ICMP 2000
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LONDON

SMALL EIGENVALUES OF LARGE HANKEL MATRICES

1. Introduction, Motivation History

2. Two Theorems

3. Example

$$w(x) = \text{Exp}(-x^{\beta})$$

$$\beta > 0, \quad 0 \leq x < \infty$$

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Hermitean Random Matrices

D_N = partition function

$$= \int e^{-\mathrm{tr}\, V(M)}\, dM \qquad \left(M \in \begin{array}{c} N\times N \\ \text{Hermitean} \\ \text{Matrices} \end{array} \right)$$

$$= \frac{\Omega_N}{N!} \int \cdots \int \prod_{1\le i<j\le N} |x_i - x_j|^2 \prod_k w(x_k)\, dx_k$$

$$= \det \left(S_{i+j} \right)_{i,j=0}^{N-1}$$

$$S_j = \int_{\mathbb{R}} x^j \, d\alpha(x) \,, \quad d\alpha(x) = e^{-V(x)} dx := w(x)\, dx$$

(moments) $\qquad j = 0, 1, 2, \ldots$

* $\mathcal{H}_N = \left(S_{i+j} \right)_{i,j=0}^{N+1}$, $(N+1)\times(N+1)$ Hankel or moments matrix

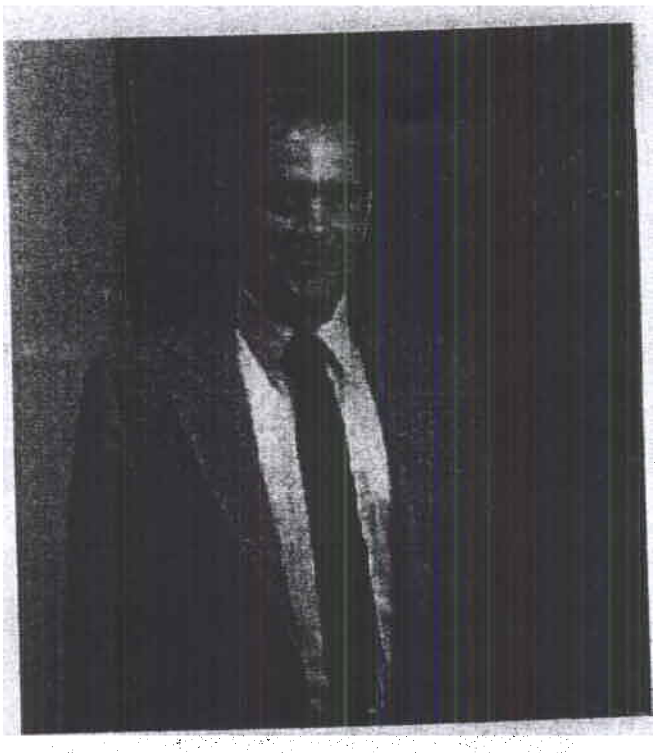
* λ_N : the smallest eigenvalue of $\mathcal{H}_N$

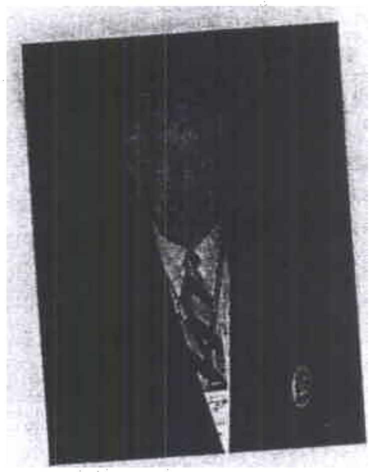
Szegö (1936) Computed λ_N, for Large N

$$* \begin{cases} \lambda_N \sim 2^{9/4} \pi^{3/2} \dfrac{N^{1/2}}{(\sqrt{2}+1)^{2N+3}} \,, & w(x) = \begin{cases} 1, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases} \\[2em] \lambda_N \sim 2^{15/4} \pi^{3/2} \dfrac{N^{1/2}}{(\sqrt{2}+1)^{4N+4}} \,, & w(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

$\frac{1}{\ldots+k+1}$

Harold Widom & Herbert Wilf





(1966). Suppose $w(x) > 0$, $x \in [a,b]$

Szegö Conditions

(i) w, absolutely cont.

(ii) $\displaystyle\int_a^b \frac{\ln w}{\sqrt{(b-x)(x-a)}}\,dx > -\infty$

$\lambda_N \sim A N^{\frac{1}{2}} B^{-N}$

A depends on w

B depends on a, b.

Szegö also studied two examples of infinite interval:

$$\lambda_N \sim 2^{\frac{13}{4}} \pi^{3/2} e N^{\frac{1}{4}} e^{-2\sqrt{2}\, N^{\frac{1}{2}}}, \quad w(x) = e^{-x^2}, \ x \in (-\infty, \infty),$$

$$\lambda_N \sim 2^{7/2} \pi^{3/2} e N^{\frac{1}{4}} e^{-4N^{\frac{1}{2}}}, \quad w(x) = e^{-x}, \ x \in [0, \infty)$$

Infinite interval cases are (more) interesting because the moment problems may become indeterminate.

Given $\{s_j\}_{j=0}^{\infty}$ find $d\alpha(x)$ or $w(x)$

if $d\alpha$ is uniquely determined, determinate

if there are more than one, indeterminate

* Rayleigh principle

$$\lambda_N = \min \left\{ \sum_{0 \le j,k \le N} s_{j+k} v_j v_k : \sum_{j=0}^{N} v_j^2 = 1, \ v_j \in \mathbb{R} \right\}$$

NOTE: λ_N decreases with N.

* Theorem. The moment problem assoc. with

$$s_j = \int_{\mathbb{R}} x^j \, d\alpha(x)$$

is determinate if and only if $\lim_{N \to \infty} \lambda_N = 0$

Consider

$$\mu_N = \min\left\{ \sum_{j,k} S_{j+k} v_j v_k \ : \ v_0 = 1,\ v_j \in \mathbb{R} \right\}$$

Theorem (Hamburger) moment problem is determinate iff $\lim_{N\to\infty} \mu_N = 0$

NOTE: $\mu_N > \lambda_N$

Theorem' The moment problem is indeterminate iff $\lim_{N\to\infty} \lambda_N > 0$

$$\pi_N(x) = \sum_{j=0}^{N} v_j x^j, \quad v_j \in \mathbb{R}$$

$$\sum_{0\le j,k\le N} S_{j+k} v_j v_k = \int \pi_N^2(x)\, d\alpha(x)$$

$$\sum_{j=0}^{N} v_j^2 = \int_0^{2\pi} \pi_N(e^{i\theta})\, \pi_N(e^{-i\theta}) \frac{d\theta}{2\pi}$$

So far, λ_N is small for large N; it is easier (perhaps) to compute large numbers:

$$\frac{1}{\lambda_N} = \max\left\{ \sum_{j=0}^{N} v_j^2 : \sum_{j,k} S_{j+k} v_j v_k = 1 \right\}$$

$$= \max\left\{ \int_0^{2\pi} |\pi_N(e^{i\theta})|^2 \frac{d\theta}{2\pi} : \pi_N,\ \int \pi_N^2(x)\, d\alpha(x) = 1 \right\}$$

* Build $P_k(x)$, orthonormal w.r.t. $d\alpha(x)$

$$P_k(x) = (\#)\, x^k + (\blacksquare)\, x^{k-1} + \cdots$$

where $\# > 0$, $\int P_k(x) P_\ell(x)\, d\alpha = \delta_{k\ell}$

* Expand π_N in terms of P_k

$$\pi_N(x) = \sum_{j=0}^{N} c_j P_j(x)$$

$$\int_0^{2\pi} |\pi_N(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{0 \le j,k \le N} K_{jk}\, c_j c_k$$

where $K_{jk} = \int_0^{2\pi} P_j(e^{i\theta})\, P_k(e^{-i\theta}) \frac{d\theta}{2\pi}$

$$\int \pi_N^2\, d\alpha = \sum_{j=0}^{N} c_j^2 \quad , \left(\overline{K_{jk}} = K_{kj}\right)$$

$$\frac{1}{\lambda_N} = \max \left\{ \sum_{j,k} K_{jk}\, c_j c_k : c_j ,\ \sum_{j=0}^{N} c_j^2 = 1 \right\}$$

Since eigenvalues of $(K_{jk})_{j,k=0}^{N}$ are are positive

$$\frac{1}{\lambda_N} \le \sum_{j=0}^{N} K_{jj} = \int_0^{2\pi} \sum_{k=0}^{N} |P_k(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

If the moment problem is indeterminate $\sum_{j=0}^{\infty} |P_j(z_0)|^2$ converges

for z_0 in compact subset and $\operatorname{Im} z_0 \neq 0$.

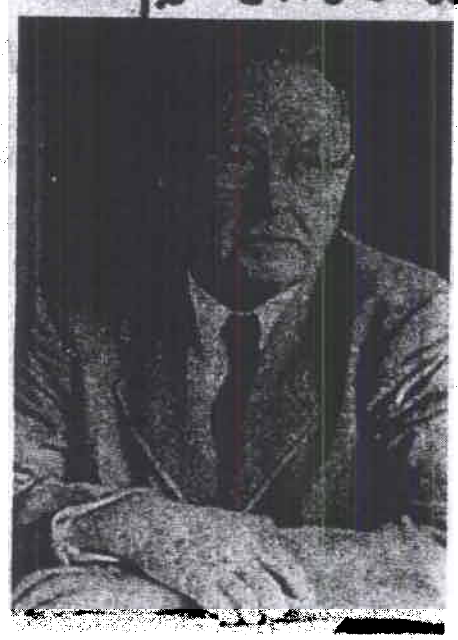
(see Akhiezer's Classical moment problems)

$$\frac{1}{\int_0^{2\pi} \frac{1}{\rho(e^{i\theta})} \frac{d\theta}{2\pi}} \leq \lambda_N$$

where $\rho(z_0) = \frac{1}{\sum_{j=0}^{\infty} |P_j(z_0)|^2}$

$\frac{\rho(z)}{|z-\bar{z}|}$ radius of the Weyl circle at z.

Akhiezer



Suppose $\lambda_N \geq \gamma$ for all N and where $\gamma > 0$, $\Rightarrow \frac{1}{\lambda_N} \leq \frac{1}{\gamma}$

Since $\frac{1}{\lambda_N}$ is the largest eigenvalue of $(K_{jk})_{(N+1)\times(N+1)}$,

$$\sum_{0 \leq j,k \leq N} K_{jk} c_j \bar{c}_k \leq \frac{1}{\gamma} \sum_{j=0}^{N} |c_j|^2$$

(or introducing $p(x) = \sum_{k=0}^{N} c_k P_k(x)$)

$$\int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq \frac{1}{\gamma} \int |p(x)|^2 d\alpha$$

Take $|z_0| < 1$,

$$p(z_0) = \int_0^{2\pi} \frac{p(e^{i\theta})}{e^{i\theta} - z_0} e^{i\theta} \frac{d\theta}{2\pi}$$

$$\Rightarrow |p(z_0)|^2 \leq \int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\theta} - z_0|^2} \frac{d\theta}{2\pi}$$

$$= \frac{1}{1 - |z_0|^2} \int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

$$\leq M \int |p(x)|^2 d\alpha$$

where $M = \dfrac{1}{\gamma(1 - |z_0|^2)}$

Apply above inequality to

$$p(x) = \sum_{k=0}^{N} P_k(\bar{z}_0)\, P_k(x)$$

$$p(z_0) = \sum_{k=0}^{N} |P_k(z_0)|^2$$

$$|p(z_0)|^2 = \left(\sum_{k=0}^{N} |P_k(z_0)|^2\right)^2$$

$$\leq M \sum_{0 \leq j,k \leq N} P_k(\bar{z}_0)\, P_j(z_0) \cdot \int P_k(x)\, P_j(x)\, d\alpha$$

$$= M \sum_{k=0}^{N} |P_k(z_0)|^2$$

$$\Rightarrow \sum_{k=0}^{N} |P_k(z_0)|^2 < M$$

Since N is arbitrary indeterminacy follows.

An example, $d\alpha = w\,dx$

$$w(x) = \exp(-x^{\beta})$$

$$\beta > 0, \quad 0 \le x < \infty \qquad S_{j+k} = \frac{1}{\beta}\Gamma\left(\frac{j+k+1}{\beta}\right)$$

* Need $K_{jk} = \int_0^{2\pi} P_j(e^{i\theta})\, P_k(e^{-i\theta})\, \frac{d\theta}{2\pi}$

Remark $\quad 0 < \beta < \frac{1}{2}$, moment problem indeterminate

$\beta = \frac{1}{2}$, 'Critical Point' (still determinate)

$\beta > \frac{1}{2}$, determinate

$\beta > \frac{1}{2}$ and $\beta \neq n + \frac{1}{2}$, $n = 1, 2, \ldots$

$$P_N(t) \sim \frac{(-1)^N}{\sqrt{2\pi}} \frac{1}{(-t\,C N^{\frac{1}{\beta}})^{1/4}} \quad \text{(Coulomb gas)}$$

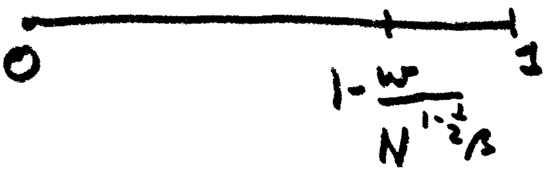
$$\cdot \exp\left(\frac{(-t)^{\beta}}{2} \sec \pi\beta + \frac{N^{1-\frac{1}{2\beta}}}{\sqrt{\pi C}} \sum_{r=0}^{[\beta - \frac{1}{2}]} \frac{(-1)^r A_r (-t)^{r+\frac{1}{2}}}{(C N^{\frac{1}{\beta}})^r}\right)$$

$t \notin [0, \infty)$, $N \to \infty$

A_r, $C > 0$ independent of N.

*** $N - \omega N^{\frac{1}{2\beta}} \leq j, k \leq N$, $\omega > 0$ fixed

$$\left(1 - \frac{\omega}{N^{1-\frac{1}{2\beta}}} \leq \frac{j}{N}, \frac{k}{N} \leq 1 \right)$$



** In the computation of

$$K_{jk} = \int_0^{2\pi} p_j(e^{i\theta}) \overline{p_k(e^{i\theta})} \frac{d\theta}{2\pi},$$

the dominant contribution comes from a small arc around $\theta = \pi$ (or $t = -1$), $[\pi - \varepsilon, \pi + \varepsilon]$

$$K_{j,k} \sim \frac{(-1)^{j+k}}{(\pi C)^{1/4}} \frac{1}{A_0^{1/2}} \frac{e^{\operatorname{Sec} \pi\beta}}{N^{\frac{1}{2}+\frac{1}{4\beta}}}, \quad A_0 > 0$$

$$\operatorname{Exp}\left(\frac{1}{\sqrt{\pi C}} \sum_{r=0}^{[\beta - \frac{1}{2}]} \frac{(-1)^r A_r}{C^r} \left(j^{1-\frac{1}{2\beta}-\frac{r}{\beta}} + k^{1-\frac{1}{2\beta}-\frac{r}{\beta}} \right)\right),$$

where for $\beta > \frac{1}{2}$, $0 \leq r \leq [\beta - \frac{1}{2}]$, $1 - \frac{1}{2\beta} - \frac{r}{\beta} \geq 0$. rapidly increasing "Kernels"

H.W., 1963

** $K_{jk} \sim (-1)^{j+k} K_{jj}^{1/2} K_{kk}^{1/2}$

(Kernel factors)

recall: $\frac{1}{\lambda_N} = \max\left\{\sum_{j,k} K_{jk}\, c_j \bar{c}_k \;:\; \sum_j |c_j|^2 = 1\right\}$

choose $c_j = \begin{cases} \sigma (-1)^j K_{jj}^{1/2}, & [N - \omega N^{\frac{1}{2\rho}}] \le j \le N \\ 0, & j < [N - \omega N^{\frac{1}{2\rho}}] \end{cases}$

where $\sum_{j=0}^{N} |c_j|^2 = 1 = \sigma^2 \sum_{j=[N-\omega N^{\frac{1}{2\rho}}]}^{N} K_{jj}$

* $\sum_{0 \le j,k \le N} K_{jk}\, c_j \bar{c}_k = \sum_{[N-\omega N^{\frac{1}{2\rho}}] \le j,k \le N} \sigma^2 (-1)^{j+k} K_{jj}^{1/2} K_{kk}^{1/2} K_{jk}$

$$\approx \sigma^2 \sum_{j,k=[N-\omega N^{\frac{1}{2\rho}}]}^{N} K_{jj} K_{kk}$$

$$= \sigma^2 \left(\sum_{j=[N-\omega N^{\frac{1}{2\rho}}]}^{N} K_{jj} \right)^2 = \sum_{j=[N-\omega N^{\frac{1}{2\rho}}]}^{N} K_{jj}$$

$$\Rightarrow \frac{1}{\lambda_N} \sim \sum_{j=0}^{N} K_{jj}$$

Replace sum by integral & Laplace

$\beta > \frac{1}{2}$ and $\beta \neq n + \frac{1}{2}$,

$$\frac{1}{\lambda_N} \sim \frac{1}{8\pi^{3/4}} \frac{1}{A_0^{1/2}} \frac{e^{\sec \pi\beta}}{N^{\frac{1}{2}-\frac{1}{4\beta}}}$$

$$\cdot \operatorname{Exp}\left(\frac{2N}{\sqrt{\pi C}}^{1-\frac{1}{2\beta}} \sum_{r=0}^{[\beta-\frac{1}{2}]} \frac{(-1)^r A_r}{C} N^{1-\frac{r}{\beta}} \right)$$

$* \ \beta = \frac{1}{2}$, $\quad p_N(t) \sim \frac{(-1)^N}{2\pi} \frac{1}{(-t)^{1/4}} \frac{1}{\sqrt{N}}$

$$\cdot \operatorname{Exp}\left(\frac{\sqrt{-t}}{\pi} \ln\left(\frac{4\pi N e}{\sqrt{-t}}\right)\right)$$

But kernel decreases, previous argument no longer holds

$$\left(\begin{array}{l} \beta = \frac{1}{2}, \\ \lambda_N \sim \dfrac{8\pi\sqrt{\ln(4\pi N e)}}{(4\pi N e)^{2/\pi}} \quad ? \end{array} \right)$$

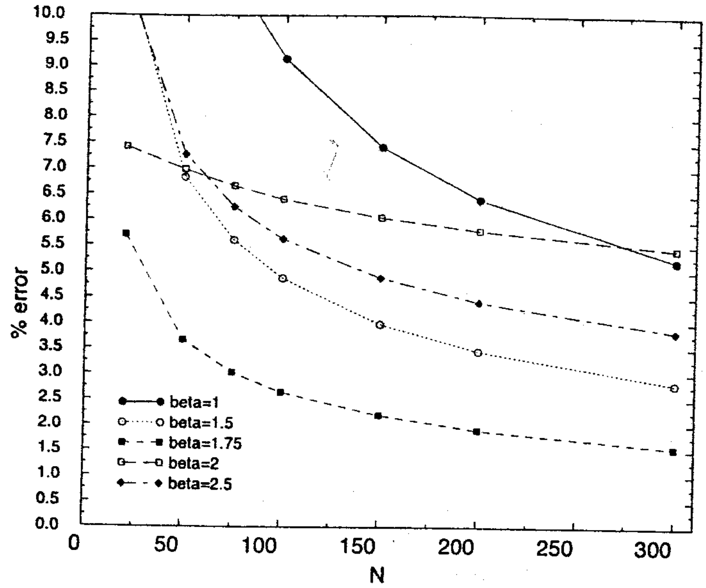


Figure 1. The percentage error of the theoretical values of λ_N when compared with those obtained numerically, for various β.

Table 1. Numerical and theoretical values of λ_N for various β.

β	N	Numerical λ_N	Theoretical λ_N
1	50	2.0948×10^{-10}	2.3695×10^{-10}
	100	2.1079×10^{-15}	2.3006×10^{-15}
	150	2.9551×10^{-19}	3.1743×10^{-19}
	200	1.6387×10^{-22}	1.7437×10^{-22}
	300	5.5215×10^{-28}	5.8090×10^{-28}
$\frac{3}{2}$	50	6.4066×10^{-22}	6.8438×10^{-22}
	100	6.2353×10^{-36}	6.5384×10^{-36}
	150	9.9476×10^{-48}	1.0343×10^{-47}
	200	2.8132×10^{-58}	2.9101×10^{-58}
	300	4.6009×10^{-77}	4.7300×10^{-77}
$\frac{7}{4}$	50	6.4483×10^{-27}	6.6844×10^{-27}
	100	1.6976×10^{-45}	1.7424×10^{-45}
	150	1.5193×10^{-61}	1.5525×10^{-61}
	200	3.9265×10^{-76}	4.0009×10^{-76}
	300	1.4844×10^{-102}	1.5074×10^{-102}
2	50	2.7356×10^{-31}	2.5449×10^{-31}
	100	3.8907×10^{-54}	3.6415×10^{-54}
	150	2.9557×10^{-74}	2.7769×10^{-74}
	200	8.9775×10^{-93}	8.4574×10^{-93}
	300	9.5593×10^{-127}	9.0396×10^{-127}
$\frac{5}{2}$	50	2.2384×10^{-38}	2.4010×10^{-38}
	100	1.2580×10^{-68}	1.3288×10^{-68}
	150	5.3195×10^{-96}	5.5789×10^{-96}
	200	1.2155×10^{-121}	1.2691×10^{-121}
	300	1.5236×10^{-169}	1.5819×10^{-169}