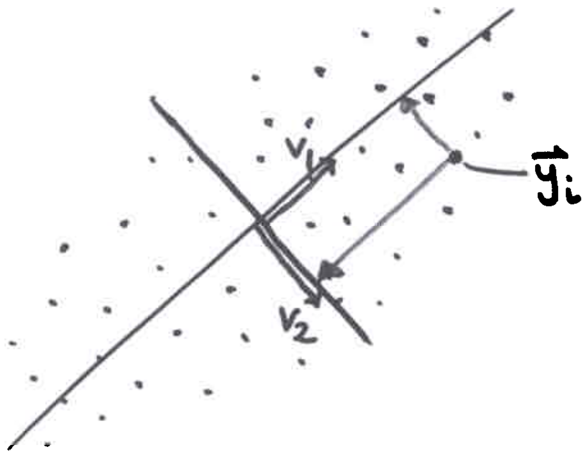
1st p.c. $z_1(t_i)$ $i = 1, \ldots, 1095$



MAX VARIANCE DIRECTION

n points in
p dimensions

$p = 2$



$$Y = \begin{bmatrix} \vec{y}_1 \\ \vdots \\ \vec{y}_n \end{bmatrix}$$

$$\mathrm{Var}\{ v'\vec{y}_i \} = v' S v$$

First p.c. variance + eigenvector

$$l_1 = \max \{ v' S v \; : \; v'v = 1 \}$$

v_1 = 1^{st} eigenvector / p.c. loadings

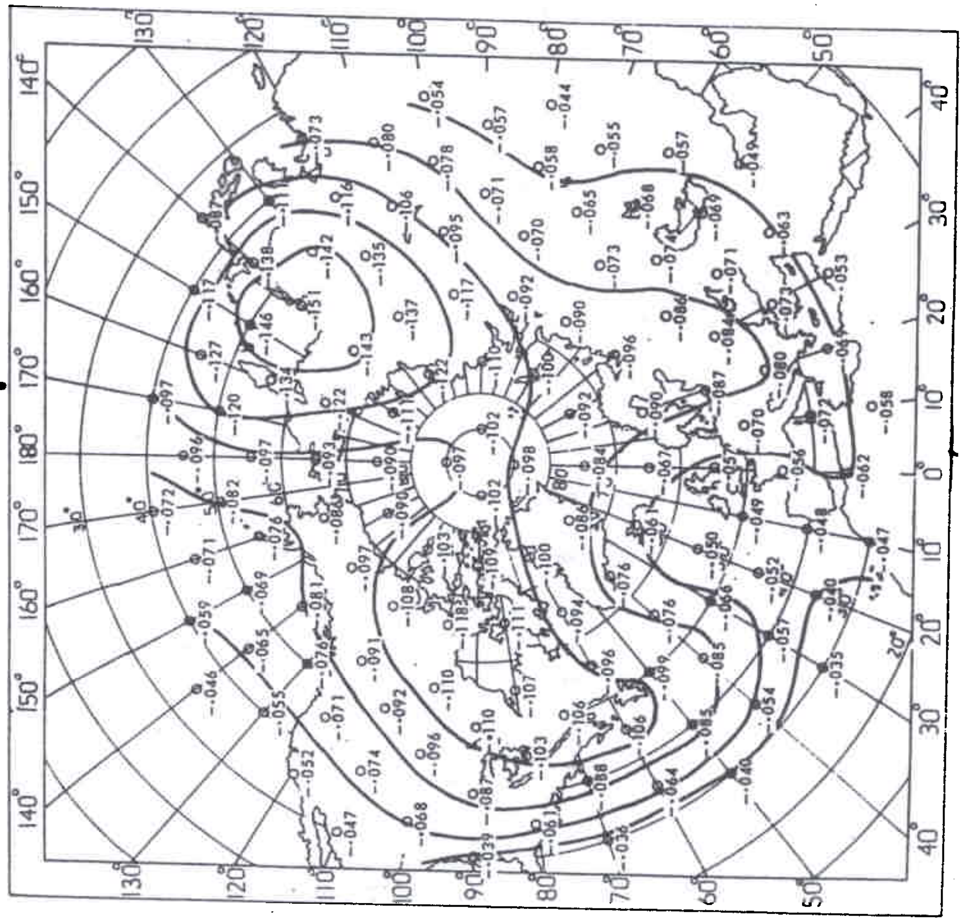
First p.c. :

$$z_1 = \sqrt{l_1}\, u_1 = Y v_1$$

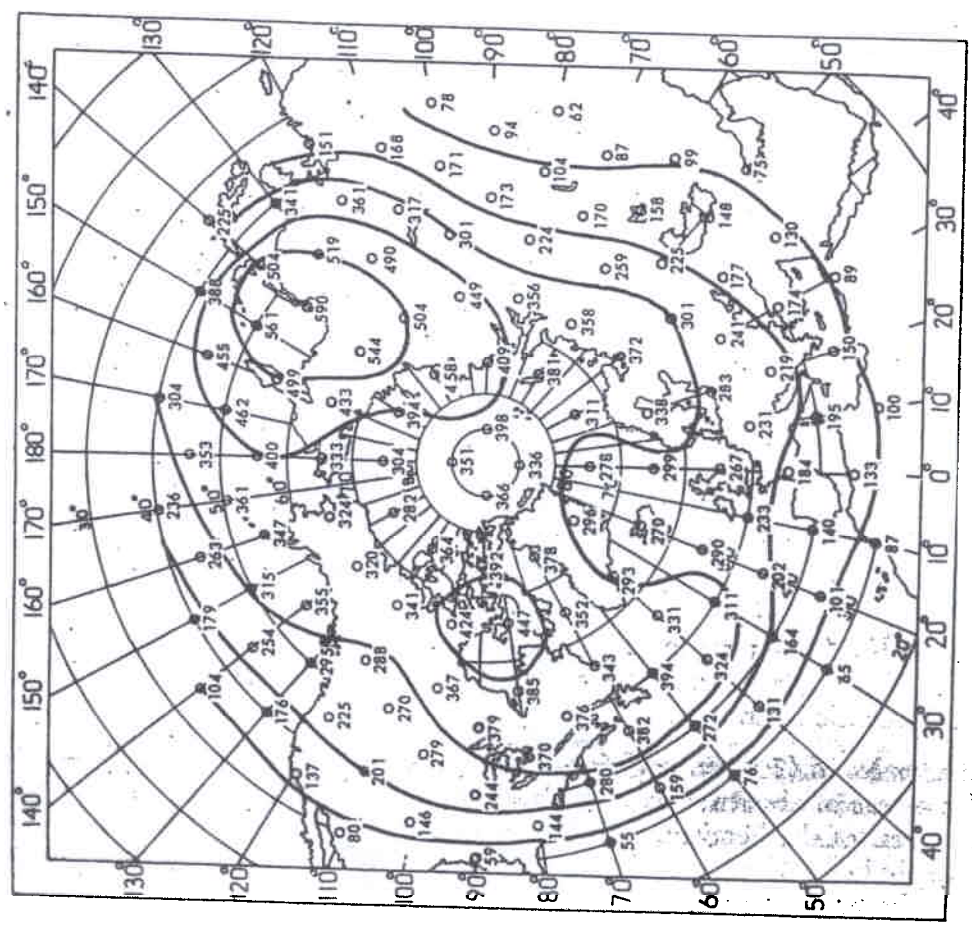
i.e. $\quad z_1(t_i) = \sum_j Y(t_i, x_j)\, v_1(x_j)$

Seasonal nature of 1st p.c.

1st eigenvector



Variances by location



ORTHOGONAL MAX VARIANCE DIRECTIONS

Iterate on p.c. variances

$$l_k = \max \{ v' S v : \quad v'v = 1, \quad v' v_m = 0 \quad m < k \}$$

→ $v_k = k^{th}$ eigenvector

$$z_k = \sqrt{l_k}\, u_k = Y v_k = k^{th} \text{ p.c.}$$

→ Singular Value Decomposition of data:

$$Y = U L V' = \sum_k \sqrt{l_k}\, u_k v_k' \qquad l_1 \geq l_2 \geq \dots$$

→ Best (linear!) r term approx^n to data Y:

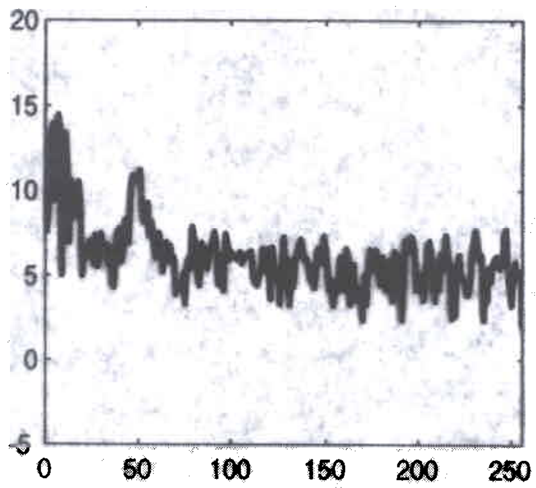
$$\hat{Y}_r(t_i, x_j) = \sum_{k=1}^{r} \sqrt{l_k}\, u_k(t_i)\, v_k(x_j)$$

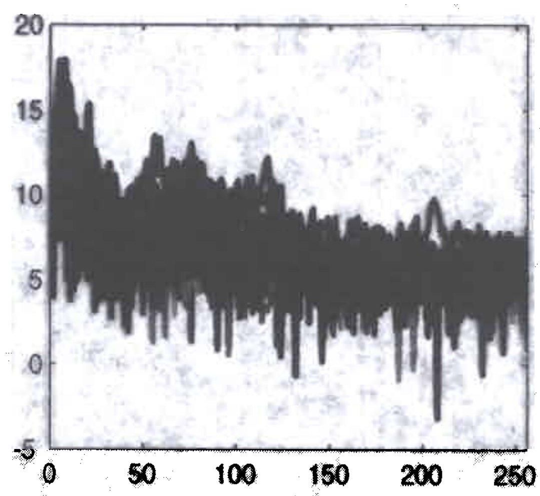
$$\| Y - \hat{Y}_r \|_F^2 = \sum_{r+1}^{p} l_k$$

→ wide use in data reduction/compression

- key role of eigenvalues $\{ l_k \}$

Example (via T. Hastie) Phoneme 'dcL'

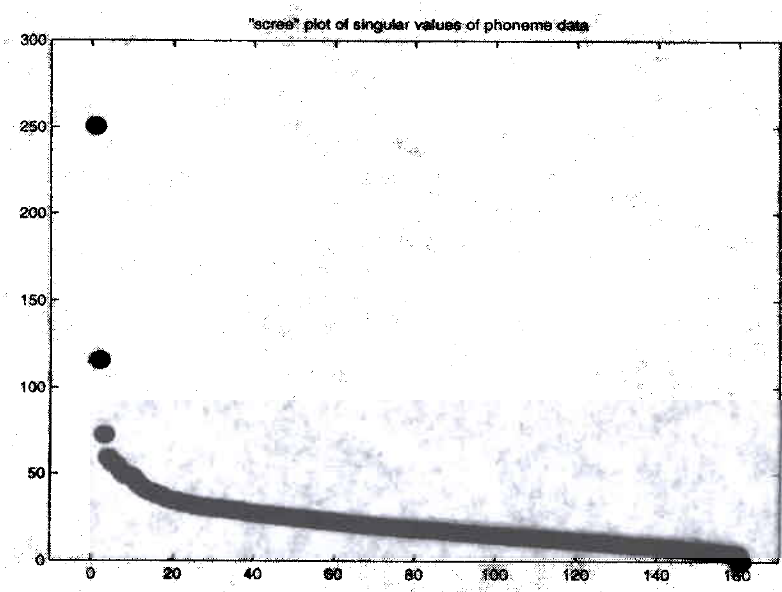




p = 256 log periodogram; n = 162 instances (50 spkrs)

GRAPHICAL METHODS : 'SCREE' PLOT

Cattell, 1966



CANONICAL CORRELATIONS ANALYSIS (CCA)

$$X = [\, x_1 \;\; x_2 \;\; \cdots \;\; x_p \,] \qquad Y = [\, y_1 \;\; y_2 \;\; \cdots \;\; y_q \,]$$

Find $a'x$ most correlated with $b'y$

$$\mathrm{Corr}(a'x,\, b'y) = \frac{a' S_{xy}\, b}{\sqrt{a' S_{xx}\, a}\,\sqrt{b' S_{yy}\, b}} \qquad \begin{array}{l} S_{xy} = X'Y \\ S_{xx} = X'X \\ \vdots \qquad \vdots \end{array}$$

$$r_k^2 = \max \left\{ a' S_{xy}\, b \;:\; \begin{array}{l} a' S_{xx}\, a = b' S_{yy}\, b = 1 \\ a' S_{xx}\, a_j = b' S_{yy}\, b_j = 0 \quad j < k \end{array} \right\}$$

→ determinantal equation

$$(*) \qquad \det\left(S_{xy} S_{yy}^{-1} S_{yx} - r^2 S_{xx} \right) = 0$$

→ $\quad r_1^2 \geq r_2^2 \geq \cdots \geq r_p^2 \qquad \left(\begin{array}{l} \text{and } a_1, \ldots, a_p \\ \qquad\; b_1, \ldots, b_p \end{array} \right)$

→ how many r_k^2 are "significant" ?

$$(*) \Leftrightarrow \qquad \det\left(A - r^2 (A+B) \right) = 0$$

$$A = S_{xy} S_{yy}^{-1} S_{yx} \qquad B = S_{xx} - A$$

1942, Econometrica

REGRESSIONS BETWEEN SETS OF VARIABLES

By Frederick V. Waugh, Agricultural Marketing Administration, Washington D.C.

Professor Hotelling's paper, "Relations between Two Sets of Variates,"[1] should be widely known and his method used by practical statisticians. Yet, few practical statisticians seem to know of the paper, and

X = "wheat characteristics" (raw)

x_1 = kernel texture

x_2 = test weight

x_3 = damaged kernels

x_4 = foreign material

x_5 = crude protein in wheat

Y = "flour characteristics" (finished)

y_1 = wheat per bbl. of flour

y_2 = ash in flour

y_3 = crude protein in flour

y_4 = gluten quality index

$a_1'x$ index of wheat quality

$b_1'y$ ... flour ...

GOAL: highly correlated grading of raw + finished products

$p = 5 \qquad q = 4 \qquad n = 138$

Monthly Weather Review

Origins and Levels of Monthly and Seasonal Forecast Skill for United States Surface Air Temperatures Determined by Canonical Correlation Analysis

T. P. BARNETT

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R. PREISENDORFER

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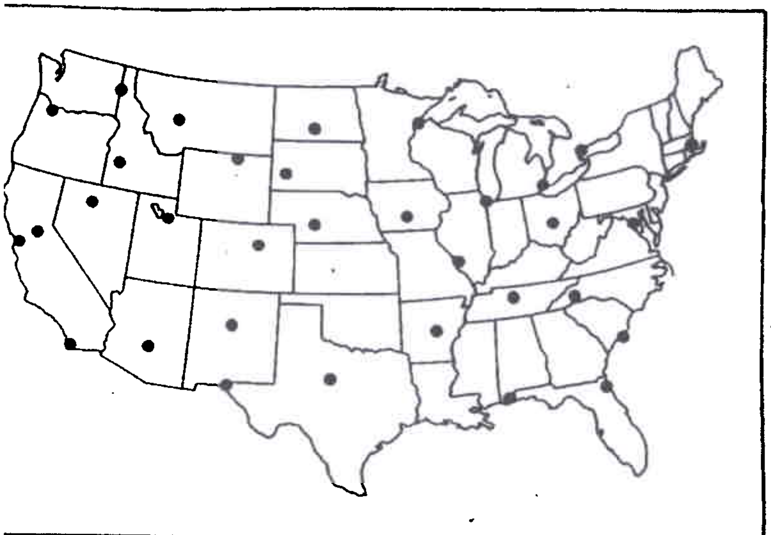


FIG. 1. Locations of stations/districts providing surface air temperature predictand data.

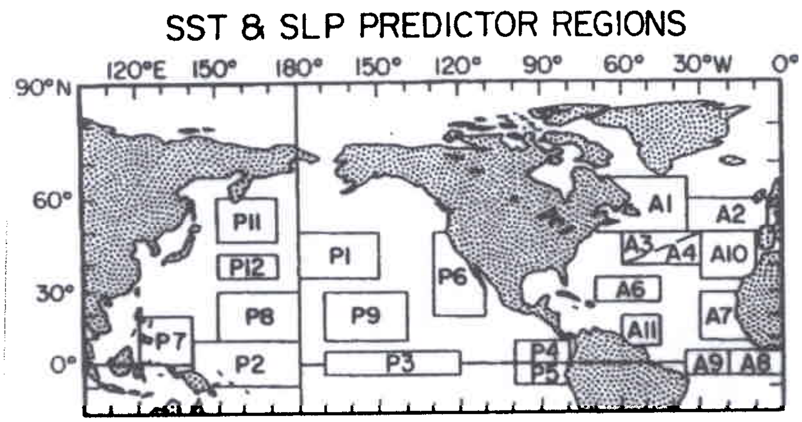


FIG. 2. SST from the large averaging areas shown above were used as predictor information. SLP predictor data came from the region 20°–70°N, 140°E to the Greenwich Meridian.

Y vars : surface air temps at 33 U.S. locations monthly data, 1931 – 1980

X vars : sea surface temp (SST) in 21 regions for 3 prior months in 2 seasons

$p = 126$ $q = 33$ $n = 600$

$(21 \times 3 \times 2)$ (50×12)

ROOTS OF A DETERMINANTAL EQUATION

$$|A - r^2(A+B)| = 0$$

where $\quad A = X'PX \qquad\qquad P = Y(Y'Y)^{-1}Y'$

$\qquad\quad\; B = X'P^{\perp}X \qquad\quad P^{\perp} = I - P$

Introduce probability: $\underset{n \times (p+q)}{[X : Y]} \sim N(0, I_n \otimes \Sigma) \qquad \Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$

<u>Null</u> distribution $\quad \Sigma_{xy} = 0 \quad \leftrightarrow \quad \rho_1 = \cdots = \rho_p = 0$

(and transform to $\Sigma_{xx} = I$)

BASIC SETTING : 2 INDEPENDENT WISHARTS

$$\left.\begin{aligned} A &\sim W_p(q, I) \\ B &\sim W_p(n-q, I) \end{aligned}\right\rangle \text{ indep., Wishart}$$

Multivariate Beta roots $:= (u_i)_{i=1}^p$

$$|u(A+B) - A| = 0$$

$\Leftrightarrow$ Multivariate F roots $:= (w_i)_1^p \qquad w = u/(1-u)$

$$|wB - A| = 0$$

MULTIVARIATE LINEAR MODEL

$$\underset{n\times p}{Y} = \underset{n\times q}{X}\ \underset{q\times p}{\beta} + \underset{n\times p}{E} \qquad E \sim N(0, I_n \otimes \Sigma)$$

q predictors $\qquad$ p responses

Let P = orthoprojection on column span of X

$$\begin{aligned} A &= Y'PY && \text{model SS} && \sim W_p(q, \Sigma, \Omega) \\ B &= Y'P^{\perp}Y && \text{error SS} && \sim W_p(n-q, \Sigma) \end{aligned} \Big\} \text{ inde}$$

Invariant tests of $H_0: \beta = 0$ $\qquad$ $(p \le q,\ p \le n-q$

use latent roots of AB^{-1}

E.g. Chemometrics (Skagerberg et.al. 92 in Breiman Friedman 9

$p = 6$ $\qquad$ output characteristics of polymers (correlated)

$q = 22$ $\qquad$ predictors: temps, feed rate

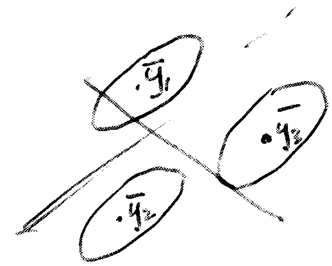
$n = 56$ $\qquad$ observations

MULTIPLE DISCRIMINANT ANALYSIS

$i = 1, \dots, q$ populations

n_i observations / pop'n on p variables

$$\vec{y}_{i1}, \dots, \vec{y}_{in_i} \sim N_p(\mu_i, \Sigma)$$



Between class:

$$A = \sum_i n_i(\bar{y}_i - \bar{y})(y_i - \bar{y})' \sim W_p(q-1, \Sigma, \Omega)$$

Within class:

$$B = \sum_{i,j} (y_{ij} - \bar{y}_i)(y_{ij} - \bar{y}_i)' \sim W_p(n-q, \Sigma)$$

Discriminant functions l_i solve

$$(A - w_i B) l_i = 0$$

- How many l_i are useful ($\leftrightarrow$ w_i signif)
- under H_0, roots of AB^{-1}

2. GENERALITIES: (non-central) WISHART

$$\underset{n\times p}{Y} = \begin{bmatrix} \vec{y}_1 + i\vec{y}_1' \\ \vdots \quad \vdots \\ \vec{y}_n + i\vec{y}_n' \end{bmatrix} \qquad \vec{y}_i + i\vec{y}_i' \overset{ind}{\sim} \mathbb{C}N_p(\vec{\mu}_i, \Sigma) \qquad M = \begin{bmatrix} \vec{\mu}_1 \\ \vdots \\ \vec{\mu}_n \end{bmatrix}$$

$$\underset{p\times p}{A} = Y^T Y \sim \mathbb{C}W_p(n, \Sigma; \Omega) \qquad \Omega = \Sigma^{-1} M^T M$$

- $M = 0 \rightarrow$ central Wishart
- $p = 1 \rightarrow \chi^2_n(\omega) \qquad \omega = M^T M / \sigma^2$

Roots of determinantal equations

"Wishart" $\det[A - l I] = 0$

— principal components, MDS

$B \sim W_p(n', \Sigma')$ independent of A:

"M. Beta" $\det[A - u(A+B)] = 0$

— canonical correlations

"M. F" $\det[A - wB] = 0$

— multivariate linear model

— mult discriminant; equality of covariances

Role of Eigenvalues in Some classical questions

Hypothesis testing: e.g. $H_0: \Sigma = I$ or $\Sigma_X = \Sigma_Y$

$H_0: M = 0$

$H_0: \mathrm{Cov}(X, Y) = 0$

$\rightarrow$ group invariant tests based on (l_i), (u_i), (w_i)

Estimation: e.g. of Σ; using

'mle': $\hat{\Sigma} = n^{-1} A = U L U^T$ $\quad U \in O(p)$, $L = \mathrm{diag}(l_i)$

Need to <u>shrink</u> (l_i):

e.g. $n = p = 10$ BUT $(l_i) = \underline{3.07}$ 1.40 1.12 0.78 0.51

$\Sigma = I$ (one sample!) 0.30 0.16 .095 .036 $\underline{.00}$

$\Rightarrow$ $\hat{\Sigma} = U \varphi(L) U^T$ Stein, Perlman-Lin, Haff, Berger, Loh, ...

$\rightarrow$ difficult to analyze for fixed n, p.

Classification, Regression, Clustering, ...

Non-null densities E.g. (l_i) from $A \sim W_p(n, \Sigma)$

$$f(l) = c\,|\Sigma|^{-\frac{n}{2}}\; {}_0\tilde{F}_0(-\tfrac{1}{2}\Sigma^{-1}, A) \prod_i l_i^{\frac{n-m-1}{2}} \prod_{i<j} (l_i - l_j)^{2-1}$$

$${}_0F_0(-\Sigma^{-1}, A) = \int_{\substack{U(p)\\ O(p)}} \operatorname{etr}\{-\Sigma^{-1} U A U^*\}\, dU$$

[$\Sigma = I \rightarrow$ Laguerre {U,O} E]

James, 1964 Hypergeom. functions of matrix argument

${}_0F_0$	exponential	χ^2, Wishart, $\lambda(A)$
${}_1F_0$	binomial	F, $\lambda(AB^{-1})$
${}_0F_1$	Bessel	non-central χ^2, Wishart, mea
${}_1F_1$	confluent hyper.	non-central F, $\lambda(AB^{-1})$
${}_2F_1$	Gaussian hyper.	multiple R^2, CCA

- influence on practice ?
- useful approximations via RMT ?

TRADITIONAL ASYMPTOTICS p fixed, $n \to \infty$

$$A = X^T X = U L U^T \sim W_p(n, \Sigma) \qquad \Sigma = \mathrm{diag}(\lambda_1 > \cdots > \lambda_r > 1 = \lambda_{r+1} = \cdots = \lambda_p)$$

$$\sqrt{\tfrac{n}{2}} \left(\tfrac{L}{n} - \lambda \right) \xrightarrow{\mathcal{D}} D \qquad \text{Girshick, 39} \to \text{Anderson, 63}$$

$$\begin{cases} D_1, \ldots, D_r \overset{ind}{\sim} N(0, \lambda_i^2) \\ D' = (D_{r+1}, \ldots, D_p) \sim GOE(p-r) \end{cases}$$

$$f_{D'}(d) = c\, e^{-\frac{1}{2}\sum_{r+1}^{p} d_j^2} \prod_{r<j<k\le p} |d_j - d_k|$$

Null case: $r = 0$; → not (traditionally) useful

Large n, p (+ q) asymptotics:

	Craddock Flood	Phoneme	Barnett-Preisendorfer	Waugh	Chemometrics
n	1095	161	600	137	56
p	130	256	126	4	6
q			33	5	22

First try: $p/n \to c_1 > 0$, $q/n \to c_2 > 0$

3. 'CASE STUDY' : **Largest eigenvalues in null cases:**

[PCA etc.] $\quad A = X^T X \sim \mathbb{C}W_p(n, I) \rightarrow l_1 \geq l_2 \geq \cdots \geq l_p$

$$f(l) = c \prod_i e^{-\frac{1}{2} l_i} \, l_i^{\frac{n-p-1}{2}} \prod_{i<j} (l_i - l_j)^{2-1}$$

Laguerre $\{U, O\}$ E

Empirical spectrum: Marčenko-Pastur + ...

Strong law for l_1 : Geman, + ... Bai (99)

Tracy-Widom limit for rescaled l_1 : Johansson 2000 $\mathbb{C}$

IMJ 2001 $\mathbb{R}$.

[CCA etc.] $\quad A \sim W_p(q, I)$, $B \sim W_p(n-q, I)$ } indep, $\quad p \leq q, n-q$

$u_1 \geq u_2 \geq \cdots \geq u_p$ solve $\det[A - u(A+B)] = 0$

$$f(u) = c \prod_i u_i^{\frac{1}{2}(q-p-1)} (1-u_i)^{\frac{1}{2}(n-q-p-1)} \prod_{i<j} (u_i - u_j)^{2-1}$$

Jacobi $\{U, O\}$ E

Fisher, Girshick, Hsu, Mood, Roy 1939

LIMITING EMPIRICAL SPECTRUM

Wachter 1980

$$A \sim W_p(q, I) \perp\!\!\!\perp B \sim W_p(n-q, I) \qquad p \le q, n-q$$

$$|A - u_i(A+B)| = 0 \qquad u_1 \ge u_2 \ge \cdots \ge u_p \ge 0$$

NOTE: $u_i = r_i^2$ squared correlation scale

ASSUME: $0 < \sin^2 \frac{\gamma_0}{2} \leftarrow \frac{p - \frac{1}{2}}{n-1} \le \frac{q - \frac{1}{2}}{n-1} \rightarrow \sin^2 \frac{\phi_0}{2}$

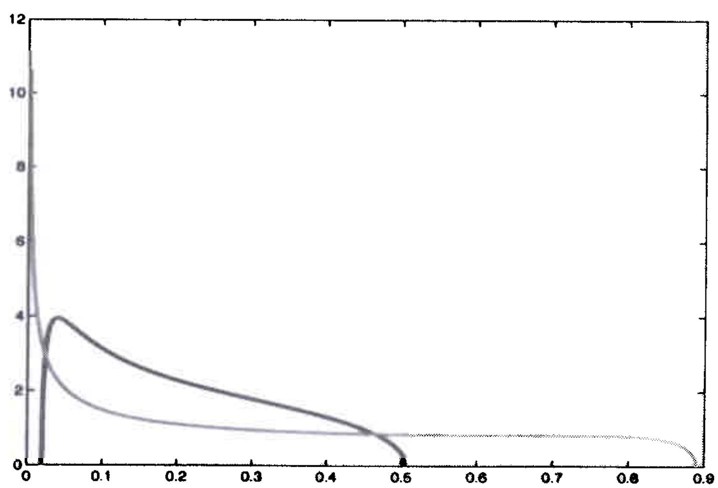
THEN: $F_p(u) = p^{-1} \# \{ i : u_i \le u \} \longrightarrow \int_0^u f(u')\, du'$

$$f(u) = \frac{c_u \sqrt{(\mu_+ - u)(u - \mu_-)}}{u(1-u)} \qquad c_u = 2\pi \sin^2 \frac{\gamma_0}{2}$$

where

$$\mu_\pm = \cos^2\left(\frac{\pi}{2} - \frac{\phi_0 \pm \gamma_0}{2}\right)$$

$f(u)$



u

$\uparrow$ μ_-

$\uparrow$ $\mu_+ \doteq (.7)^2$

$\uparrow$

$n = 200, \quad p = 20$
$q = 40$

"evil RA"

c.f. also

Bai - Yin - Krishnaiah
Silverstein ('85)
Bai (99)

LIMITING DISTRIBUTION FOR u_1

$$A \sim W_p(q, I) \perp\!\!\!\perp B \sim W_p(n-q, I)$$ (resp complex) Wishart

$$|A - u_i(A+B)| = 0 \qquad u_1 \geq u_2 \geq \cdots \geq u_p \qquad p \leq q, n-q$$

Let $p, q(p), n(p) \to \infty$

Set $$\sin^2 \frac{\gamma_p}{2} = \frac{p - 1/2}{n-1} \qquad \sin^2 \frac{\varphi_p}{2} = \frac{q - 1/2}{n-1}$$

$$\mu_{p+} = \cos^2\left(\frac{\pi}{2} - \frac{\varphi_p + \gamma_p}{2}\right)$$

$$\sigma_{p+}^3 = \frac{1}{(2n-2)^2} \frac{\sin^4(\varphi_p + \gamma_p)}{\sin \varphi_p \cdot \sin \gamma_p}$$

Assume $(\gamma_p, \varphi_p) \to (\gamma_0, \varphi_0)$ with $0 < \gamma_0 \leq \varphi_0$

Theorem $$\frac{u_1 - \mu_{p+}}{\sigma_{p+}} \xrightarrow{\mathcal{D}} W_1 \quad (\text{resp } W_2)$$

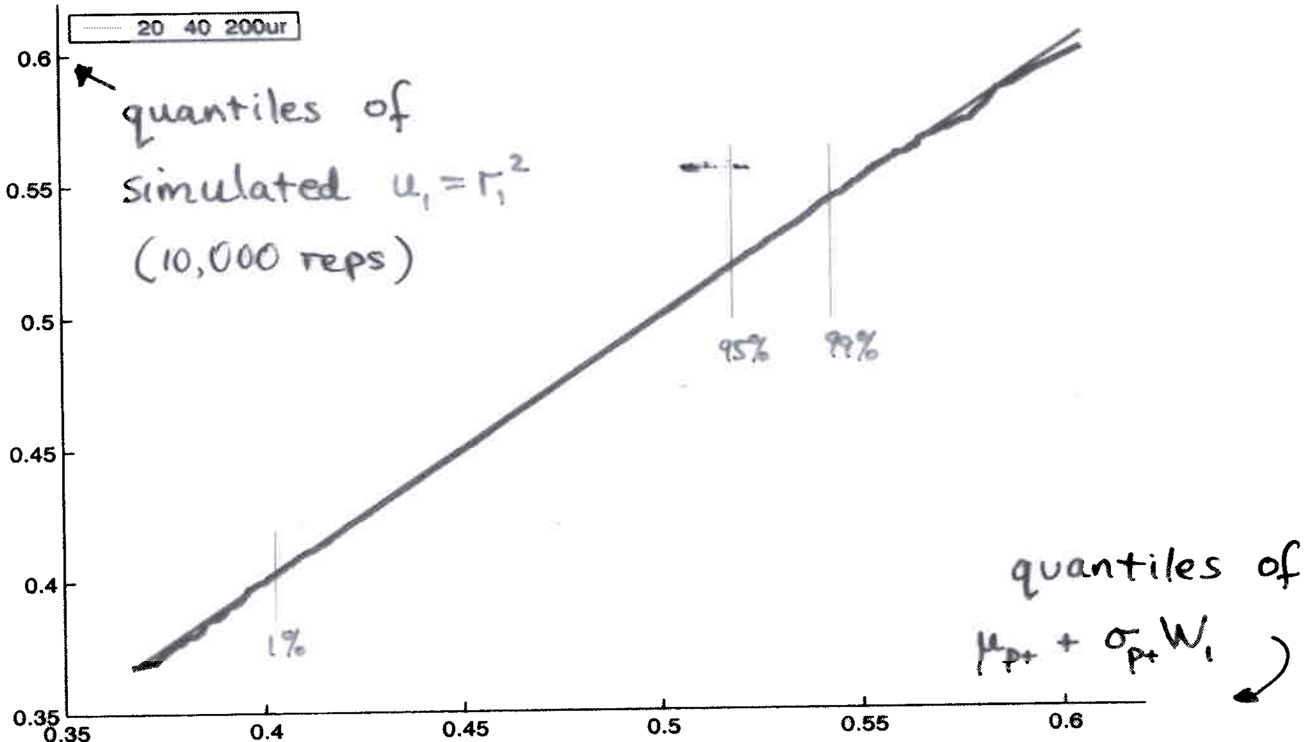
(Tracy–Widom laws)

|| Real case: $p - \frac{1}{2}$, $q - \frac{1}{2}$, $n-1$ v. important for numerics

– reduces error from $O(p^{-1/3})$ to $O(p^{-2/3})$

Complex case: tweak $\mu_{p+}^c, \sigma_{p+}^c$ to get $O(p^{-2/3})$

Finite (p,q,n) simulations of $u_1 = r_1^2$



Percentile for W_1	TW (μ, σ)	$\boxed{20,40,200}$ (.494, .024)	5,10,50 (.478, .061)	p, q, n 2,4,20 (.442, .117)	2 * SE
-3.90	.01	.010	.008	.000	(.002)
-3.18	.05	.052	.049	.009	(.004)
-2.78	.10	.104	.099	.046	(.006)
-1.91	.30	.311	.304	.267	(.009)
-1.27	.50	.507	.506	.498	(.010)
-0.59	.70	.706	.705	.711	(.009)
0.45	.90	**.904**	**.910**	**.911**	(.006)
0.98	.95	**.950**	**.955**	**.958**	(.004)
2.02	.99	**.990**	**.992**	**.995**	(.002)

$\hat{P}\left[\frac{u_1 - \mu_p}{\sigma_p} \le 0.45 \;\middle|\; (n,p,q) = (20,40,200)\right]$

REMARKS

- $p^{-2/3}$ scale of variability for u_1

- 95 % tile $\doteq$ $\mu_{p+} + \sigma_{p+}$

  99 % tile $\doteq$ $\mu_{p+} + 2\sigma_{p+}$

- if $\mu_{p+} > .8$ better approx on logit scale:

$$v_i = \log u_i/(1-u_i)$$

$$\mu_{v+} = \log \frac{\mu_{p+}}{1-\mu_{p+}} \qquad \sigma_{v+} = v'(\mu_{p+})\,\sigma_{p+} = \frac{\sigma_{p+}}{\mu_{p+}(1-\mu_{p+})}$$

- <u>Smallest</u> eigenvalue : previous assumptions <u>and</u> $\gamma_0 < \varphi_0$

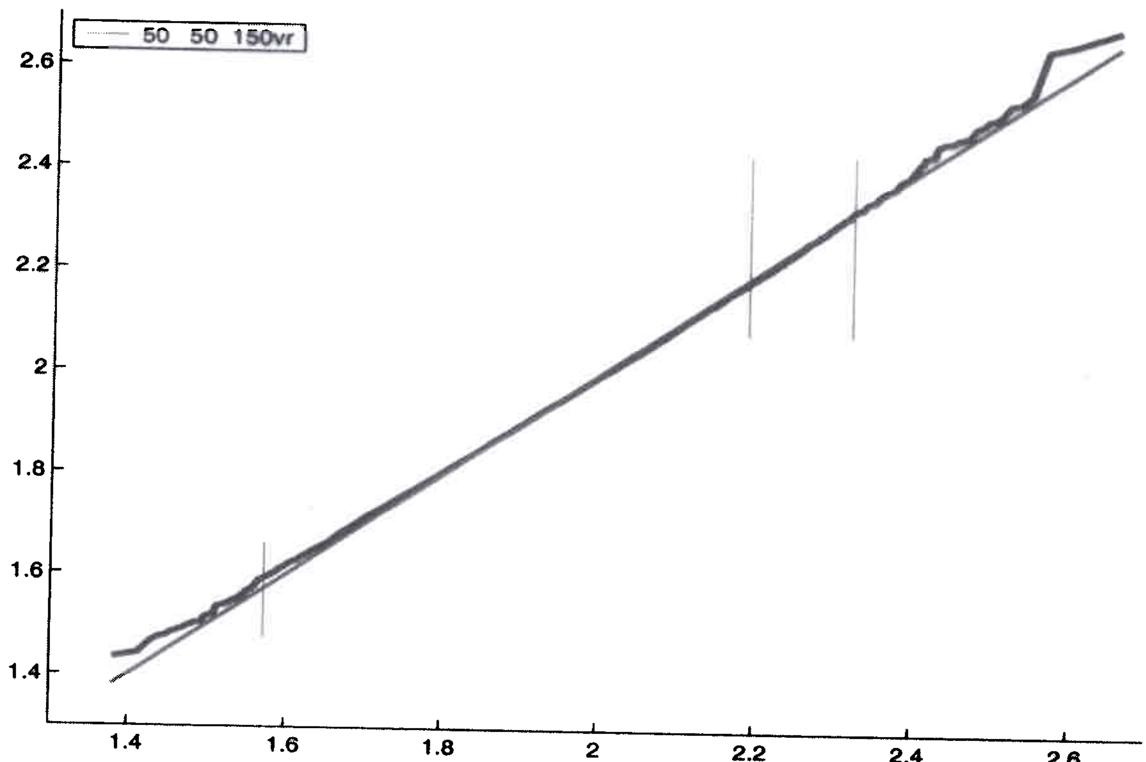
$$\sigma_{p-}^3 = \frac{1}{(2n-2)^2} \frac{\sin^4(\varphi_p - \gamma_p)}{\sin\varphi_p \cdot \sin\gamma_p}$$

then $\dfrac{\mu_{p-} - u_p}{\sigma_{p-}} \xrightarrow{\mathcal{D}} W_1 \quad (W_2)$

- Corresponding limit dist^{ns} for $u_2 \geq \cdots \geq u_k$

  $u_{p-k} \geq \cdots \geq u_{p-1}$

  k <u>fixed</u>.



Percentile	TW (μ, σ)	50,50,150 (2.06, .127)	5,5,15 (1.93, .594)	p, q, n, 2,2,6 (1.69, 1.11)	2 * SE
-3.90	.01	.007	.002	.010	(.002)
-3.18	.05	.042	.023	.037	(.004)
-2.78	.10	.084	.062	.074	(.006)
-1.91	.30	.289	.262	.264	(.009)
-1.27	.50	.499	.495	.500	(.010)
-0.59	.70	.708	.725	.730	(.009)
0.45	.90	.905	**.919**	**.931**	(.006)
0.98	.95	.953	**.959**	**.966**	(.004)
2.02	.99	.990	**.991**	**.993**	(.002)

QUALITY OF APPROXIMATION: AN ANALOGY

	t-statistic $\sqrt{n}\bar{x}/s$	largest root l_1 of S
Model	$x_i \overset{ind}{\sim} N(\mu, \sigma^2)$ $H_0: \mu = 0$	$\vec{X}_i \sim N(0, \Sigma)$ $H_0: \Sigma = I$
Exact Law	$t \sim t_{n-1}$	$l_1 \sim LOE(p, n-p)$
Approx Law	$\Phi(x) = \int_{-\infty}^{x} \phi(s)\,ds$	$F_1(x) = \exp\{-\frac{1}{2}\int_x^\infty q(s) + (x-s)^2 q(s)\, d$

Second-Order Accuracy

$$P\{t_{(n-1)} \le x\} = \Phi(x) + O\left(\left(\tfrac{1}{\sqrt{n}}\right)^2\right)$$

'Correlation' functions

$$\sigma_{np} S_p(\mu_{np} + \sigma_{np} x, \mu_{np} + \sigma_{np} y)$$

$$\to S(x, y) + O(p^{-2/3})$$

<u>Conj:</u> $P\{l_1 \le \mu_{np} + \sigma_{np} x\}$

$$= F_1(x) + O(p^{-2/3})$$

Robustness to Model

- analogous issues
- Soshnikov (2001)

Convergence Strategy for Extreme Eigenvalues

$$f(x_1, \dots, x_N) = \frac{1}{N!} (q) \det \left[K_{N(1)} (x_i, x_j) \right]$$

Correlation functions:

$$\varrho_{N,k} (x_1, \dots, x_k) = (q) \det \left[K_{N(1)} (x_i, x_j) \right]$$

[Soshnikov] Convergence of moments $\int_{I_k} \varrho_{N,k} \to \int_{I_k} \varrho_k \quad \forall I_k$

(+ ...) implies, for determinantal point fields F_N, F

$$\mathcal{L}\{x_{(1)}, \dots, x_{(k)} \mid F_N\} \xrightarrow[N \to \infty]{} \mathcal{L}\{x_{(1)}, \dots, x_{(k)} \mid F\} \qquad \underline{k \text{ fixed}}$$

so, need (uniform) convergence of rescaled kernel

$$\sigma_N K_N(\mu_N + \sigma_N s, \mu_N + \sigma_N t) \to K_A(s,t)$$

AND to reduce error from $O(N^{-1/3})$ to $O(N^{-2/3})$

need good approx^{ns} to center μ_N

scale σ_N

JACOBI $\to$ AIRY To show

$$\sigma_N K_N(\mu_N+\sigma_N s, \mu_N+\sigma_N t) \to K_A(s,t) = \frac{1}{s-t}\begin{vmatrix} Ai(s) & Ai(t) \\ Ai'(s) & Ai'(t) \end{vmatrix}$$

use Christoffel - Darboux property

$$K_N(x,y) = \sum_{k=0}^{N-1} \phi_k(x)\,\phi_k(y) = \frac{a_N}{x-y}\begin{vmatrix} \varphi_N(x) & \varphi_N(y) \\ \varphi_{N-1}(x) & \varphi_{N-1}(y) \end{vmatrix}$$

So, need to show $\boxed{\zeta N^{-1/6}\varphi_N(\mu_N+\sigma_N s) \simeq Ai(s)} + O(N^{-2/3})$

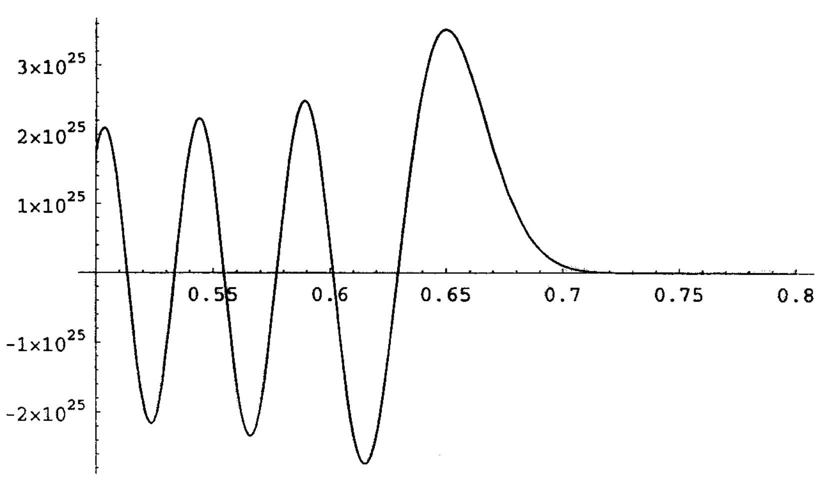
where $\varphi_N(x) = h_N^{-1/2}(1-x)^{\alpha/2}(1+x)^{\beta/2}P_N^{\alpha,\beta}(x)$,

and, treble asymptotics: $\begin{pmatrix} \alpha \\ \beta \\ N \end{pmatrix} = \begin{pmatrix} n-p-q \\ q-p \\ p \end{pmatrix} \to \infty$

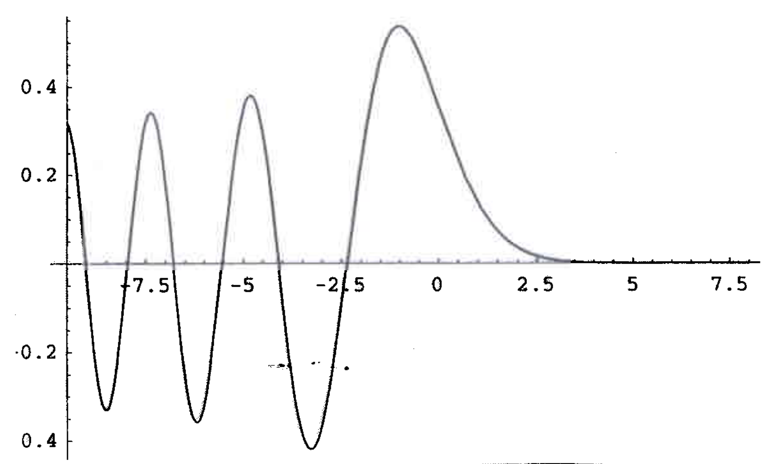
<u>Example</u>:

$(\alpha, \beta, N) = (200, 100, 100) \iff (p, q, n) = (100, 200, 500)$

$(1-x)^{\alpha/2}(1+x)^{\beta/2}P_N^{\alpha,\beta}(x)$



$Ai(s)$



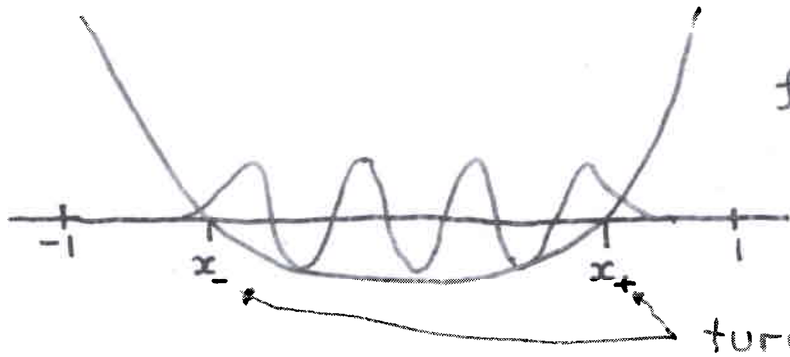
$\boxed{Ai''(s) = s\,Ai(s)}$

LIOUVILLE - GREEN asymptotics for ortho polynomials

e.g. Laguerre $L_N^{\alpha}(x)$, Jacobi $P_N^{\alpha,\beta}(x)$ α,β,N large

Satisfy 2nd order o.d.e.

$$(*) \qquad \frac{d^2 w}{dx^2} = \{\kappa^2 f(x) + g(x)\} w(x) \qquad \kappa = \kappa(N, \alpha, \beta) \text{ large}$$



$$f(x) = \frac{(x - x_-)(x - x_+)}{4(1-x^2)^2}$$

Transform $\quad \dfrac{d^2 W}{d^2 \zeta} = [\kappa^2 \zeta + \psi(\zeta)]\, W(\zeta)$

$$\zeta^{3/2}(x) = \int_{x_+}^{x} \sqrt{f}\,(y)\, dy$$

ERROR BOUND (Olver, 74) $\exists$ solution of $(*)$ with

$$w(x,\kappa) = \hat{f}^{-1/4}(x) \left\{ Ai(\kappa^{2/3}\zeta) + \varepsilon(x,\kappa) \right\} \qquad \hat{f}(x) = \frac{f(x)}{\zeta}$$

$$|\varepsilon(x,\kappa)| \leq |Ai|(\kappa^{2/3}\zeta) \left\{ \exp\left[\frac{c_0}{\kappa} V(\zeta)\right] - 1 \right\}$$

so long as $\quad V(\zeta) = \displaystyle\int_{\zeta}^{\infty} |\psi(v)|\, v^{-1/2}\, dv < \infty$

Virtues of Error Bound

- constrains approx^n : $V(\zeta)$ convergent

$$\Rightarrow \quad g(x) \sim \frac{-1/4}{(1-x)^2} \quad \Rightarrow \quad \kappa_N = 2N + \alpha + \beta + 1$$

$\Rightarrow$ uniform error bounds in x : $|\varepsilon(x,\kappa)| = O(1/N)$

Centering + Scaling constants: $x_N = x_+$

σ_N by matching: $\kappa^{2/3}\,\zeta(x_N + \sigma_N s) \simeq s$ $\quad s$ small

$$\Rightarrow \quad \sigma_N = \frac{1}{\kappa^{2/3}\,\dot{\zeta}(x_N)} \quad \Rightarrow \quad \sigma_N^3 = \frac{2\sin^4(\varphi+\gamma)}{\kappa^2 \sin\varphi \,\sin\gamma}$$

$\Rightarrow$ Jacobi $\rightarrow$ Airy approx^n: with $x = x_N + \sigma_N s$

$$\phi_N(x) = \left(\frac{\sigma_N \kappa_N}{1-x^2}\right)^{1/2} \left[Ai(s) + O(N^{-2/3}) \right]$$

Similarly for ϕ_{N-1} :

$$x_{N-1} = x(N-1, \alpha, \beta)$$
$$\sigma_{N-1} = \sigma(N-1, \alpha, \beta)$$

Lazy Higher Order Approximation Seek <u>good</u> μ, σ with

$$\sigma K_N(\mu + s\sigma, \mu + t\sigma) = K_A(s,t) + R_N \qquad (\dagger)$$

From Christoffel-Darboux,

$$K_N(x,y) = a_N \frac{\phi_N(x)\phi_{N-1}(y) - \phi_N(y)\phi_{N-1}(x)}{x-y}, \quad \text{with}$$

$$\phi_N(x) = \left(\frac{\sigma_N K_N}{1-x^2}\right)^{1/2}\left[Ai(s_N) + O(N^{-2/3})\right], \qquad s_N = \frac{x - x_N}{\sigma_N}$$

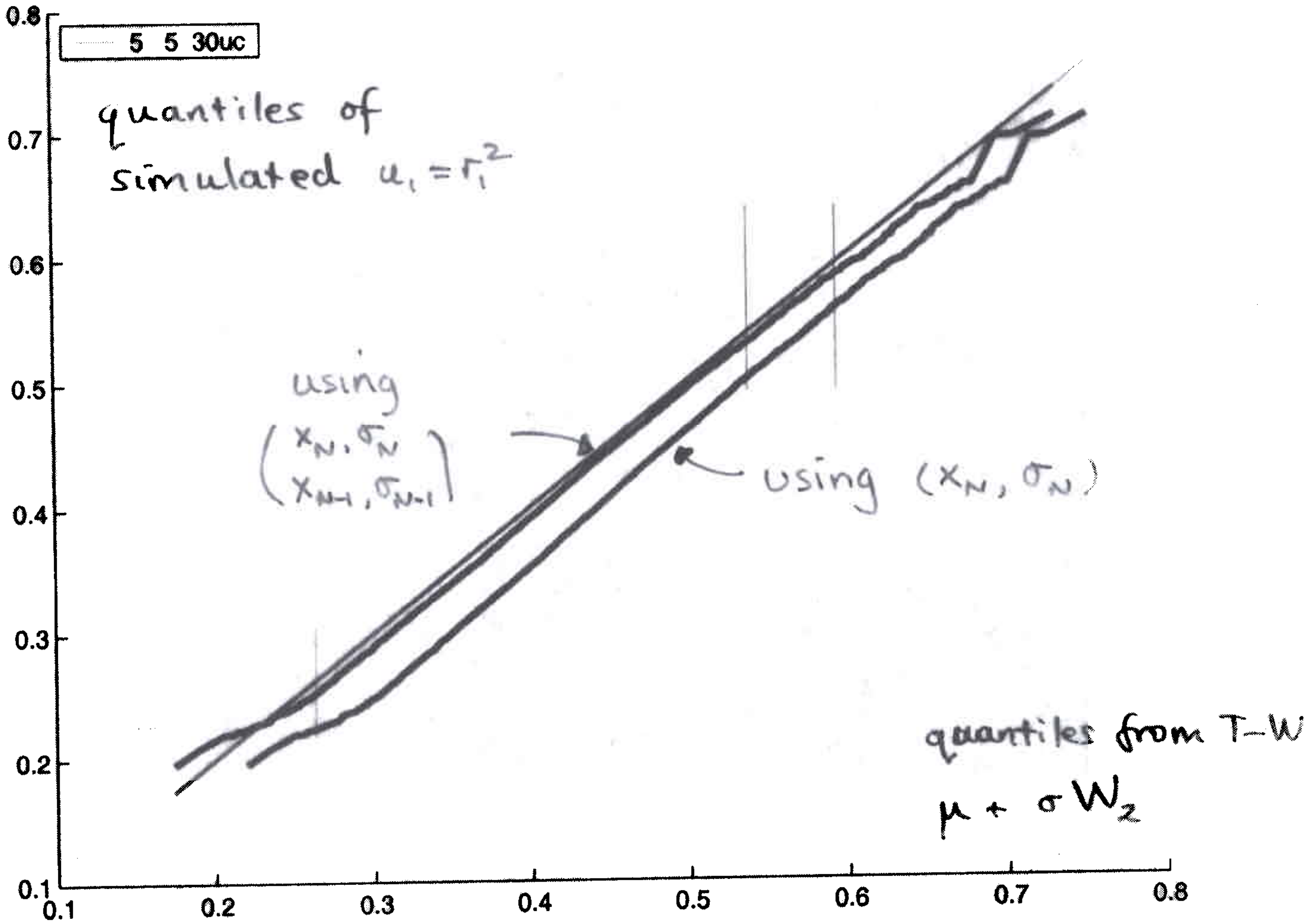
For ϕ_N use center x_N, scale σ_N,

but using (x_N, σ_N) in $(\dagger)$ $\Rightarrow R_N = O(N^{-1/3})$

For ϕ_{N-1} use center x_{N-1}, scale σ_{N-1}, $s_{N-1} = \dfrac{x - x_{N-1}}{\sigma_{N-1}}$

and with $s = \dfrac{s_N + s_{N-1}}{2}, \ldots$ $\Rightarrow \boxed{R_N = O(N^{-2/3})}$

$$\Rightarrow \quad \mu = \frac{x_N/\sigma_N + x_{N-1}/\sigma_{N-1}}{1/\sigma_N + 1/\sigma_{N-1}}, \qquad \frac{1}{\sigma} = \frac{1}{2}\left(\frac{1}{\sigma_N} + \frac{1}{\sigma_{N-1}}\right)$$



5 5 30uc
quantiles of simulated $u_1 = r_1^2$
using $\begin{pmatrix} x_N, \sigma_N \\ x_{N-1}, \sigma_{N-1} \end{pmatrix}$
using (x_N, σ_N)
quantiles from T-W
$\mu + \sigma W_2$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

THE REAL CASE + higher order approximation

$$\rho_{N,k}(x_1, \dots, x_k) = \underset{1 \le i,j \le k}{\mathrm{q\,det}} \left[f_{N,1}(x_i, x_j) \right]$$

$$f_{N,1}(x,y) = \begin{bmatrix} K_{N,1}(x,y) & I_{N,1}(x,y) \\ D_{N,1}(x,y) & S_{N,1}(x,y) \end{bmatrix}$$

Laguerre OE : closed form representation in Widom, 99

Jacobi OE : use Adler - Forrester - Nagao - van Moerbeke, OC

$$K^{\alpha-1,\beta-1}_{N,1}(x,y) = \underbrace{\sqrt{\frac{1-x^2}{1-y^2}}\, K^{\alpha,\beta}_{N-1,2}(x,y)} + \underbrace{a_{N-1} \frac{k_{N-1}}{2} \psi_{N-1}(y)(\epsilon * \psi_{N-2})(x)}$$

$\psi_N(x) = \phi_N(x) / \sqrt{1-x^2}$

$\epsilon(x) = \frac{1}{2} \operatorname{sgn} x$

$x = x_{N-1} + s\,\sigma_{N-1}$

$y = \dots \; t \dots$

$$K_A(s,t) - \frac{\Delta_N}{2} Ai(s) Ai(t) + O(N^{-2/3})$$

$$\frac{1}{2} Ai(t) \int_{-\infty}^{s} Ai(u)\, du + \frac{\Delta_N}{2} Ai(s) Ai(t) + O(N^{-2/3})$$

- 'miraculous' cancellation of $O(N^{-1/3})$ terms

$\Rightarrow$ much improved small-(n,p,q) approx^{n}.

SOME OPEN QUESTIONS

E.g. 'Non-white' Wishart $\qquad A \sim \mathbb{C}W_p(n, \Sigma)$

$$f(l;\Sigma) = c|\Sigma|^{-n} \prod_i l_i^{n-p} \int_{U(p)} \mathrm{etr}(-\Sigma^{-1} U A U^*)\, dU\, \Delta^2(l)$$

(James)

$$= c \prod_i \lambda_i \prod_i l_i^{n-p} \frac{\det[e^{-l_k/\lambda_j}]\, \Delta(l)}{\Delta(\lambda_j^{-1})}$$

(HCIZ)

$\rightarrow$ determinantal correlation functions

Distribution of l_1 (or $l_2, \dots$) as $n/p \to \gamma$. ?

E.g. a) $\Sigma = (\lambda_1, \dots, \lambda_\eta > 1 = \lambda_{\eta+1} = \dots = \lambda_p)$ $\qquad \eta$ FIXED

$$\lambda_\eta \simeq n^{-1}(\sqrt{n} + \sqrt{p})^2$$

b) power law $\quad \lambda_k = c k^{-\beta}$

Progress :
- Jinho Baik on (a)
- S. Peché + G. Ben Arous

A Vaguer List:

- Extreme eigenvalues:
  - 'Bootstrap' C.I's + validity as $n/p \to \gamma$?
  - strong law behavior for general Σ

- Eigenvectors:
  - Consistency, asymptotic dist^n as $n/p \to \gamma$
  - Effect of regularization (FDA)
  - 'Sparse' versions of PCA (Lu)

- Empirical dist^n of eigenvalues
  - (further) statistical uses of Marčenko-Pastur
  - use of CLT for $\Sigma\, h(l_i)$

- Estimation of Large covariance matrices
  - sparsity for non-unit variances
  - Bayes estimates (Guionnet)

- Classification / Clustering
  - models for large covariance structures (Bickel-Levina)

- Issues from application domains
  - climate studies, face recognition, document retrieval ..