Z-MEASURES ON PARTITIONS AND THEIR SCALING LIMITS

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Abstract. We study certain probability measures on partitions of $n = 1, 2, \ldots$, originated in representation theory, and demonstrate their connections with random matrix theory and multivariate hypergeometric functions.

Our measures depend on three parameters including an analog of the β parameter in random matrix models. Under an appropriate limit transition as $n \to \infty$, our measures converge to certain limit measures, which are of the same nature as one–dimensional log–gas with arbitrary $\beta > 0$.

The first main result says that averages of products of "characteristic polynomials" with respect to the limit measures are given by the multivariate hypergeometric functions of type (2,0). The second main result is a computation of the limit correlation functions for the even values of β.

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Introduction

The goal of this paper is to study certain measures on partitions which are in many ways similar to log–gas (random matrix) models with arbitrary $\beta = 2\theta$. The measures give rise to discrete (lattice) models. They admit nontrivial scaling limits which have representation theoretic origin. The limit objects can be viewed as random point processes on the real line.

In our earlier works [P.I–P.V], [BO1–3], [Bor], we thoroughly studied the simplest case $\theta = 1$. In that case, the correlation functions in the discrete and continuous pictures were explicitly computed in terms of the Gauss hypergeometric function and the Whittaker function. Our goal is to see to what extent these results can be carried over to the general θ.

As for the log–gas models, it seems to be very hard to compute the correlation functions for general θ. However, one can evaluate other quantities of interest. In [Aom], [Ka], [BF] the authors computed the averages of products of characteristic

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polynomials in random matrix type ensembles for general θ. The answer is always given in terms of a multivariate hypergeometric function.

Our first result is of the same kind: we show that in our model, the averaged product of the natural analogs of characteristic polynomials is given by the multivariate hypergeometric functions of type (2,1) or (2,0).

The main difference of our situation, as compared to random matrices, is that we are dealing with the infinite number of particles. In a degenerate situation, our model turns into the Laguerre ensemble of the random matrix theory, and we recover known results of [Ka], [BF].

Our second result states that for integral θ we can extract the correlation functions of our measures from the averages of the "characteristic polynomials". The correlation functions are given by hypergeometric functions with repeated arguments. For similar results in the random matrix context, see [BF], [F1, section 4], [Ok1], and references therein.

Finally, our third result is a computation of a scaling limit of the correlation functions for integral θ. This limit transition is similar to the bulk scaling limit in the random matrix ensembles. The limit correlation functions are translation invariant and are given in terms of the A–type spherical function of Heckman–Opdam.

The paper is organized as follows. In §1 we introduce a family of measures on partitions depending on two parameters and explain that these measures must have a scaling limit as the size of partitions tends to infinity. In §2 we compute, in terms of hypergeometric functions, the averages of products of "characteristic polynomials" with respect to the limit measures. In §3 we relate, for the integral values of θ, the lattice correlation functions and averages of analogs of characteristic polynomials for partitions. In §4 we prove that the lattice correlation functions converge to the correlation functions of the limit measure in the appropriate scaling limit. In §5 we express the limit correlation functions through the hypergeometric functions. In §6 we compute the "tail asymptotics" of the limit correlation functions, which leads to a translation invariant answer.

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1. Z–measures

Let $\mathbb{Y}_n$ be the set of all partitions of a natural number n (equivalently, the set of all Young diagrams with n boxes). For any $n = 1, 2, \ldots$, we consider a three-parameter family of probability measures $M_{z,z',\theta}^{(n)}$ on $\mathbb{Y}_n$ given by

$$M_{z,z',\theta}^{(n)}(\lambda) = \frac{n! \, (z)_{\lambda,\theta}(z')_{\lambda,\theta}}{(t)_n H(\lambda,\theta) H'(\lambda,\theta)} , \qquad (1.1)$$

where we use the following notation:

$z, z' \in \mathbb{C}$ and $\theta > 0$ are parameters (admissible values of (z, z') are described below) and $t = zz'/\theta$;

λ is a Young diagram with n boxes;

$$(t)_n = t(t+1) \cdots (t+n-1) = \frac{\Gamma(t+n)}{\Gamma(t)}$$

is the Pochhammer symbol;

$$(z)_{\lambda,\theta} = \prod_{(i,j)\in\lambda} (z + (j-1) - (i-1)\theta) = \prod_{i=1}^{\ell(\lambda)} (z - (i-1)\theta)_{\lambda_i}$$

is a multidimensional analog of the Pochhammer symbol (here $(i,j) \in \lambda$ stands for the box in the ith row and jth column of the Young diagram λ, and $\ell(\lambda)$ denotes the number of rows of λ);

$$H(\lambda,\theta) = \prod_{(i,j)\in\lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1),$$

$$H'(\lambda,\theta) = \prod_{(i,j)\in\lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta),$$

where λ' denotes the transposed diagram.

One can easily see that

$$M^{(n)}_{z,z',\theta}(\lambda) = M^{(n)}_{-z/\theta,-z'/\theta,1/\theta}(\lambda').$$

Note that for any fixed λ, $M^{(n)}_{z,z',\theta}(\lambda)$ is a rational function in z, z', θ.

Proposition 1.1.
$$\sum_{\lambda\in\mathbb{Y}_n} M^{(n)}_{z,z',\theta}(\lambda) \equiv 1.$$

Proof. See [Ke2], [BO4].

Proposition 1.2. *The expression* (1.1) *for* $M^{(n)}_{z,z',\theta}(\lambda)$ *is strictly positive for all* $n = 1, 2, \ldots$ *and all* $\lambda \in \mathbb{Y}_n$ *if and only if*:

• *either* $z \in \mathbb{C} \setminus (\mathbb{Z}_{\leq 0} + \mathbb{Z}_{\geq 0}\,\theta)$ *and* $z' = \bar{z}$ (*the principal series*)

• *or, under the additional assumption that* θ *is rational, both* z, z' *are real numbers lying in one of the intervals between two consecutive numbers from the lattice* $\mathbb{Z} + \mathbb{Z}\theta$ (*the complementary series*).

Proof. We have to find necessary and sufficient conditions under which

$$\frac{\prod_{(i,j)\in\lambda}(z + c_\theta(i,j))(z' + c_\theta(i,j))}{(zz')(zz' + \theta)\ldots(zz' + (n-1)\theta)} > 0, \qquad \text{where } c_\theta(i,j) := (j-1) - (i-1)\theta,$$

for any $n = 1, 2, \ldots$ and any $\lambda \in \mathbb{Y}_n$. In the particular case $\theta = 1$ this was proved in [P.I, Proposition 2.2]. The same argument works with minor modifications.

Sufficiency: Our conditions imply that $(z + c_\theta(i,j))(z' + c_\theta(i,j)) > 0$ for any (i,j), so that the numerator is always strictly positive. They also imply $zz' > 0$, so that the denominator is strictly positive, too.

Necessity: For any (i,j) and any n large enough there exist diagrams $\lambda \in \mathbb{Y}_n$ and $\mu \in \mathbb{Y}_{n-1}$ such that $\mu \subset \lambda$ and $\lambda \setminus \mu = \{(i,j)\}$. Dividing the expression corresponding to λ by that corresponding to μ we see that

$$\frac{(z+c)(z'+c)}{(zz' + n\theta)} > 0, \qquad c = c_\theta(i,j).$$

Note that c can take any value from the set $(\mathbb{Z}_{\geq 0} + \mathbb{Z}_{\leq 0}\,\theta) \subset \mathbb{R}$.

Letting $n \to \infty$ we conclude that the numerator $(z+c)(z'+c)$ must be real and strictly positive for any c from the set indicated above. It follows that both zz' and $z+z'$ are real. Hence, either z, z' are complex conjugate to each other or they are both real.

In the former case, the inequality $z + c \neq 0$ implies that $z \notin (\mathbb{Z}_{\leq 0} + \mathbb{Z}_{\geq 0}\,\theta)$. Hence, z, z' are in the principal series.

In the latter case, we may assume that $z \neq z'$ (otherwise z, z' are in the principal series). We use the fact that $z + c$ and $z' + c$ must be of the same sign for any c. If θ is irrational then the numbers c form an everywhere dense subset in $\mathbb{R}$, so that there exists c such that $-c$ is strictly between z and z', which leads to a contradiction. Thus, θ is rational. Then $\mathbb{Z}_{\geq 0} + \mathbb{Z}_{\leq 0}\,\theta$ coincides with the lattice $\mathbb{Z} + \mathbb{Z}\theta$. Since z, z' cannot be separated by a point of this lattice, we conclude that (z, z') is in the complementary series. $\square$

In addition to the principal and complementary series of couples (z, z') there also exist (z, z') such that the expression (1.1) vanishes on a nonempty subset of diagrams λ and is strictly positive on the remaining diagrams. By definition, such couples (z, z') form the *degenerate series*. In the next two propositions we provide examples of (z, z') belonging to the degenerate series.

Proposition 1.3. *Let $m = 1, 2, \ldots$, and assume that z, z' satisfy one of the following two conditions* (1), (2):

(1) $(z = m\theta,\ z' > (m-1)\theta)$ *or* $(z' = m\theta,\ z > (m-1)\theta)$;
(2) $(z = -m,\ z' < -m+1)$ *or* $(z' = -m,\ z < -m+1)$.

Then (z, z') is in the degenerate series. The set of diagrams λ such that the expression (1.1) *is strictly positive looks, respectively, as follows*:

(1) *all diagrams with at most m rows*;
(2) *all diagrams with at most m columns*.

Proof. We leave the proof to the reader. $\square$

Given $k, l \in \{1, 2, \ldots\}$, let $\Gamma(k, l)$ denote the set of all boxes (i, j) such that at least one of the inequalities $i \leq k$, $j \leq l$ holds (a "fat hook shape").

Proposition 1.4. *If θ is irrational, let $k, l \in \{1, 2, \ldots\}$ be arbitrary. If θ is a rational number not equal to 1, write it as the ratio $\theta = s/r$ of relatively prime natural numbers, and then assume that at least one of the inequalities $k < r$, $l < s$ holds. Finally, if $\theta = 1$ then assume $k = l = 1$.*

Under these assumptions, assume further that both parameters z, z' are real, one of them equals $-(k - l\theta)$, and the difference $|z - z'|$ is small enough.

Then (z, z') is in the degenerate series, and the expression (1.1) *is strictly positive exactly on whose diagrams that are contained in the "fat hook shape" $\Gamma(k, l)$ as defined above.*

Proof. We leave the proof to the reader. $\square$

Thus, if the parameters z, z' are in the principal, complementary, or degenerate series then $M_{z,z',\theta}^{(n)}$ is a probability measure on $\mathbb{Y}_n$ for any $n = 1, 2, \ldots$. These measures deserve a special name. We call them the *z–measures*.

When both z, z' go to infinity, the expression (1.1) has a limit

$$M_{\infty,\infty,\theta}^{(n)}(\lambda) = \frac{n!\,\theta^n}{H(\lambda, \theta)\,H'(\lambda, \theta)},$$

which we call the *Plancherel measure* on $\mathbb{Y}_n$. The Plancherel measure with $\theta = 1$ was considered in many works, see [LS], [VK1], [VK3], [BDJ1], [BDJ2], [BDR], [BOO], [J1], [J2], [Ok3].

The z–measures with $\theta = 1$ first originated in [KOV] in connection with the problem of harmonic analysis on the infinite symmetric group. The limits of the measures $M_{z,z',1}^{(n)}$ as $n \to \infty$ govern the spectral decomposition of the so–called generalized regular representations. The z–measures with $\theta = 1$ and their limits were studied in detail in [P.I–P.V], [BO1–2], [Bor], [Ok2].

Various special cases and degenerations of the z–measures with $\theta = 1$ also arise in a number of problems not related to representation theory: see [J1], [J2], [TW], [GTW], and our survey [BO3]. Special cases of z-measures with $\theta = 1/2, 2$ were considered in [AvM], [BR1], [BR2].

The z–measures with general $\theta > 0$ were first defined in [Ke2] (see also [BO4] for another derivation). Besides $\theta = 1$, there exists one more special value of the parameter θ when the z–measures admit a representation–theoretic interpretation: specifically, the case $\theta = 1/2$ is related to a certain Gelfand pair associated with the infinite symmetric group. No such interpretation exists for general θ. Nevertheless, introducing the general parameter θ seems to be a reasonable generalization. It is quite similar to Heckman–Opdam's generalization of noncommutative spherical Fourier analysis. Another motivation comes from comparison with log–gas (or random matrix) models with general parameter $\beta = 2\theta$.

The z–measures with different n are related to each other by a *coherency relation*, see Proposition 1.5 below. To state it, we need more notation.

Let P_μ be the Jack symmetric function with parameter θ and index μ (see [Ma2, VI.10]; note that Macdonald uses $\alpha = \theta^{-1}$ as the parameter). The simplest case of Pieri's formula for the Jack functions reads as follows:

$$P_\mu P_{(1)} = \sum_{\lambda:\, \lambda \searrow \mu} \varkappa_\theta(\mu, \lambda) P_\lambda\,,$$

where $\lambda \searrow \mu$ means that μ can be obtained from λ by removing one box, $\varkappa_\theta(\mu, \lambda)$ are certain positive numbers. For the sake of completeness, we give an explicit formula for $\varkappa_\theta(\mu, \lambda)$, although we will not use it in the sequel. We have

$$\varkappa_\theta(\mu, \lambda) = \prod_b \frac{\bigl(a(b) + (l(b) + 2)\theta\bigr)\bigl(a(b) + 1 + l(b)\theta\bigr)}{\bigl(a(b) + (l(b) + 1)\theta\bigr)\bigl(a(b) + 1 + (l(b) + 1)\theta\bigr)}\,,$$

where $b = (i, j)$ ranges over all boxes in the jth column of the diagram μ, provided that the new box $\lambda \setminus \mu$ belongs to the jth column of λ, see [Ma2, VI.10, VI.6],

$$a(b) = a(i, j) = \mu_i - j, \quad l(b) = l(i, j) = \mu_j' - i.$$

For any $\mu \in \mathbb{Y}_{n-1}$ and $\lambda \in \mathbb{Y}_n$ set

$$q_\theta(\mu, \lambda) = \begin{cases} \dfrac{H(\lambda, \theta)}{n\, H(\mu, \theta)}\, \varkappa_\theta(\mu, \lambda), & \lambda \searrow \mu, \\ 0, & \text{otherwise.} \end{cases}$$

For any $\lambda \in \mathbb{Y}_n$ we have

$$\sum_{\mu \in \mathbb{Y}_{n-1}} q_\theta(\mu, \lambda) = 1.$$

This relation readily follows from the Pieri formula for the Jack functions above and the relation

$$P_{(1)}^n = \sum_{\lambda \in \mathbb{Y}_n} \frac{n!}{H(\lambda, \theta)} \, P_\lambda.$$

Later on we will also use the notation

$$C_\lambda = \frac{n!}{H(\lambda, \theta)} \, P_\lambda.$$

Proposition 1.5. *For any $n = 1, 2, \ldots$ and any $\mu \in \mathbb{Y}_{n-1}$ we have*

$$M_{z,z',\theta}^{(n-1)}(\mu) \equiv \sum_{\lambda \in \mathbb{Y}_n} q_\theta(\mu, \lambda) \, M_{z,z',\theta}^{(n)}(\lambda),$$

where we agree that $\mathbb{Y}_0 = \{\varnothing\}$ and $M_{z,z',\theta}^{(0)}(\varnothing) = 1$.

Proof. See [Ke2], [BO4]. $\square$

It is convenient to view $\{q_\theta(\mu, \lambda)\}$ as probabilities of a transition from $\mathbb{Y}_n$ to $\mathbb{Y}_{n-1}$. Under this transition, the nth measure $M_{z,z',\theta}^{(n)}$ transforms into the $(n-1)$st measure $M_{z,z',\theta}^{(n-1)}$. Thus, the nth measure is a refinement of the $(n-1)$st one.

We are interested in the asymptotic behavior of the measures $M_{z,z',\theta}^{(n)}$ as $n \to \infty$. Since these measures live on different spaces, we need to explain in what sense we understand the limit.

Let $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ be the product of countably many copies of the real line. We equip $\mathbb{R}^\infty$ with the product topology. Set $\mathbb{R}^{2\infty} = \mathbb{R}^\infty \times \mathbb{R}^\infty$. Let Ω be a subset of $\mathbb{R}^{2\infty}$ consisting of pairs of sequences

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \qquad \beta_1 \geq \beta_2 \geq \cdots \geq 0,$$

subject to the condition

$$\sum_{i=1}^\infty (\alpha_i + \beta_i) \leq 1.$$

This is a metrizable compact topological space. Note that the subset of Ω with $\sum_i (\alpha_i + \beta_i) = 1$ is dense in Ω.

For any $n = 1, 2, \ldots$, we define an embedding $\iota_n : \mathbb{Y}_n \hookrightarrow \Omega$ as follows. For any $\lambda \in \mathbb{Y}_n$, let $d = d(\lambda)$ be the number of diagonal boxes of λ. Set

$$a_i(\lambda) = \begin{cases} \lambda_i - i + 1/2, & i \leq d, \\ 0, & i > d, \end{cases} \qquad b_i(\lambda) = \begin{cases} \lambda_i' - i + 1/2, & i \leq d, \\ 0, & i > d, \end{cases}$$

These are the *modified Frobenius coordinates* of λ first introduced in [VK2]. Set

$$\alpha_i(\lambda) = a_i(\lambda)/n, \qquad \beta_i(\lambda) = b_i(\lambda)/n.$$

Note that $\sum_i (\alpha_i(\lambda) + \beta_i(\lambda)) = 1$. We define

$$\iota_n(\lambda) = (\alpha_1(\lambda), \alpha_2(\lambda), \ldots; \beta_1(\lambda), \beta_2(\lambda), \ldots).$$

(In [KOO], the definition of ι_n was slightly different. This does not affect, however, the following important claim, which is a special case of one of the main results of [KOO]. This follows, for instance, from Remark 1.7 below.)

Theorem 1.6. *There exists a weak limit $\underline{M}_{z,z',\theta}$ of the pushforwards of the measures $M^{(n)}_{z,z',\theta}$ under ι_n:*

$$\underline{M}_{z,z',\theta} = \text{w-lim}_{n\to\infty}\, \iota_n^*\left(M^{(n)}_{z,z',\theta} \right).$$

Proof. See [KOO]. Note that the claim holds for *any* system of measures on $\mathbb{Y}_n$'s which satisfy the coherency relation of Proposition 1.5. $\square$

Remark 1.7. Consider the probability spaces $(\mathbb{Y}_n, M^{(n)}_{z,z',\theta})$ and consider the functions $\alpha_i(\,\cdot\,)$ and $\beta_i(\,\cdot\,)$ as random variables $\alpha_i^{(n)}$, $\beta_i^{(n)}$ defined on these spaces. Similarly, we view the coordinate functions α_i, β_i on Ω as random variables defined on the probability space $(\Omega, M_{z,z',\theta})$. Then Theorem 1.6 is equivalent to saying that for any positive integers m, l,

$$\{\alpha_1^{(n)}, \ldots, \alpha_m^{(n)}, \beta_1^{(n)}, \ldots, \beta_l^{(n)}\} \xrightarrow{d} \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l\},$$

where $\xrightarrow{d}$ denotes the convergence in distribution.

Our main goal is to study the limit measures $\underline{M}_{z,z',\theta}$.

The finite level measures $M^{(n)}_{z,z',\theta}$ can be reconstructed from the limit measure by means of an analog of the Poisson integral representation of the harmonic functions. Let us briefly state this result. A more detailed exposition can be found in [KOO].

Let Λ be the algebra of symmetric functions over $\mathbb{R}$. Following [KOO], we will view the elements of Λ as continuous functions on Ω. Namely, the values of the power sums p_k are defined by

$$p_k(\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots) = \begin{cases} 1, & k = 1, \\ \sum_i \left(\alpha_i^k + (-\theta)^{k-1}\beta_i^k \right), & k \geq 2. \end{cases}$$

Since $\{p_k\}$ are free generators of the commutative algebra Λ, this defines an algebra homomorphism $\Lambda \to C(\Omega)$. In particular, the Jack symmetric functions $\{P_\lambda\}$ can also be viewed as elements of $C(\Omega)$.

Theorem 1.8. *For any $n = 1, 2, \ldots$ and any $\lambda \in \mathbb{Y}_n$, we have*

$$M^{(n)}_{z,z',\theta}(\lambda) = \frac{n!}{H(\lambda, \theta)} \int_{\omega = (\alpha,\beta)\in\Omega} P_\lambda(\omega)\underline{M}_{z,z',\theta}(d\omega).$$

Proof. See [KOO]. Again, the claim holds for any system of measures satisfying the coherency relation. $\square$

Theorem 1.8 can also be interpreted in a different way, namely, as providing the values of integrals of $\{P_\lambda\}$ with respect to the measure $\underline{M}_{z,z',\theta}$ on Ω. This set of integrals defines the limit measure uniquely, because the functions $\{P_\lambda(\omega)\}$ span a dense linear subspace of $C(\Omega)$. We view these integrals as "moments" of $\underline{M}_{z,z',\theta}$.

Both descriptions of the measure $\underline{M}_{z,z',\theta}$, as the weak limit (Theorem 1.6) and through the moments (Theorem 1.8), are rather abstract. Our goal is to find yet another description which would allow us to obtain probabilistic information about random points $\omega = (\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots)$ distributed according to $\underline{M}_{z,z',\theta}$.

It turns out to be very hard to compute directly the joint distribution functions of finitely many α_i's or/and β_i's regarded as random variables. Instead of that, we will focus on computing the *correlation functions* of the measures $\underline{M}_{z,z',\theta}$. Informally, the nth correlation function of $\{\alpha_i\}$ measures the probability to find one α_i near each of the n given locations $x_1,\ldots,x_n > 0$:

$$
\rho_n(x_1,\ldots,x_n)
$$
$$
= \lim_{\Delta x_1,\ldots,\Delta x_n \to +0} \frac{\mathrm{Prob}\{\{\alpha_i\} \cap (x_j, x_j + \Delta x_j) \neq \varnothing \text{ for all } j = 1,\ldots,n\}}{\Delta x_1 \cdots \Delta x_n}.
$$

The correlation functions $\rho_n(x)$ should be viewed as densities of the *correlation measures* $\rho_n(dx)$ with respect to the Lebesgue measure dx. The knowledge of the correlation functions allows to evaluate averages of the *additive* functionals on $\{\alpha_i\}$. Namely, for any continuous function $F : \mathbb{R}_{>0}^n \to \mathbb{C}$ with compact support, we have

$$
\int_{\omega=(\alpha;\beta)\in\Omega} \sum_{\substack{i_1,\ldots,i_n \\ \text{pairwise distinct}}} F(\alpha_{i_1},\ldots,\alpha_{i_n}) \underline{M}_{z,z',\theta}(d\omega) = \int_{\mathbb{R}_{>0}^n} F(x_1,\ldots,x_n)\rho_n(dx).
$$

This equality can be viewed as a rigorous definition of $\rho_n(dx)$. A detailed discussion of the correlation measures/functions can be found in [Len], [DVJ].

Note that the correlation measure $\rho_n(dx)$ is supported by the simplex

$$
\{(x_1,\ldots,x_n) \in (\mathbb{R}_{\geq 0})^n : x_1 + \cdots + x_n \leq 1\}.
$$

More generally, one can similarly define *joint* correlations of $\{\alpha_i\}$ and $\{\beta_i\}$. In the case $\theta = 1$ these joint correlation functions have been computed in [P.II].

This definition of $\rho_n(dx)$ makes sense for an arbitrary probability measure M on Ω. Indeed, observe that for any point $\omega = (\alpha,\beta) \in \Omega$, we have the estimate

$$
\alpha_{m+1} < \frac{1}{m}, \qquad m = 1, 2, \ldots, \tag{1.2}
$$

which follows from the fact that $\alpha_1 \geq \alpha_2 \geq \ldots$ and $\sum_i \alpha_i < 1$. For any nonnegative $F \in C_0((\mathbb{R}_{>0})^n)$, choose m so large that $\mathrm{supp}\, F \subset (\mathbb{R}_{\geq 1/m})^n$. Then in the above formula for $\langle F, \rho_n \rangle$ the summands involving indices $i_k > m$ vanish. Thus, the integrand is bounded by

$$
\sup F \cdot m(m-1)\cdots(m-n+1).
$$

This fact ensures the very existence of the correlation measures, see [Len]. It also implies a useful bound

$$
\rho_n((\mathbb{R}_{\geq 1/m})^n) \leq m(m-1)\cdots(m-n+1) \leq m^n, \qquad m = 1, 2, \ldots. \tag{1.3}
$$

In the case $\theta = 1$ it was shown in [P.II] that the expressions for the correlation functions are substantially simplified by a one-dimensional integral transform, see also [P.III, P.V], [BO1-3], [Bor]. This integral transform corresponds to a simple modification of the initial measure on Ω. The modified measure for general θ is defined as follows.

Let us denote by $\widetilde{\Omega}$ the set of triples $\omega = (\alpha, \beta, \delta) \in \mathbb{R}^{2\infty} \times \mathbb{R}_{\geq 0}$, where $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0)$, $\delta \in \mathbb{R}_{\geq 0}$, and $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) \leq \delta$. We will also use the notation $\gamma = \delta - \sum_i (\alpha_i + \beta_i) \geq 0$.

Note that $\widetilde{\Omega}$ is a locally compact space with respect to the topology induced from the product topology on $\mathbb{R}^{\infty} \times \mathbb{R}_{\geq 0}$. It is metrizable, the metric can be defined in the standard fashion:

$$\operatorname{dist}(\omega, \omega') = |\delta - \delta'| + \sum_i \frac{\min(|\alpha_i - \alpha'_i|, 1)}{2^i} + \sum_i \frac{\min(|\beta_i - \beta'_i|, 1)}{2^i} \,.$$

The subsets of $\widetilde{\Omega}$ of the form $\{\omega \in \widetilde{\Omega} : \delta(\omega) \leq \mathrm{const}\}$ are compact (here $\delta(\omega)$ is the δ–coordinate of ω). The set $\{\omega \in \widetilde{\Omega} : \gamma(\omega) = 0\}$ is everywhere dense in $\widetilde{\Omega}$.

The space $\widetilde{\Omega}$ is homeomorphic to $\Omega \times \mathbb{R}_{\geq 0}$ modulo contracting $\Omega \times \{0\}$ to a single point, the corresponding map looks as follows

$$((\alpha, \beta), \delta) \in \Omega \times \mathbb{R}_{\geq 0} \mapsto (\delta\alpha, \delta\beta, \delta) \in \widetilde{\Omega}.$$

The modified measure $\widetilde{\underline{M}}_{z,z',\theta}$ is the pushforward under this map of the measure

$$\underline{M}_{z,z',\theta} \otimes \left(\frac{s^{t-1}}{\Gamma(t)} e^{-s} ds \right)$$

on $\Omega \times \mathbb{R}_{\geq 0}$ (recall that $t = zz'/\theta$).

The correlation measures/functions $\widetilde{\rho}_n$ of $\widetilde{\underline{M}}_{z,z',\theta}$ are defined in the same way as those of $\underline{M}_{z,z',\theta}$. The definition of $\widetilde{\underline{M}}_{z,z',\theta}$ immediately implies that for any test function $F \in \mathbb{C}_0((\mathbb{R}_{>0})^n)$,

$$\langle F, \widetilde{\rho}_n \rangle = \int_0^{\infty} \frac{s^{t-1}e^{-s}}{\Gamma(t)} \langle F_s, \rho_n \rangle ds,$$

where $F_s(x_1, \ldots, x_n) = F(sx_1, \ldots, sx_n)$. In terms of the correlation functions (which may always be viewed as generalized functions), we have

$$\widetilde{\rho}_n(x_1, \ldots, x_n) = \int_0^{\infty} \frac{s^{t-1}e^{-s}}{\Gamma(t)} \rho_n(x_1 s^{-1}, \ldots, x_n s^{-1}) \frac{ds}{s^n} \qquad (1.4)$$

for any $n = 1, 2, \ldots$. The convergence of the integral follows from (1.3). This transform is easily reduced to the one–dimensional Laplace transform along the rays $\{(\delta x_1, \ldots, \delta x_n), \delta > 0\}$. Hence, it is invertible. The passage from $\underline{M}_{z,z',\theta}$ to $\widetilde{\underline{M}}_{z,z',\theta}$ is called *lifting*.

The following proposition will be used in §5.

Proposition 1.9. *Let $F \in C_0((\mathbb{R}_{>0})^n)$ and $\delta \in \mathbb{R}_{>0}$. Then the expression $\langle F_\delta, \widetilde{\rho}_n \rangle$, where $F_\delta(x) = F(\delta \cdot x)$ as above, is a real–analytic function of δ.*

Proof. We have

$$\langle F_\delta, \widetilde{\rho}_n \rangle = \int_0^{\infty} \frac{s^{t-1}e^{-s}}{\Gamma(t)} \langle F_{s\delta}, \rho_n \rangle ds = \delta^{-t} \int_0^{\infty} \frac{s^{t-1}e^{-s/\delta}}{\Gamma(t)} \langle F_s, \rho_n \rangle ds.$$

Pick $\epsilon > 0$ such that $\operatorname{supp} F \subset (\mathbb{R}_{\geq \epsilon})^n$. The claim follows from the following two facts:

1. $\langle F_s, \rho_n \rangle$ vanishes for $s < \epsilon$.

2. $\langle F_s, \rho_n \rangle$ has at most polynomial growth in s when $s \to \infty$.

The vanishing follows from the fact that $\operatorname{supp} \rho_n \subset \{ \sum_{i=1}^n x_i \leq 1 \}$.

For the second fact, observe that by (1.3) we have

$$|\langle F_s, \rho_n \rangle| \leq \sup |F| \cdot \rho_n((\mathbb{R}_{\geq s^{-1}\epsilon})^n) \leq \sup |F| \cdot ([s\epsilon^{-1}] + 1)^n. \quad \square$$

Remark 1.10. In the case when (z, z') belong to the degenerate series (see the definition above), the measures $M_{z,z',\theta}^{(n)}$ and their limit $\underline{M}_{z,z',\theta}$ were studied by Kerov [Ke1]. To be concrete, assume that $z = m\theta$, $m = 1, 2, \dots$, and $z' > (m-1)\theta$. Then the limit measure $\underline{M}_{z,z',\theta}$ is concentrated on the $(m-1)$-dimensional face

$$\{ (\alpha, \beta) \in \Omega : \alpha_1 + \cdots + \alpha_m = 1, \, \alpha_{m+1} = \alpha_{m+2} = \cdots = \beta_1 = \beta_2 = \cdots = 0 \}.$$

Its density with respect to the Lebesgue measure on this simplex is equal to

$$\operatorname{const} \cdot (\alpha_1 \cdots \alpha_m)^{z' - (m-1)\theta - 1} \cdot \prod_{1 \leq i < j \leq m} |\alpha_i - \alpha_j|^{2\theta}. \tag{1.5}$$

The lifting $\underline{M}_{z,z',\theta}$ of this measure lives on $(\mathbb{R}_{\geq 0})^m$ and has the density (with respect to the Lebesgue measure) equal to

$$\operatorname{const} \cdot (\alpha_1 \cdots \alpha_m)^{z' - (m-1)\theta - 1} \cdot e^{-\alpha_1 - \cdots - \alpha_m} \cdot \prod_{1 \leq i < j \leq m} |\alpha_i - \alpha_j|^{2\theta}. \tag{1.6}$$

This is the distribution function for the m-particle Laguerre ensemble, see [F1], [F2].

2. AVERAGES OF $E_\theta(\,\cdot\,; u_1) \cdots E_\theta(\,\cdot\,; u_l)$ AS HYPERGEOMETRIC FUNCTIONS

Set

$$E_\theta(\omega; u) = e^{\gamma/u} \frac{\prod_{i=1}^\infty (1 + \alpha_i/u)}{\prod_{i=1}^\infty (1 - \theta\beta_i/u)^{1/\theta}}, \qquad \omega \in \widetilde{\Omega}, \; u \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}.$$

Let us comment on this definition. Consider the algebra homomorphism $\Lambda \to C(\widetilde{\Omega})$ defined on the power sums by

$$p_1(\omega) = \delta; \qquad p_k(\omega) = \sum \alpha_i^k + (-\theta)^{k-1} \sum \beta_i^k, \quad k \geq 2.$$

This is an algebra embedding generalizing the homomorphism $\Lambda \to C(\Omega)$ as defined in section 1. Then $E_\theta(\omega; u)$ is nothing but the image of the generating function $\sum e_k u^{-k}$, where $e_k \in \Lambda$ are the elementary symmetric functions.

We view $E_\theta(\omega; u)$ as the analog of the characteristic polynomial of a matrix, the roles of "eigenvalues" are played by α_i's and β_i's.

One can show that for any $u \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, the function $E_\theta(\,\cdot\,; u)$ is a continuous function on $\widetilde{\Omega}$, cf. [KOO], and for any $\omega \in \widetilde{\Omega}$, $E_\theta(\omega; \,\cdot\,)$ is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

Observe that E_θ is homogeneous of degree 0:

$$E_\theta(s \cdot \omega; s \cdot u) = E_\theta(\omega; u), \qquad s > 0.$$

We will also consider $E_\theta(\,\cdot\,; u)$ as a function on $\Omega.$[1] Then the domain of u can be expanded to $\mathbb{C} \setminus [0, \theta]$.

The goal of this section is to express the averages $(l = 1, 2, \dots)$

$$\int_\Omega E_\theta(\,\cdot\,; u_1) \cdots E_\theta(\,\cdot\,; u_l) \underline{M}_{z,z',\theta}(d\omega), \qquad \int_{\widetilde\Omega} E_\theta(\,\cdot\,; u_1) \cdots E_\theta(\,\cdot\,; u_l) \widetilde{\underline{M}}_{z,z',\theta}(d\omega)$$

in terms of multivariate hypergeometric functions.

Recall that in the previous section we introduced the renormalized Jack polynomials $C_\lambda = C_\lambda^{(\nu)}(x)$. Here we deliberately included the parameter ν in the notation of the Jack polynomials. In §1 this parameter was equal to θ, and in this section we will need $\nu = \theta^{-1}$.

For $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, set

$${}_2\widehat{F}_1^{(\nu)}(a, b; c; x) = \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq l}} \frac{(a)_{\lambda,\nu}(b)_{\lambda,\nu}}{(c)_{|\lambda|}|\lambda|!} C_\lambda^{(\nu)}(x), \quad x = (x_1, \dots, x_l).$$

Note that the normalized series

$$\frac{{}_2\widehat{F}_1^{(\nu)}(a, b; c; x)}{\Gamma(c)} = \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq l}} \frac{(a)_{\lambda,\nu}(b)_{\lambda,\nu}}{\Gamma(c + |\lambda|)\,|\lambda|!} C_\lambda^{(\nu)}(x), \quad x = (x_1, \dots, x_l)$$

makes sense for any $c \in \mathbb{C}$.

When $l = 1$, the definition of ${}_2\widehat{F}_1^{(\nu)}(a, b; c; x)$ above coincides with that of the classical Gauss hypergeometric function. When $l > 1$ our series differs from the standard multivariate generalization of the Gauss function, see [Mu], [Ma1], [Ko], [FK], [Y]. Indeed, in the standard definition one has $(c)_{\lambda,\nu}$ instead of $(c)_{|\lambda|}$ in the denominator. However, our function ${}_2\widehat{F}_1^{(\nu)}(a, b; c; x)$ shares many properties of the standard hypergeometric functions.

Proposition 2.1. (i) *The defining series for ${}_2\widehat{F}_1^{(\nu)}(a, b; c; x)$ converges in the polydisk $\{|x_1| < 1, \dots, |x_l| < 1\}$ and defines a holomorphic function in this domain.*

(ii) *${}_2\widehat{F}_1^{(\nu)}(a, b; c; x)/\Gamma(c)$ is an entire function in the parameters $(a, b, c) \in \mathbb{C}^3$. As a function in x, it can be analytically continued to a domain in $\mathbb{C}^l$ containing the tube $\{(x_1, \dots, x_l) \in \mathbb{C}^l : \Re x_i < 0, i = 1, \dots, l\}$.*

(iii) *As $x_1, \dots, x_l \to -\infty$ inside $\mathbb{R}$, $|{}_2\widehat{F}_1^{(\nu)}(a, b; c; x)|$ has at most polynomial growth in x.*

Idea of proof. (i) Compare the series ${}_2\widehat{F}_1^{(\nu)}(a, b; c; x)$ with the series

$${}_1F_0^{(\nu)}(a; x) = \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq l}} \frac{(a)_{\lambda,\nu}}{|\lambda|!} C_\lambda^{(\nu)}(x), \quad x = (x_1, \dots, x_l).$$

---

[1] In what follows we view Ω as a subset of $\widetilde\Omega$ defined by the condition $\delta = 1$.

By virtue of the well–known binomial theorem (see, e.g., [Ma1], [OO])

$$_1F_0^{(\nu)}(a;x) = \prod_{i=1}^{l}(1-x_i)^{-a},$$

which implies that the latter series converges in the polydisk in question. Since the ratio $(b)_{\lambda,\nu}/(c)_{|\lambda|}$ has at most polynomial growth in $|\lambda|$, the former series also converges in the same polydisk.

(ii) An argument is given below after Proposition 2.2.

(iii) This can be derived from a Mellin–Barnes integral representation for $_2\widehat{F}_1^{(\nu)}(a,b;c;x)$, which will be given elsewhere. $\square$

Consider the multivariate hypergeometric function of type $(1,0)$ in two sets of variables $x = (x_1, \ldots, x_l)$ and $y = (y_1, \ldots, y_l)$:

$$_1\mathcal{F}_0^{(\nu)}(a;x,y) = \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq l}} \frac{(a)_{\lambda,\nu}}{|\lambda|!} \frac{C_\lambda^{(\nu)}(x)C_\lambda^{(\nu)}(y)}{C_\lambda^{(\nu)}(1^l)}, \qquad a \in \mathbb{C}, \ \nu > 0,$$

see [Ma1], [Y, (37)]. When $\nu = 1/2, 1, 2$, this function admits a simple matrix integral representation. For instance, in the case $\nu = 1/2$

$$_1\mathcal{F}_0^{(\nu)}(a;x,y) = \int_{U \in O(l))} \det(1 - XUYU^{-1})^{-a}dU,$$

where $O(l)$ is the group of $l \times l$ orthogonal matrices, dU is the normalized Haar measure on $O(l)$, and X and Y stand for the diagonal matrices with diagonal entries (x_i) and (y_i).

The next statement gives an Euler-type integral representation of $_2\widehat{F}_1^{(\nu)}(a,b;c;x)$ in terms of $_1\mathcal{F}_0^{(\nu)}$. For the three particular values of the parameter, $\nu = 1/2, 1, 2$, it can be written as a matrix integral involving elementary functions only.

Proposition 2.2. *For any $\nu > 0$, assume that $\Re b > (l-1)\nu$, $\Re c > l\,\Re b$. Then*

$$\frac{_2\widehat{F}_1^{(\nu)}(a,b;c;x)}{\Gamma(c)} = \frac{1}{\Gamma(c-lb)} \prod_{j=1}^{l} \frac{\Gamma(\nu+1)}{\Gamma(b-(j-1)\nu)\Gamma(j\nu+1)}$$

$$\times \int_{\substack{\tau_1,\ldots,\tau_l>0 \\ \sum_i \tau_i < 1}} \prod_{i=1}^{l} \tau_i^{b-(l-1)\nu-1} \left(1 - \sum_{i=1}^{l} \tau_i\right)^{c-lb-1} \prod_{1 \leq i < j \leq l} |\tau_i - \tau_j|^{2\nu} {}_1\mathcal{F}_0^{(\nu)}(a;x,\tau)d\tau.$$

$$(2.1)$$

Proof. We use the following integral representation of the ratio $(b)_{\lambda,\nu}/\Gamma(c+|\lambda|)$

$$\frac{(b)_{\lambda,\nu}}{\Gamma(c+|\lambda|)} = \frac{1}{\Gamma(c-lb)} \prod_{j=1}^{l} \frac{\Gamma(\nu+1)}{\Gamma(b-(j-1)\nu)\Gamma(j\nu+1)}$$

$$\times \int_{\substack{\tau_1,\ldots,\tau_l>0 \\ \sum_i \tau_i < 1}} \prod_{i=1}^{l} \tau_i^{b-(l-1)\nu-1} \left(1 - \sum_{i=1}^{l} \tau_i\right)^{c-lb-1} \prod_{1 \leq i < j \leq l} |\tau_i - \tau_j|^{2\nu} \frac{C_\lambda^{(\nu)}(\tau)}{C_\lambda^{(\nu)}(1^l)} d\tau.$$

$$(2.2)$$

A derivation of (2.2) is given in [Ma2, ch. VI, §10, Example 7 (b)]. Multiplying both sides of (2.2) by

$$\frac{(a)_{\lambda,\nu}}{|\lambda|!} C_\lambda^{(\nu)}(x),$$

taking the sum over λ and interchanging summation and integration, one obtains the required equality. $\square$

Note that the l–dimensional integral (2.2) is a consequence of the following integral over an $(l-1)$–dimensional simplex

$$
\int_{\substack{t_1+\cdots+t_l=1 \\ t_1,\ldots,t_l \geq 0}} \prod_{j=1}^{l} t_j^{A-1} \prod_{1\leq i<j\leq l} |t_i - t_j|^{2\nu} \frac{C_\lambda^{(\nu)}(t_1,\ldots,t_l)}{C_\lambda^{(\nu)}(1^l)} \, dt
$$
$$
= \frac{1}{\Gamma(|\lambda| + Al + l(l-1)\nu)} \prod_{j=1}^{l} \frac{\Gamma(\lambda_j + A + (l-j)\nu)\Gamma(j\nu + 1)}{\Gamma(\nu + 1)},
\tag{2.3}
$$

where $\Re A > 0$ and dt is Lebesgue measure on the simplex.

The integral (2.2) can be derived from the integral (2.3) as follows: Set $\tau = ts$, where $s = \sum \tau_j$. Since the integrand of (2.2) is a homogeneous function, the integral splits into the product of an $(l-1)$–dimensional integral over t (which is the integral (2.3) with $A = b - (l-1)\nu$) and a one–dimensional beta–integral over s.

As for the integral (2.3), it is a simplex version of the generalized Selberg integral over the unit cube $[0,1]^l$, see [Ma2, ch. VI, §10, example 7]. Once one knows the integral over the cube, it is easy to pass to the simplex. On the other hand, the integral (2.3) can be obtained directly by making use of degenerate z–measures, see Kerov [Ke1, §12].

Sketch of proof of Proposition 2.1 (ii). Our argument is based on the Euler–type integral representation (2.1). We will prove that the integral (2.1), as a function in x, can be continued to the tube $\{x \in \mathbb{C}^l : \Re x_i < 1/2, \, i = 1,\ldots,l\}$. This result is not optimal: when $\nu = 1/2$, 1, 2, use of the matrix integral representation for ${}_1\mathcal{F}_0^{(\nu)}(x,\tau)$ allows one to extend the domain to the tube $\{x \in \mathbb{C}^l : \Re x_i < 1, \, i = 1,\ldots,l\}$ (cf. [FK, Prop. XV.3.3]).

Assume first $\Re b > (l-1)\nu$ and $\Re(c - lb) > 0$ so that the integrand in (2.1) is an integrable function (then we will explain how to get rid of these restrictions).

The idea is to apply the transformation formula

$$
{}_1\mathcal{F}_0^{(\nu)}(x,y) = \prod_{j=1}^{l}(1-x_j)^{-a} \cdot {}_1\mathcal{F}_0^{(\nu)}\left(\frac{x}{x-1}, 1-y\right),
\tag{2.4}
$$

established in Macdonald [Ma1, section 6]. Here we abbreviate

$$
\frac{x}{x-1} = \left(\frac{x_1}{x_1-1}, \ldots, \frac{x_l}{x_l-1}\right), \qquad 1-y = (1-y_1, \ldots, 1-y_l).
$$

When $\nu = 1/2$, 1, 2, the transformation (2.4) is immediate from the matrix integral representation of ${}_1\mathcal{F}_0^{(\nu)}$. But in the general case, when we dispose of the series

expansion only, (2.4) is not evident. (Note that Macdonald's argument uses some properties of generalized binomial coefficients and Jack polynomials, admitted as conjectures. But nowadays these are well-established facts.)

Since $\zeta \mapsto \zeta(\zeta - 1)^{-1}$ transforms the half–plane $\Re\zeta < 1/2$ into the unit disk $|\zeta| < 1$, the transformation (2.4) can be used to correctly define ${}_1\mathcal{F}_0^{(\nu)}(x,y)$ when x ranges over the tube $\Re x_i < 1/2$ and $y = \tau$.

Thus, we checked that the required analytic continuation in x exists under an additional restriction on the parameters b, c. Let us show how to get rid of this restriction. Take a large constant $C > 0$ and assume first that $\Re c > lC$. Then, as a function in (a,b), our integral admits a continuation to the tube domain $\{(a,b) \in \mathbb{C}^2 : \Re a < C, (l-1)\nu < \Re b < C\}$. By virtue of symmetry $a \leftrightarrow b$, the same holds for the tube $\{(a,b) \in \mathbb{C}^2 : (l-1)\nu < \Re a < C, \Re b < C\}$. Applying a general theorem about "forced" analytic continuation on tube domains (see, e.g., [H, Theorem 2.5.10]) we obtain a continuation to the tube $\{(a,b) \in \mathbb{C}^2 : \Re a < C, \Re b < C\}$. Finally, to remove the restriction on c, we use the relation

$$(c - 1 + \mathcal{D})\left(\frac{{}_2\widehat{F}_1^{(\nu)}(a,b;c;x)}{\Gamma(c)}\right) = \frac{{}_2\widehat{F}_1^{(\nu)}(a,b;c-1;x)}{\Gamma(c-1)},$$

where $\mathcal{D}$ is the Euler operator,

$$\mathcal{D} = \sum_{j=1}^{l} x_j \frac{\partial}{\partial x_j},$$

which follows from the initial series expansion for ${}_2\widehat{F}_1^{(\nu)}(a,b;c;x)/\Gamma(c)$ and the fact that $C_\lambda^{(\nu)}(x)$ is a homogeneous function of degree $|\lambda|$. $\square$

We return to our main subject.

Theorem 2.3. *Let $l = 1, 2, \ldots,$ and let $\Re u_i < 0$, $i = 1, \ldots, l$. Then*

$$\int_\Omega E_\theta(\omega; u_1) \cdots E_\theta(\omega; u_l)\underline{M}_{z,z',\theta}(d\omega) = {}_2\widehat{F}_1^{(\nu)}(a,b;c;\theta/u),$$

where

$$\nu = \theta^{-1}, \quad \theta/u = (\theta/u_1, \ldots, \theta/u_l),$$
$$a = -z\theta^{-1}, \quad b = -z'\theta^{-1}, \quad c = zz'\theta^{-1}.$$

Proof. Observe that Ω is compact and $E_\theta(\,\cdot\,; u) \in C(\Omega)$, thus, the integral is well-defined. Since both sides of the equality in question are holomorphic in $u_1, \ldots, u_l$, we may assume that $|u_i| \gg 0$.

The dual Cauchy identity for the ordinary Jack polynomials (see [Ma2, Ch. VI, (5.4)]) implies the expansion

$$E_\theta(\omega; u_1) \ldots E_\theta(\omega; u_l) = \sum_{\lambda: \ell(\lambda) \leq l} P_{\lambda'}^{(\theta)}(\omega) P_\lambda^{(\theta^{-1})}(u_1^{-1}, \ldots, u_l^{-1}), \quad \omega \in \Omega.$$

Let us integrate the series over Ω termwise. By Theorem 1.8 and (1.1), for any $\lambda \in \mathbb{Y}_n$,

$$\int\limits_{\omega \in \Omega} P_{\lambda'}^{(\theta)}(\omega) \underline{M}_{z,z',\theta}(d\omega) = \frac{H(\lambda',\theta)}{n!} M_{z,z',\theta}^{(n)}(\lambda') = \frac{(z)_{\lambda',\theta}(z')_{\lambda',\theta}}{(t)_n H'(\lambda',\theta)}.$$

An easy computation shows that

$$(z)_{\lambda',\theta}(z')_{\lambda',\theta} = \theta^{2n}(-z\theta^{-1})_{\lambda,\theta^{-1}}(-z'\theta^{-1})_{\lambda,\theta^{-1}},$$
$$H'(\lambda',\theta) = \theta^n H(\lambda,\theta^{-1}).$$

Since $C_\lambda^{(\theta^{-1})} = n!P_\lambda^{(\theta^{-1})}/H(\lambda,\theta^{-1})$, the claim follows. $\square$

We would like to obtain an analog of Theorem 2.3 when Ω is replaced by $\widetilde{\Omega}$ and $\underline{M}_{z,z',\theta}$ is replaced by the lifted measure $\widetilde{\underline{M}}_{z,z',\theta}$. By definition of $\widetilde{\underline{M}}_{z,z',\theta}$ and Fubini's theorem, we have

$$\int_{\widetilde{\Omega}} E_\theta(\omega; u_1) \ldots E_\theta(\omega; u_l) \widetilde{\underline{M}}_{z,z',\theta}(d\omega)$$
$$= \int_0^\infty \frac{s^{t-1}}{\Gamma(t)} e^{-s} \left(\int_\Omega E_\theta(s \cdot \omega; u_1) \ldots E_\theta(s \cdot \omega; u_l) \underline{M}_{z,z',\theta}(d\omega) \right) ds,$$

provided that the integral exists. By the 0-homogeneity property of $E_\theta(\omega; u)$ we can rewrite the integral as

$$\int_0^\infty \frac{s^{t-1}}{\Gamma(t)} e^{-s} \left(\int_\Omega E_\theta(\omega; u_1/s) \ldots E_\theta(\omega; u_l/s) \underline{M}_{z,z',\theta}(d\omega) \right) ds.$$

Hence, by Theorem 2.3, this equals

$$\int_0^\infty \frac{s^{t-1}}{\Gamma(t)} e^{-s} {}_2\widehat{F}_1^{(\nu)}(a,b;c;\, s\theta/u) ds.$$

Recall that $t = c = zz'\theta^{-1}$. This computation suggests the following definition.

For $a, b \in \mathbb{C}$, set

$${}_2F_0^{(\nu)}(a,b;x) = \int_0^\infty \frac{s^{c-1}}{\Gamma(c)} e^{-s} {}_2\widehat{F}_1^{(\nu)}(a,b;c;\, s \cdot x) ds, \qquad c > 0, \quad x = (x_1, \ldots, x_l).$$

(2.5)

As will be shown below, see Proposition 2.4, the right–hand side does not depends on the choice of c. By Proposition 2.1(iii), the integral above makes sense at least when $x_1, \ldots, x_l < 0$.

The notation ${}_2F_0^{(\nu)}$ is justified by the following formal argument: applying the integral transform to the series expansion of ${}_2\widehat{F}_1^{(\nu)}$ we obtain the series

$${}_2F_0^{(\nu)}(a,b;x) = \sum_{\substack{\lambda \in \mathbb{Y} \\ \ell(\lambda) \leq l}} \frac{(a)_{\lambda,\nu}(b)_{\lambda,\nu}}{|\lambda|!} C_\lambda^{(\nu)}(x).$$

Note that the series in the right–hand side does not depend on c. However, if a, b are not equal to $0, -1, -2, \ldots$, this series is *everywhere divergent* (except the origin).[2] Such phenomenon is well known already in the classical one–dimensional case, see [Er, section 5.1]. Our definition is one possibility to circumvent this difficulty in making sense of ${}_2F_0$.

---

[2]If one of the parameters a and b is equal to $0, -1, -2, \ldots$, then the series terminates and defines a polynomial, which can also be written through ${}_1F_1$ series, see Remark 2.6 below.

Proposition 2.4. *For any $\nu > 0$, assume that $\Re b > (l-1)\nu$. Then*

$$_2F_0^{(\nu)}(a,b;x) = \prod_{j=1}^{l} \frac{\Gamma(\nu+1)}{\Gamma(b-(j-1)\nu)\Gamma(j\nu+1)}$$

$$\times \int_{\tau_1,\ldots,\tau_l>0} \prod_{i=1}^{l} \tau_i^{b-\nu(l-1)-1} e^{-\tau_i} \prod_{1 \le i < j \le l} |\tau_i - \tau_j|^{2\nu} {}_1\mathcal{F}_0^{(\nu)}(a;x;\tau)d\tau.$$

Proof. By the homogeneity, $_1\mathcal{F}_0^{(\nu)}(a; s \cdot x; \tau) = {}_1\mathcal{F}_0^{(\nu)}(a; x; s \cdot \tau)$. Using Theorem 2.3 and changing the variables $s \cdot \tau_i = \sigma_i$, we obtain

$$_2F_0^{(\nu)}(a,b;x) = \prod_{j=1}^{l} \frac{\Gamma(\nu+1)}{\Gamma(b-(j-1)\nu)\Gamma(j\nu+1)} \int_{\sigma_1,\ldots,\sigma_l>0} \prod_{i=1}^{l} \sigma_i^{b-\nu(l-1)-1}$$

$$\times \prod_{1 \le i < j \le l} |\sigma_i - \sigma_j|^{2\nu} {}_1\mathcal{F}_0^{(\nu)}(a;x;\sigma) \left(\int_0^{\infty} \frac{(s-\sum_i \sigma_i)^{d-1}}{\Gamma(d)} ds \right) d\sigma,$$

which immediately gives the desired formula. $\square$

Similarly to the one–dimensional case, the function $_2F_0^{(\nu)}(a,b;x)$ can be analytically continued to tube $\{x \in \mathbb{C}^l : \Re x_i < 0, i = 1, \ldots, l\}$. The divergent series for $_2F_0$ given above is, in fact, the asymptotic expansion of $_2F_0$ near $x = 0$.

When $l = 1$, we have $_1\mathcal{F}_0^{(\nu)}(a;x;\tau) = (1 - x\tau)^{-a}$, so that the dependence on ν disappears and Proposition 2.4 takes the form

$$_2F_0(a,b;x) = \frac{1}{\Gamma(b)} \int_0^{\infty} \tau^{b-1}(1-x\tau)^{-a} e^{-\tau} d\tau.$$

This is equivalent to the classical integral representation for the Whittaker function Ψ, see [Er, 6.5(2)] (note that $_2F_0$ and Whittaker's Ψ are essentially the same functions, see [Er, 6.6(3)]).

Again, when $\nu = 1/2, 1, 2$ (and l is arbitrary), we dispose of a matrix integral representation for $_2F_0(a,b;x)$. In the case $\nu = 1/2$, the integral was studied in detail in [MP1], [MP2].

Theorem 2.5. *For any $l = 1, 2, \ldots,$ and $u_1, \ldots, u_l < 0$, the product $E_\theta(\omega; u_1) \cdots E_\theta(\omega; u_l)$ as a function on $\widetilde{\Omega}$ is integrable with respect to the measure $\underline{\widetilde{M}}_{z,z',\theta}$ on $\widetilde{\Omega}$, and*

$$\int_{\widetilde{\Omega}} E_\theta(\omega; u_1) \cdots E_\theta(\omega; u_l) \underline{\widetilde{M}}_{z,z',\theta}(d\omega) = {}_2F_0^{(\nu)}(a,b;\theta/u), \qquad (2.6)$$

where

$$\nu = \theta^{-1}, \quad \theta/u = (\theta/u_1, \ldots, \theta/u_l),$$
$$a = -z\theta^{-1}, \quad b = -z'\theta^{-1}.$$

Proof. If we take the integrability for granted then the statement follows from Theorem 2.3 and definition of $_2F_0$ as was explained above. To prove the integrability, it suffices to show that

$$\int_{\widetilde{\Omega}} |E_\theta(\omega; u_1) \cdots E_\theta(\omega; u_l)|^2 \, \widetilde{\underline{M}}_{z,z',\theta}(d\omega)$$

$$= \int_0^\infty \frac{s^{t-1}}{\Gamma(t)} e^{-s} \left(\int_\Omega |E_\theta(s \cdot \omega; u_1) \dots E_\theta(s \cdot \omega; u_l)|^2 \underline{M}_{z,z',\theta}(d\omega) \right) ds < \infty,$$

because the total measure of the whole space $\widetilde{\Omega}$ is finite. By Theorem 2.3, the integral over Ω equals

$$_2\widehat{F}_0(a, b; c; s \cdot \theta/u, s \cdot \theta/u),$$

which grows at most polynomially as $s \to \infty$. $\square$

Remark 2.6. Assume, as in Remark 1.10, that $z = m\theta$, $m = 1, 2, \dots$, so that $a = -m$ in Theorem 2.5 above. In this case $E_\theta(\omega; u)$ reduces to

$$E_\theta(\omega; u) = u^{-m} \prod_{i=1}^m (u + \alpha_i).$$

Then the integral in the left–hand side of (2.6) takes the form

$$\text{const} \cdot (u_1 \cdots u_l)^{-m}$$

$$\times \int_{(\mathbb{R}_{\geq 0})^m} \prod_{j=1}^l \prod_{i=1}^m (u_j + \alpha_i) \cdot \prod_{1 \leq i < j \leq m} |\alpha_i - \alpha_j|^{2\theta} \cdot \prod_{i=1}^m \alpha_i^{z'-(m-1)\theta-1} e^{-\alpha_i} d\alpha_i.$$

On the other hand, one can prove the general identity: for $m = 1, 2, \dots$,

$$_2F_0^{(\nu)}(-m, b; x_1^{-1}, \cdots, x_l^{-1}) = \prod_{i=1}^l (b - (i-1)\nu)_m \cdot (x_1 \cdots x_l)^{-m}$$

$$\times {}_1F_1^{(\nu)}(-m; -b - m + 1 + (l-1)\nu; -x_1, \dots, -x_l).$$

(Note that the series for $_1F_1$ in the right–hand side terminates.)

Thus, (2.6) turns into (using the notation $A = z' - (m-1)\theta > 0$)

$$\int_{(\mathbb{R}_{\geq 0})^m} \prod_{j=1}^l \prod_{i=1}^m (u_j + \alpha_i) \cdot \prod_{1 \leq i < j \leq m} |\alpha_i - \alpha_j|^{2\theta} \cdot \prod_{i=1}^m \alpha_i^{A-1} e^{-\alpha_i} d\alpha_i$$

$$= \text{const} \cdot {}_1F_1^{(1/\theta)} \left(-m; \frac{A+l-1}{\theta}; -\frac{u_1}{\theta}, \dots, -\frac{u_l}{\theta} \right).$$

This agrees with the results of [Ka] and [BF].

Remark 2.7. The formula

$$_1\mathcal{F}_0^{(\nu)}(a; \underbrace{x, \dots, x}_{l \text{ times}}; \tau_1, \dots, \tau_l) = \prod_{i=1}^l (1 - x\tau_1)^{-a}$$

shows that the integral representations of Propositions 2.2 and 2.4 in the case when $x_1 = \dots = x_l = x$ involve elementary functions only.

3. Lattice correlation functions

The lifting transform introduced at the end of §1 has a natural discrete counterpart. Starting with probability measures $M_{z,z',\theta}^{(n)}$ on $\mathbb{Y}_n$, $n = 0, 1, \ldots$, we define a probability measure $\widetilde{M}_{z,z',\theta;\xi}$ on the set $\mathbb{Y} = \mathbb{Y}_0 \sqcup \mathbb{Y}_1 \sqcup \mathbb{Y}_2 \sqcup \ldots$ of all Young diagrams with an additional parameter $\xi \in (0,1)$ by

$$\widetilde{M}_{z,z',\theta;\xi}(\lambda) = (1-\xi)^t \frac{(t)_n}{n!} \xi^n \cdot M_{z,z',\theta}^{(n)}(\lambda), \qquad n = |\lambda|.$$

That is, we mix the measures on $\mathbb{Y}_n$'s using the *negative binomial distribution* $\{(1-\xi)^t \frac{(t)_n}{n!} \xi^n\}$ on nonnegative integers n.

In the particular case $\theta = 1$, these mixed measures on $\mathbb{Y}$ were introduced in [BO2]. They are a special case of Okounkov's Schur measures defined in [Ok4]. For general $\theta > 0$, the measures $\widetilde{M}_{z,z',\theta;\xi}$ are a special case of "Jack measures" — a natural extension of Okounkov's concept.

In the next section we will show that the lifted measure $\underline{\widetilde{M}}_{z,z',\theta}$ on $\widetilde{\Omega}$ can be obtained as a limit of the discrete mixed measures $\widetilde{M}_{z,z',\theta;\xi}$ as $\xi \nearrow 1$.

For the rest of this section we assume that θ is a positive integer: $\theta = 1, 2, 3, \ldots$.

To a Young diagram λ we assign a semiinfinite point configuration $\mathcal{L} = \mathcal{L}(\lambda)$ on $\mathbb{Z}$, as follows

$$\mathcal{L} = \{l_1, l_2, \ldots\}, \qquad l_i := \lambda_i - i\theta.$$

In particular,

$$\mathcal{L}(\varnothing) = \{l_1^\varnothing, l_2^\varnothing, l_3^\varnothing, \ldots\} = \{-\theta, -2\theta, -3\theta, \ldots\}.$$

Proposition 3.1. *A sequence of integers $\mathcal{L} = (l_1, l_2, \ldots)$ is of the form $\mathcal{L} = \mathcal{L}(\lambda)$ for some Young diagram λ if and only if the following conditions hold:*

(i) $l_i - l_{i+1} \geq \theta$ *for all i.*

(ii) *If i is large enough then $l_i - l_{i+1} = \theta$.*

(iii) *The stable value of the quantity $l_i + i\theta$, whose existence follows from (ii), equals 0.*

Proof. The above conditions are clearly necessary. Let us check that they are sufficient. Set $\lambda_i = l_i + i\theta$. Condition (i) implies that $\lambda_i \geq \lambda_{i+1}$. Conditions (ii) and (iii) imply that $\lambda_i = 0$ for all i large enough. Hence $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition. $\square$

Let $\mathcal{L}$ satisfy the conditions (i)–(iii) from Proposition 3.1. Let $a \in \mathcal{L}$. If one removes a from $\mathcal{L}$ then the new configuration $\mathcal{L} \setminus \{a\}$ will satisfy (i) and (ii) but not (iii). Indeed, in $\mathcal{L} \setminus \{a\}$, the stable value of the quantity $l_i + i\theta$ will be equal to $-\theta$, not 0. To compensate, we shift the whole $\mathcal{L} \setminus \{a\}$ by θ (that is, we add θ to all members of the sequence). Then (i) and (ii) remain intact while the stable value in (iii) becomes equal to 0, as required. Let us denote the resulting configuration by $\mathcal{D}_a(\mathcal{L})$.

Observe that $\mathcal{D}_a(\mathcal{L})$ does not intersect $\{a+1, \ldots, a+2\theta-1\}$. Conversely, any configuration that satisfies this property together with (i)–(iii) has the form $\mathcal{D}_a(\mathcal{L})$ for a certain configuration $\mathcal{L}$ satisfying (i)–(iii).

One could also define the inverse operation: given a configuration satisfying (i)–(iii) and not intersecting $\{a+1, \ldots, a+2\theta-1\}$, we add to it the point $a+\theta$ and then shift all the points by $-\theta$.

We use the same symbol $\mathcal{D}_a$ to denote the corresponding operation on Young diagrams. In diagram notation, this operation looks as follows. Given $\lambda \in \mathbb{Y}$, let j be such that $\lambda_j - j\theta = a$, which is equivalent to $l_j = a$ (if there is no such j then the operation is not defined). Then

$$\mathcal{D}_a(\lambda) = (\lambda_1 + \theta, \ldots, \lambda_{j-1} + \theta, \lambda_{j+1}, \lambda_{j+2}, \ldots).$$

Note that

$$|\mathcal{D}_a(\lambda)| = |\lambda| - a - \theta.$$

More generally, let $A = \{a_1, \ldots, a_k\}$ be a k–tuple of integral points such that the pairwise distances between them are at least θ. Given a diagram λ such that $\mathcal{L}(\lambda)$ contains A we define a new diagram $\mathcal{D}_A(\lambda)$ as follows: $\mathcal{L}(\mathcal{D}_A(\lambda))$ is obtained from $\mathcal{L}(\lambda)$ by removing A and shifting the remaining points by $k\theta$. Clearly,

$$\mathcal{D}_A = \mathcal{D}_{a_k+(k-1)\theta} \circ \cdots \circ \mathcal{D}_{a_2+\theta} \circ \mathcal{D}_{a_1}.$$

It follows, in particular, that

$$|\mathcal{D}_A(\lambda)| = |\lambda| - a_1 - \cdots - a_k - \frac{k(k+1)}{2}\theta.$$

Proposition 3.2. *Fix a k-point subset A of $\mathbb{Z}$. A Young diagram μ can be represented as $\mathcal{D}_A(\lambda)$ for a Young diagram λ if and only if $\mathcal{L}(\mu)$ does not intersect the set*

$$\bigcup_{j=1}^{k} [a_j + (k-1)\theta + 1, \, a_j + (k+1)\theta - 1].$$

Proof. Evident. $\square$

For any Young diagram λ we introduce a rational function

$$E_\theta^*(\lambda; u) = \prod_{i=1}^{\infty} \frac{u + \lambda_i - i\theta + \theta}{u - i\theta + \theta} = \prod_{i=1}^{\infty} \frac{u + l_i + \theta}{u - i\theta + \theta}.$$

Both these products are, in fact, finite, because the ith factor turns into 1 as soon as $i > \ell(\lambda)$. This function has no poles in $\{u \in \mathbb{C} : \Re u < 0\}$. As we will see later, $E_\theta^*(\lambda; u)$ is a discrete counterpart of the function $E_\theta(\omega; u)$ introduced in §2.

We also define

$$E_\theta^\#(\lambda; u) = \frac{E_\theta^*(\lambda; u)}{\Gamma(-u/\theta)}. \tag{3.1}$$

Proposition 3.3. *For any Young diagram λ, $E_\theta^\#(\lambda; u)$ is an entire function in u. It has simple zeros at the points $u = -l_i - \theta = -\lambda_i + i\theta - \theta$, where $i = 1, 2, \ldots$. Moreover, these are the only zeros of $E_\theta^\#(\lambda; u)$.*

Proof. Fix λ and let r be a large enough integer. We have

$$E_\theta^\#(\lambda; u) = \frac{1}{\Gamma(-u/\theta)} \prod_{i=1}^{r} \frac{u + l_i + \theta}{u - i\theta + \theta}$$

$$= \frac{1}{\Gamma(-u/\theta)} \prod_{i=1}^{r} \frac{-u/\theta - l_i/\theta - 1}{-u/\theta + i - 1}$$

$$= \frac{1}{\Gamma(-u/\theta + r)} \prod_{i=1}^{r} (-u/\theta - l_i/\theta - 1)$$

This expression is clearly an entire function in u. Restrict u to a left half–plane of the form $\Re u \le c$ where $c \gg 0$. The above argument with large enough r shows that the factor $\frac{1}{\Gamma(-u/\theta+r)}$ does not vanish in that half–plane. Thus, the only zeros come from the product. But these are simple zeros at $u = -l_i - \theta$. $\square$

For any function F on the set $\mathbb{Y}$ of all Young diagrams we denote by $\langle F \rangle_{z,z',\theta;\xi}$ the average value of F with respect to $\widetilde{M}_{z,z',\theta;\xi}$:

$$\langle F \rangle_{z,z',\theta;\xi} = \sum_{\lambda \in \mathbb{Y}} F(\lambda) \widetilde{M}_{z,z',\theta;\xi}(\lambda).$$

The next statement expresses the correlation functions of the mixed measures $\widetilde{M}_{z,z',\theta;\xi}$ through the averages of products of $E_\theta^\#$ with appropriate arguments.

Theorem 3.4. *Let $A = \{a_1, \ldots, a_k\}$ be a k-point subset of $\mathbb{Z}$. We have*

$$\widetilde{M}_{z,z',\theta;\xi}\left(\{\lambda \in \mathbb{Y} \mid \mathcal{L}(\lambda) \supset A\}\right) = C \left\langle \prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} E_\theta^\#(\,\cdot\,;u_{j,\sigma}^+)E_\theta^\#(\,\cdot\,;u_{j,\sigma}^-) \right\rangle_{z-k\theta,z'-k\theta,\theta;\xi}$$

where the prefactor C is given by

$$C = (2\pi)^{k(\theta-1)}(\Gamma(\theta))^k \theta^{-2(a_1+\cdots+a_k)-\theta k(2k+1)}(1-\xi)^{k(z+z')-k^2\theta}\,\xi^{a_1+\cdots+a_k+k(k+1)\theta/2}$$

$$\times \prod_{j=1}^{k} \frac{\Gamma(z+a_j+\theta)\Gamma(z'+a_j+\theta)}{\Gamma(z-j\theta+\theta)\Gamma(z'-j\theta+\theta)} \cdot \prod_{1 \le j < j' \le k} \prod_{\sigma=0}^{\theta-1} ((a_j - a_{j'})^2 - \sigma^2).$$

and

$$u_{j,\sigma}^\pm = -a_j \pm \sigma - (k+1)\theta, \qquad j = 1, \ldots, k, \quad \sigma = 0, 1, \ldots, \theta - 1.$$

Proof. The claim is equivalent to

$$\sum_{\lambda : \mathcal{L}(\lambda) \supset A} \widetilde{M}_{z,z',\theta;\xi}(\lambda) = C \sum_{\mu \in \mathbb{Y}} \prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} E_\theta^\#(\mu;u_{j,\sigma}^+)E_\theta^\#(\mu;u_{j,\sigma}^-) \cdot \widetilde{M}_{z-k\theta,z'-k\theta,\theta;\xi}(\mu).$$

If $\mathcal{L}(\mu)$ intersects

$$\bigcup_{j=1}^{k} [a_j + (k-1)\theta + 1,\, a_j + (k+1)\theta - 1]$$

then one of the factors $E_\theta^\#(\mu;u_{j,\sigma}^\pm)$ vanishes by Proposition 3.3. Hence, we may consider only those μ which are of the form $\overline{\lambda} := \mathcal{D}_A(\lambda)$.

Thus, it suffices to prove that for any λ such that $\mathcal{L}(\lambda)$ contains A,

$$\widetilde{M}_{z,z',\theta;\xi}(\lambda) = C \prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} E_\theta^\#(\overline{\lambda};u_{j,\sigma}^+)E_\theta^\#(\overline{\lambda};u_{j,\sigma}^-) \cdot \widetilde{M}_{z-k\theta,z'-k\theta,\theta;\xi}(\overline{\lambda}).$$

By the definition of $\widetilde{M}_{z,z',\theta;\xi}$, we have

$$\widetilde{M}_{z,z',\theta;\xi}(\lambda) = \underbrace{(1-\xi)^{zz'/\theta}\xi^{|\lambda|}}_{(1)} \cdot \underbrace{(z)_{\lambda,\theta}(z')_{\lambda,\theta}}_{(2)} \cdot \underbrace{\frac{1}{H(\lambda;\theta)H'(\lambda;\theta)}}_{(3)} \cdot$$

Similarly,

$$\widetilde{M}_{z-k\theta,z'-k\theta,\theta;\xi}(\overline{\lambda})$$

$$= \underbrace{(1-\xi)^{(z-k\theta)(z'-k\theta)/\theta}\xi^{|\overline{\lambda}|}}_{(1)} \cdot \underbrace{(z-k\theta)_{\overline{\lambda},\theta}(z'-k\theta)_{\overline{\lambda},\theta}}_{(2)} \cdot \underbrace{\frac{1}{H(\overline{\lambda};\theta)H'(\overline{\lambda};\theta)}}_{(3)} \cdot$$

The ratio of the first factors is

$$\frac{(1-\xi)^{zz'/\theta}\,\xi^{|\lambda|}}{(1-\xi)^{(z-k\theta)(z'-k\theta)/\theta}\,\xi^{|\overline{\lambda}|}} = (1-\xi)^{k(z+z')-k^2\theta}\,\xi^{a_1+\cdots+a_k+k(k+1)\theta/2} \cdot$$

We used the fact that $|\overline{\lambda}| = |\lambda| - (a_1 + \cdots + a_k) - \frac{k(k+1)}{2}\theta$ mentioned above.

To handle the second factors, let us rewrite these factors in terms of $\mathcal{L}(\lambda)$, $\mathcal{L}(\overline{\lambda})$. Denote

$$\mathcal{L}(\lambda) = \{l_1, l_2, \dots\}, \qquad \mathcal{L}(\overline{\lambda}) = \{\bar{l}_1, \bar{l}_2, \dots\}.$$

With this notation, for any integral r large enough we can write

$$(z)_{\lambda,\theta}(z')_{\lambda,\theta} = \prod_{i=1}^{r} \frac{\Gamma(z+l_i+\theta)}{\Gamma(z-i\theta+\theta)}\frac{\Gamma(z'+l_i+\theta)}{\Gamma(z'-i\theta+\theta)},$$

$$(z-k\theta)_{\overline{\lambda},\theta}(z'-k\theta)_{\overline{\lambda},\theta} = \prod_{i=1}^{r-k} \frac{\Gamma(z-k\theta+\bar{l}_i+\theta)}{\Gamma(z-k\theta-i\theta+\theta)}\frac{\Gamma(z'-k\theta+l_i+\theta)}{\Gamma(z'-k\theta-i\theta+\theta)}.$$

Observe that for a large integer r the numbers $\bar{l}_1, \dots, \bar{l}_{r-k}$ are obtained from the numbers $l_1, \dots, l_r$ by removing $a_1, \dots, a_k$ and adding $k\theta$ to each of the $r-k$ remaining numbers. This implies that

$$(z)_{\lambda,\theta}(z')_{\lambda,\theta} = \prod_{j=1}^{k} \frac{\Gamma(z+a_j+\theta)\Gamma(z'+a_j+\theta)}{\Gamma(z-j\theta+\theta)\Gamma(z'-j\theta+\theta)} \cdot (z-k\theta)_{\overline{\lambda},\theta}(z'-k\theta)_{\overline{\lambda},\theta}.$$

The ratio of the third factors is computed in

Lemma 3.5. *For any large enough integer r, we have*

$$\frac{H(\overline{\lambda};\theta)H'(\overline{\lambda};\theta)}{H(\lambda;\theta)H'(\lambda;\theta)} = (\Gamma(\theta))^k \prod_{1\le j<j'\le k}\prod_{\sigma=0}^{\theta-1}((a_j-a_{j'})^2-\sigma^2)$$

$$\times \prod_{i=1}^{r-k}\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1}((\bar{l}_i-a_j-k\theta)^2-\sigma^2) \cdot \prod_{j=1}^{k}\frac{1}{\Gamma(a_j+r\theta+1)\Gamma(a_j+r\theta+\theta)} \cdot$$

Proof.

$$H(\lambda;\theta) = \prod_{1 \leq i < j \leq r} \frac{((j-i)\theta + 1 - \theta)_{\lambda_i - \lambda_j}}{((j-i)\theta + 1)_{\lambda_i - \lambda_j}} \cdot \prod_{i=1}^{r} ((r-i)\theta + 1)_{\lambda_i}$$

$$= \prod_{1 \leq i < j \leq r} \frac{\Gamma(l_i - l_j + 1 - \theta)}{\Gamma(l_i - l_j + 1)} \cdot \prod_{1 \leq i < j \leq r} \frac{\Gamma((j-i)\theta + 1)}{\Gamma((j-i)\theta + 1 - \theta)} \cdot \prod_{i=1}^{r} \frac{\Gamma(l_i + r\theta + 1)}{\Gamma((r-i)\theta + 1)},$$

The first product is equal to

$$\prod_{1 \leq i < j \leq r} \prod_{\sigma=0}^{\theta - 1} \frac{1}{l_i - l_j - \sigma}.$$

The second product is equal to

$$\prod_{1 \leq i < j \leq r} \frac{\Gamma((j-i)\theta + 1)}{\Gamma((j-i-1)\theta + 1)} = \prod_{i=1}^{r} \Gamma((r-i)\theta + 1).$$

Hence, we obtain

$$H(\lambda;\theta) = \prod_{1 \leq i < j \leq r} \prod_{\sigma=0}^{\theta - 1} \frac{1}{l_i - l_j - \sigma} \cdot \prod_{i=1}^{r} \Gamma(l_i + r\theta + 1).$$

Likewise,

$$H'(\lambda;\theta) = \prod_{1 \leq i < j \leq r} \frac{((j-i)\theta)_{\lambda_i - \lambda_j}}{((j-i)\theta + \theta)_{\lambda_i - \lambda_j}} \cdot \prod_{i=1}^{r} ((r-i)\theta + \theta)_{\lambda_i}$$

$$= \prod_{1 \leq i < j \leq r} \frac{\Gamma(l_i - l_j)}{\Gamma(l_i - l_j + \theta)} \cdot \prod_{1 \leq i < j \leq r} \frac{\Gamma((j-i)\theta + \theta)}{\Gamma((j-i)\theta)} \cdot \prod_{i=1}^{r} \frac{\Gamma(l_i + r\theta + \theta)}{\Gamma((r-i)\theta + \theta)}$$

$$= \prod_{1 \leq i < j \leq r} \prod_{\sigma=0}^{\theta - 1} \frac{1}{l_i - l_j + \sigma} \cdot \prod_{i=1}^{r} \frac{\Gamma(l_i + r\theta + \theta)}{\Gamma(\theta)}.$$

Therefore,

$$H(\lambda;\theta)H'(\lambda;\theta) = \prod_{1 \leq i < j \leq r} \prod_{\sigma=0}^{\theta - 1} \frac{1}{(l_i - l_j)^2 - \sigma^2} \cdot \prod_{i=1}^{r} \frac{\Gamma(l_i + r\theta + 1)\Gamma(l_i + r\theta + \theta)}{\Gamma(\theta)}.$$

Similarly, for $\overline{\lambda}$ we get

$$H(\overline{\lambda};\theta)H'(\overline{\lambda};\theta)$$

$$= \prod_{1 \leq i < j \leq r-k} \prod_{\sigma=0}^{\theta - 1} \frac{1}{(\overline{l}_i - \overline{l}_j)^2 - \sigma^2} \cdot \prod_{i=1}^{r-k} \frac{\Gamma(\overline{l}_i + (r-k)\theta + 1)\Gamma(\overline{l}_i + (r-k)\theta + \theta)}{\Gamma(\theta)}.$$

Using the observation made before the statement of Lemma 3.5, we readily obtain the needed result. $\square$

Lemma 3.6. *For any large enough integer r, we have*

$$\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1} E_\theta^\#(\overline{\lambda}; u_{j,\sigma}^+) E_\theta^\#(\overline{\lambda}; u_{j,\sigma}^-) = (2\pi)^{k(1-\theta)}\theta^{2(a_1+\cdots+a_k)+\theta k(2k+1)}$$

$$\times \prod_{i=1}^{r-k}\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1}((\overline{l}_i - a_j - k\theta)^2 - \sigma^2) \cdot \prod_{j=1}^{k}\frac{1}{\Gamma(a_j + r\theta + 1)\Gamma(a_j + r\theta + \theta)}$$

Proof. We have, cf. the proof of Proposition 3.3,

$$E_\theta^\#(\overline{\lambda}; u) = \frac{\displaystyle\prod_{i=1}^{r-k}(u + \overline{l}_i + \theta)}{(-\theta)^{r-k}\Gamma(-u/\theta + r - k)}.$$

Note that

$$u_{j,\sigma}^\pm + \overline{l}_i + \theta = -a_j + \overline{l}_i - k\theta \pm \sigma, \qquad -\frac{u_{j,\sigma}^\pm}{\theta} + r - k = \frac{a_j \mp \sigma}{\theta} + r + 1.$$

Hence,

$$\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1} E_\theta^\#(\overline{\lambda}; u_{j,\sigma}^+) E_\theta^\#(\overline{\lambda}; u_{j,\sigma}^-) = \frac{\theta^{-2\theta k(r-k)}\displaystyle\prod_{i=1}^{r-k}\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1}((\overline{l}_i - a_j - k\theta)^2 - \sigma^2)}{\displaystyle\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1}\Gamma\left(\frac{a_j - \sigma}{\theta} + r + 1\right)\Gamma\left(\frac{a_j + \sigma}{\theta} + r + 1\right)}.$$

Applying the multiplication formula for the gamma-function

$$\prod_{\sigma=0}^{\theta-1}\Gamma\left(x + \frac{\sigma}{\theta}\right) = (2\pi)^{\frac{\theta-1}{2}}\theta^{\frac{1}{2}-\theta x}\Gamma(\theta x) \tag{3.2}$$

in the denominator, we obtain the result. $\square$

Putting all these computations together, we arrive at the formula of Theorem 3.4. $\square$

To conclude this section, we restate Theorem 3.4 in terms of averages of $E_\theta^*(\,\cdot\,; u)$ rather than $E_\theta^\#(\,\cdot\,; u)$. Because of that, we have to restrict ourselves to subsets A of $\mathbb{Z}_{\geq 0}$, not of $\mathbb{Z}$, but the new formulation will be more convenient for the limit transition in §4.

Corollary 3.7. *Let $A = \{a_1,\ldots,a_k\}$ be a k-point subset of $\mathbb{Z}_{\geq 0}$. We have*

$$\widetilde{M}_{z,z',\theta;\xi}\left(\{\lambda \in \mathbb{Y} \mid \mathcal{L}(\lambda) \supset A\}\right) = C'\left\langle\prod_{j=1}^{k}\prod_{\sigma=0}^{\theta-1} E_\theta^*(\,\cdot\,; u_{j,\sigma}^+) E_\theta^*(\,\cdot\,; u_{j,\sigma}^-)\right\rangle_{z-k\theta, z'-k\theta, \theta; \xi}$$

where the prefactor C' is given by

$$C' = (1-\xi)^{k(z+z')-k^2\theta}\,\xi^{a_1+\cdots+a_k+k(k+1)\theta/2}\prod_{j=1}^{k}\frac{\Gamma(\theta)}{\Gamma(a_j + k\theta + 1)\Gamma(a_j + k\theta + \theta)}$$

$$\times \prod_{j=1}^{k}\frac{\Gamma(z + a_j + \theta)\Gamma(z' + a_j + \theta)}{\Gamma(z - j\theta + \theta)\Gamma(z' - j\theta + \theta)} \cdot \prod_{1 \leq j < j' \leq k}\prod_{\sigma=0}^{\theta-1}((a_j - a_{j'})^2 - \sigma^2).$$

and

$$u_{j,\sigma}^{\pm} = -a_j \pm \sigma - (k+1)\theta, \qquad j = 1, \ldots, k, \quad \sigma = 0, 1, \ldots, \theta - 1.$$

Proof. First of all, recall that $E_\theta^*(\,\cdot\,; u)$ is a meromorphic function in u which has no poles in $\{u \in \mathbb{C} : \Re u < 0\}$. Because of that, the product of E_θ^* above makes sense if all a_i are nonnegative. Indeed, then $u_{j,\sigma}^{\pm} < 0$ for all j, s.

By (3.1), we have

$$\prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} E_\theta^{\#}(\,\cdot\,; u_{j,\sigma}^+) E_\theta^{\#}(\,\cdot\,; u_{j,\sigma}^-) = \frac{\prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} E_\theta^*(\,\cdot\,; u_{j,\sigma}^+) E_\theta^*(\,\cdot\,; u_{j,\sigma}^-)}{\prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} \Gamma(-u_{j,\sigma}^+/\theta) \Gamma(-u_{j,\sigma}^-/\theta)}.$$

Applying the multiplication formula for the gamma-function (3.2), we obtain

$$\prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} \Gamma(-u_{j,\sigma}^+/\theta) \Gamma(-u_{j,\sigma}^-/\theta)$$

$$= (2\pi)^{k(\theta-1)} \theta^{-2(a_1 + \cdots + a_k) - \theta k(2k+1)} \prod_{j=1}^{k} \Gamma(a_j + k\theta + 1) \Gamma(a_j + k\theta + \theta).$$

Thus, Theorem 3.4 implies the needed claim with C' equal to C divided by the expression above. $\square$

4. CONVERGENCE OF CORRELATION FUNCTIONS

The goal of this section is to prove that the lattice correlation functions

$$M_{z,z',\theta}^{(n)}\left(\{\lambda \in \mathbb{Y}_n \mid \mathcal{L}(\lambda) \supset \{x_1, \ldots, x_k\}\}\right), \ \widetilde{M}_{z,z',\theta;\xi}\left(\{\lambda \in \mathbb{Y} \mid \mathcal{L}(\lambda) \supset \{x_1, \ldots, x_k\}\}\right)$$

converge, in the corresponding scaling limits as $n \to \infty$ or $\xi \nearrow 1$, to the correlation functions

$$\rho_k(y_1, \ldots, y_k), \qquad \widetilde{\rho}_k(y_1, \ldots, y_k)$$

defined in the end of §1.

For the random Young diagram $\lambda \in \mathbb{Y}_n$ distributed according to $M_{z,z',\theta}^{(n)}$ introduce the random variables

$$\alpha_i^{(n)} = \begin{cases} \dfrac{l_i - i\theta}{n}, & l_i - i\theta > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{l_1, l_2, \ldots\} = \mathcal{L}(\lambda)$. These $\alpha_i^{(n)}$ are different from those introduced in Remark 1.7 by $O(1/n)$. Thus, by Remark 1.7, we still have for any positive integer m the convergence

$$\{\alpha_1^{(n)}, \ldots, \alpha_m^{(n)}\} \xrightarrow{d} \{\alpha_1, \ldots, \alpha_m\}. \tag{4.1}$$

Let $r_k^{(n)}$ denote the kth correlation measure for $\left\{\alpha_i^{(n)}\right\}_{i=1}^{\infty}$. Formally, for any compactly supported continuous function F on $(\mathbb{R}_{>0})^k$,

$$\langle F, r_k^{(n)} \rangle = \mathbb{E}_n \left(\sum_{\substack{i_1, i_2, \ldots, i_k \\ \text{pairwise distinct}}} F(\alpha_{i_1}^{(n)}, \ldots, \alpha_{i_k}^{(n)}) \right), \tag{4.2}$$

where $\mathbb{E}_n$ denotes the expectation with respect to $M^{(n)}_{z,z',\theta}$.

Recall that the kth correlation measure for $\{\alpha_i\}$ was defined in a similar way in §1:

$$\langle F, \rho_k \rangle = \mathbb{E}\left(\sum_{\substack{i_1, i_2, \ldots, i_k \\ \text{pairwise distinct}}} F(\alpha_{i_1}, \ldots, \alpha_{i_k}) \right), \tag{4.3}$$

where $\mathbb{E}$ denotes the expectation with respect to $\underline{M}_{z,z',\theta}$.

Proposition 4.1. *For any $k = 1, 2, \ldots$, and any compactly supported continuous function F on $(\mathbb{R}_{>0})^k$, we have*

$$\langle F, r^{(n)}_k \rangle \longrightarrow \langle F, \rho_k \rangle, \qquad n \to \infty.$$

Proof. We rely on the convergence of the finite-dimensional distributions (4.1) and the fact that

$$\alpha^{(n)}_1 \geq \alpha^{(n)}_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} \alpha^{(n)}_i \leq 1,$$
$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} \alpha_i \leq 1. \tag{4.4}$$

These inequalities imply that

$$\alpha^{(n)}_{m+1} < 1/m, \quad \alpha_{m+1} < 1/m, \qquad m = 1, 2, \ldots, \tag{4.5}$$

cf. (1.2). Fix m so large that $\mathrm{supp}\, F \subset (\mathbb{R}_{\geq 1/m})^k$. Then the summands in (4.2) and (4.3) involving indices $i_l > m$ vanish. Thus, only finitely many summands remain, and the statement follows from (4.1). $\square$

We proceed to the mixed measures $\widetilde{M}_{z,z',\theta;\xi}$. For the random Young diagram $\lambda \in \mathbb{Y}$ distributed according to $M_{z,z',\theta;\xi}$ introduce the random variables

$$\alpha_{i,\xi} = \begin{cases} (1 - \xi)(l_i - i\theta), & l_i - i\theta > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\{l_1, l_2, \ldots\} = \mathcal{L}(\lambda)$ as above. We define the mixed correlation measures $\widetilde{r}^{(\xi)}_k$, $k = 1, 2, \ldots$, by

$$\langle F, \widetilde{r}^{(\xi)}_k \rangle = \mathbb{E}_\xi\left(\sum_{\substack{i_1, i_2, \ldots, i_k \\ \text{pairwise distinct}}} F(\alpha_{i_1,\xi}, \ldots, \alpha_{i_k,\xi}) \right),$$

where $\mathbb{E}_\xi$ denotes the expectation with respect to $M_{z,z',\theta;\xi}$. These are essentially the same objects as in Theorem 3.4, with the lattice $\mathbb{Z}$ being scaled by $(1 - \xi)$.

Recall that the lifted correlation functions (measures) $\widetilde{\rho}_k$ were defined in the end of §1.

Proposition 4.2. *For any $k = 1, 2, \ldots$, and any compactly supported continuous function F on $(\mathbb{R}_{>0})^k$, we have*

$$\langle F, \widetilde{r}_k^{(\xi)} \rangle \longrightarrow \langle F, \widetilde{\rho}_k \rangle, \qquad n \to \infty.$$

Proof. Let

$$\gamma_t = \frac{s^{t-1}}{\Gamma(t)} e^{-s} ds$$

be the gamma-distribution on $\mathbb{R}_{>0}$ with the parameter $t = zz'/\theta$, and let

$$\gamma_{t,\xi} = (1-\xi)^t \sum_{n=0}^{\infty} \frac{(t)_n}{n!} \xi^n \delta_{n(1-\xi)}$$

be a scaled version of the negative binomial distribution. Here δ_x stands for the Dirac measure at x. The similarity of notation is justified by the following statement.

Lemma 4.3. (i) *The distribution $\gamma_{t,\xi}$ weakly converges to γ_t as $\xi \nearrow 1$.*

(ii) *All moments of the distribution $\gamma_{t,\xi}$ converge to the respective moments of γ_t as $\xi \nearrow 1$.*

Proof of Lemma 4.3. (i) For any $s > 0$ we define $n(s, \xi) = [s/(1-\xi)]$. Since both $\gamma_{t,\xi}$ and γ_t are *probability* measures, it suffices to show that

$$(1-\xi)^t \frac{(t)_n}{n!} \xi^n \cdot (1-\xi)^{-1} \longrightarrow \frac{s^{t-1}}{\Gamma(t)} e^{-s}, \qquad n = n(s, \xi), \quad \xi \nearrow 1,$$

for any $s > 0$. Indeed, we have, with $n = n(s, \xi)$ and $\xi \nearrow 1$,

$$(1-\xi)^{t-1} \frac{(t)_n}{n!} \xi^n = \frac{(1-\xi)^{t-1}}{\Gamma(t)} \frac{\Gamma(n+t)}{\Gamma(n+1)} (1 - (1-\xi))^n$$

$$\sim \frac{(1-\xi)^{t-1} n^{t-1} e^{-s}}{\Gamma(t)} \sim \frac{s^{t-1} e^{-s}}{\Gamma(t)}.$$

(ii) We have to prove that for any $m = 1, 2, \ldots$,

$$\lim_{\xi \nearrow 1} \left((1-\xi)^t \sum_{n=0}^{\infty} \frac{(t)_n}{n!} \xi^n (n(1-\xi))^m \right) = \int_0^{\infty} \frac{s^{t-1}}{\Gamma(t)} s^m e^{-s} ds = (t)_m.$$

Note that

$$(n(1-\xi))^m = (1-\xi)^m n(n-1) \cdots (n-m+1) \cdot (1 + O(1-\xi))$$

uniformly in $n = 0, 1, \ldots$. Thus, it suffices to show that

$$\lim_{\xi \nearrow 1} \left((1-\xi)^{t+m} \sum_{n=0}^{\infty} \frac{(t)_n}{n!} \xi^n n(n-1) \cdots (n-m+1) \right) = (t)_m.$$

But the sum in the left–hand side is easily computed:

$$\sum_{n=0}^{\infty} \frac{(t)_n}{n!} \, \xi^n n(n-1)\cdots(n-m+1) = (t)_m \xi^m \sum_{l=0}^{\infty} \frac{(t+m)_l}{l!} \, \xi^l = (t)_m \xi^m (1-\xi)^{-t-m}.$$

The needed limit relation immediately follows. $\square$

Let us return to the proof of Proposition 4.2. We have

$$\langle F, \widetilde{r}_k^{(\xi)} \rangle = \int_0^{\infty} \mathbb{E}_{n(s,\xi)} \left(\sum_{\substack{i_1,\ldots,i_k \\ \text{pairwise distinct}}} F\left(s \cdot \alpha_{i_1}^{(n(s,\xi))}, \ldots, s \cdot \alpha_{i_k}^{(n(s,\xi))}\right) \right) \gamma_{t,\xi}(ds).$$

Note that for $s \in \mathrm{supp}(\gamma_{t,\xi})$, $n(s,\xi) = [s/(1-\xi)] = s/(1-\xi)$.

Similarly,

$$\langle F, \widetilde{\rho}_k \rangle = \int_0^{\infty} \mathbb{E} \left(\sum_{\substack{i_1,\ldots,i_k \\ \text{pairwise distinct}}} F(s \cdot \alpha_{i_1}, \ldots, s \cdot \alpha_{i_k}) \right) \gamma_t(ds).$$

Fix $\epsilon > 0$ so small that $\mathrm{supp}\, F \subset (\mathbb{R}_{\geq \epsilon})^k$. Since $\alpha_i^{(n)} \leq 1$, $\alpha_i \leq 1$, both integrals remain intact if we replace the lower limit of integration by ϵ.

Lemma 4.4. *For any $S > \epsilon$, we have*

$$\lim_{\xi \nearrow 1} \int_{\epsilon}^{S} \mathbb{E}_{n(s,\xi)} \left(\sum_{\substack{i_1,\ldots,i_k \\ \text{pairwise distinct}}} F\left(s \cdot \alpha_{i_1}^{(n(s,\xi))}, \ldots, s \cdot \alpha_{i_k}^{(n(s,\xi))}\right) \right) \gamma_{t,\xi}(ds)$$

$$= \int_{\epsilon}^{S} \mathbb{E} \left(\sum_{\substack{i_1,\ldots,i_k \\ \text{pairwise distinct}}} F(s \cdot \alpha_{i_1}, \ldots, s \cdot \alpha_{i_k}) \right) \gamma_t(ds).$$

Proof of Lemma 4.4. By the argument in the proof of Proposition 4.1, the sums above are actually finite, and it suffices to prove the limit relation for any fixed indices $i_1, \ldots, i_k$, that is, we will show that

$$\lim_{\xi \nearrow 1} \int_{\epsilon}^{S} \mathbb{E}_{n(s,\xi)} \left(F\left(s \cdot \alpha_{i_1}^{(n(s,\xi))}, \ldots, s \cdot \alpha_{i_k}^{(n(s,\xi))}\right) \right) \gamma_{t,\xi}(ds)$$

$$= \int_{\epsilon}^{S} \mathbb{E}\left(F(s \cdot \alpha_{i_1}, \ldots, s \cdot \alpha_{i_k})\right) \gamma_t(ds).$$

It is convenient to denote $F_s(x_1, \ldots, x_k) = F(s \cdot x_1, \ldots, s \cdot x_k)$. Since F is compactly supported, the map $s \mapsto F_s$ is continuous on $[\epsilon, S]$ with respect to the sup-norm in the Banach space of continuous functions. Therefore, $\{F_s, s \in [\epsilon, S]\}$ is a compact

set. Hence, by (4.1), $\mathbb{E}_n(F_s(\alpha_{i_1}^{(n)}, \ldots, \alpha_{i_k}^{(n)}))$ is close to $\mathbb{E}(F_s(\alpha_{i_1}, \ldots, \alpha_{i_k}))$ for large n uniformly in $s \in [\epsilon, S]$.

Since the variable of integration s is bounded away from zero, $n(s, \xi)$ is uniformly large as $\xi \nearrow 1$. Thus, it suffices to show that

$$\lim_{\xi \nearrow 1} \int_\epsilon^S \mathbb{E}\left(F(s \cdot \alpha_{i_1}, \ldots, s \cdot \alpha_{i_k})\right) \gamma_{t,\xi}(ds) = \int_\epsilon^S \mathbb{E}\left(F(s \cdot \alpha_{i_1}, \ldots, s \cdot \alpha_{i_k})\right) \gamma_t(ds).$$

Since the integrand is continuous in s, the convergence follows from Lemma 4.3(i). □

To complete the proof of the Proposition 4.2, it remains to prove that

$$\int_S^\infty \mathbb{E}_{n(s,\xi)}\left(\sum_{\substack{i_1,\ldots,i_k \\ \text{pairwise distinct}}} F\left(s \cdot \alpha_{i_1}^{(n(s,\xi))}, \ldots, s \cdot \alpha_{i_k}^{(n(s,\xi))}\right)\right) \gamma_{t,\xi}(ds) \longrightarrow 0$$

as $S \to \infty$, uniformly in ξ.

Observe that for any fixed s the number of terms in the sum above is $O(s^k)$ independently of ξ. Indeed, recall that $\alpha_{m+1}^{(n)} < 1/m$, see (4.5). On the other hand, we must have $s\alpha_{i_l}^{(n)} \geq \epsilon$ in order for the corresponding term not to vanish. Thus, we are only allowed to take $i_l \leq s/\epsilon$.

Thus, the absolute value of the integral is bounded by

$$\text{const} \cdot \int_S^\infty s^k \gamma_{t,\xi}(ds),$$

and the result readily follows from Lemma 4.3(ii). The proof of Proposition 4.2 is complete. □

5. LIMIT CORRELATION FUNCTIONS

The goal of this section is to derive hypergeometric-type formulas for the limit correlation functions.

Our first step is to define the limit of the right–hand side of the formula in Corollary 3.7.

We will use the notation (the function $E_\theta(\omega; u)$ was introduced in §2)

$$E^*(\lambda; u) = E_\theta^*(\lambda; u)\big|_{\theta=1} = \prod_{i=1}^\infty \frac{u - \lambda_i - i + 1}{u - i + 1},$$

$$E(\omega; u) = E(\omega; u)\big|_{\theta=1} = e^{\gamma/u} \frac{\prod_{i=1}^\infty (1 + \alpha_i/u)}{\prod_{i=1}^\infty (1 - \beta_i/u)},$$

and

$$\lambda_\theta = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\theta}, \underbrace{\lambda_2, \ldots, \lambda_2}_{\theta}, \ldots), \quad \lambda \in \mathbb{Y},$$

$$\alpha_\theta = (\underbrace{\alpha_1, \ldots, \alpha_1}_{\theta}, \underbrace{\alpha_2, \ldots, \alpha_2}_{\theta}, \ldots), \quad \theta\beta = (\theta\beta_1, \theta\beta_2, \ldots), \quad \omega_\theta = (\alpha_\theta, \theta\beta, \theta\delta).$$

Recall that in §1 we defined the modified Frobenius coordinates $\{a_i(\lambda); b_i(\lambda)\}$ of a Young diagram λ. Set

$$\iota(\lambda) = (a_1(\lambda), a_2(\lambda), \ldots; b_1(\lambda), b_2(\lambda), \ldots; |\lambda|) \in \widetilde{\Omega}.$$

Proposition 5.1.

$$E_\theta^*(\lambda; u) E_\theta^*(\lambda; u-1) \cdots E_\theta^*(\lambda; u - \theta + 1) = E^*(\lambda_\theta; u), \qquad \lambda \in \mathbb{Y},$$
$$(E_\theta(\omega; u))^\theta = E(\omega_\theta; u), \qquad \omega \in \widetilde{\Omega},$$
$$E^*(\lambda; u) = E(\iota(\lambda); u + \tfrac{1}{2}), \qquad \lambda \in \mathbb{Y}.$$

Proof. The first relation readily follows from the definition of λ_θ. The second relation is evident. The third relation is also not hard to prove, see, e.g., [ORV]. $\square$

The third relation shows that E^* and E are essentially the same, if the Young diagrams are viewed as points of $\widetilde{\Omega}$ via the embedding ι.

The next statement computes the limit of the expectation in right–hand side of Corollary 3.7. (To simplify the notation, we temporarily ignore the shift of the parameters z, z' in Corollary 3.7.)

Proposition 5.2. *For any $k = 1, 2, \ldots$, and sufficiently large $x_1, \ldots, x_k > 0$, if $a_i = a_i(\xi)$, $i = 1, \ldots, k$, are such that $a_i(1 - \xi) \to x_i$ as $\xi \nearrow 1$, then*

$$\lim_{\xi \nearrow 1} \left\langle \prod_{j=1}^k \prod_{\sigma=0}^{\theta-1} E_\theta^*(\,\cdot\,; u_{j,\sigma}^+) E_\theta^*(\,\cdot\,; u_{j,\sigma}^-) \right\rangle_{z, z', \theta; \xi} = \int_{\omega \in \widetilde{\Omega}} \prod_{j=1}^k (E_\theta(\omega; -x_j))^{2\theta} \, \widetilde{\underline{M}}_{z, z', \theta}(d\omega),$$

(5.1)

where

$$u_{j,\sigma}^{\pm} = -a_j \pm \sigma - (k+1)\theta, \qquad j = 1, \ldots, k, \quad \sigma = 0, 1, \ldots, \theta - 1.$$

We will need the following simple lemma. Recall that in §1 we introduced a metric on $\widetilde{\Omega}$ denoted by $\mathrm{dist}(\,\cdot\,,\,\cdot\,)$.

Lemma 5.3. (i) *For any $\omega \in \widetilde{\Omega}$ and $u < 0$, we have*

$$|E(\omega; u)| \le e^{\delta(\omega)/|u|}$$

where, as above, $\delta(\omega)$ denotes the δ-coordinate of ω.

(ii) *Assume that $\mathrm{dist}(\omega', \omega'') \to 0$ and $u' - u'' \to 0$. Then*

$$E(\omega'; u') - E(\omega''; u'') \to 0$$

uniformly on any set of the form $\{\omega \in \widetilde{\Omega} : \delta(\omega) \le \mathrm{const}_1\} \times \{u \le \mathrm{const}_2 < 0\}$.

Proof. (i) Without loss of generality we may assume that $\gamma(\omega) = 0$, because this condition defines a dense subset of $\widetilde{\Omega}$. By the 0-homogeneity of $E(\omega; u)$, we may also assume that $u = -1$. Let $m = m(\omega)$ be the number of $\alpha_i = \alpha_i(\omega)$ which are greater than 1. Then

$$|E(\omega; -1)| = \left| \frac{\prod_{i=1}^\infty (1 - \alpha_i)}{\prod_{i=1}^\infty (1 + \beta_i)} \right| \le \prod_{i=1}^m \alpha_i \le \left(\frac{\sum_{i=1}^m \alpha_i}{m} \right)^m \le \frac{\delta^m}{m!} \le e^\delta. \quad \square$$

(ii) By homogeneity, we have

$$E(\omega'; u') = E(\omega'/|u'|; -1), \qquad E(\omega''; u'') = E(\omega''/|u''|; -1).$$

The statement now follows from the uniform continuity of the function $E(\omega; -1)$ on the compact set $\{\omega \in \widetilde{\Omega} : \delta(\omega) \le \mathrm{const}\}$. $\square$

Remark 5.4. Even though the estimate of (i) above seems rather coarse, it cannot be substantially improved: one can show that $\sup\{E(\omega; -1) \mid \delta(\omega) = \Delta\}$ grows at least as $e^{\mathrm{const} \cdot \Delta}$ as $\Delta \to \infty$. As we will see below, this is the reason why we have to assume that x_i's are large in the proof of Proposition 5.2.

Proof of Proposition 5.2. Denote

$$F(\lambda) = \prod_{j=1}^{k} \prod_{\sigma=0}^{\theta-1} E_\theta^*(\lambda; u_{j,\sigma}^+) E_\theta^*(\lambda; u_{j,\sigma}^-).$$

By Proposition 5.1, for any $\lambda \in \mathbb{Y}$ we obtain

$$F(\lambda) = \prod_{j=1}^{k} E^*(\lambda_\theta; -a_j - k\theta - 1) E^*(\lambda_\theta; -a_j - k\theta - \theta)$$

$$= \prod_{j=1}^{k} E\big(\iota(\lambda_\theta); -a_j - k\theta - \tfrac{1}{2}\big) E\big(\iota(\lambda_\theta); -a_j - k\theta - \theta + \tfrac{1}{2}\big)$$

$$= \prod_{j=1}^{k} E\big((1-\xi)\iota(\lambda_\theta); -x_j + O(1-\xi)\big) E\big((1-\xi)\iota(\lambda_\theta); -x_j + O(1-\xi)\big),$$

where in the last equality we used the 0-homogeneity of $E(\omega; u)$.

We now split the average of $F(\lambda)$ with respect to $\widetilde{M}_{z,z',\theta;\xi}$ into two parts: over the Young diagrams λ with $(1-\xi) \cdot |\lambda| > C$ and $(1-\xi) \cdot |\lambda| \leq C$ for some constant C. The first one tends to zero as $C \to \infty$ uniformly in ξ close to 1. Indeed, by Lemma 5.3(i),

$$|F(\lambda)| \leq e^{2\theta k(1-\xi)|\lambda|/K}$$

where we assume that $\min\{x_1, \ldots, x_k\} > K$. By the hypothesis of the proposition, we may choose K as large as we need. Thus,

$$\left| \sum_{n:\,(1-\xi)n>C} \sum_{|\lambda|=n} F(\lambda) \cdot \widetilde{M}_{z,z',\theta;\xi}(\lambda) \right| \leq \sum_{n:\,(1-\xi)n>C} \sup_{|\lambda|=n} |F(\lambda)| \cdot \widetilde{M}_{z,z',\theta;\xi}(\mathbb{Y}_n)$$

$$\leq (1-\xi)^t \sum_{n:\,(1-\xi)n>C} e^{2\theta k(1-\xi)n/K} \frac{(t)_n}{n!} \xi^n.$$

For ξ close to 1, $\xi^n = (1 - (1-\xi))^n \leq e^{-\,\mathrm{const}_1 \cdot n(1-\xi)}$. Further,

$$(1-\xi)^t \frac{(t)_n}{n!} = (1-\xi)^t \frac{\Gamma(t+n)}{\Gamma(t)\Gamma(n+1)} = \frac{(1-\xi)^t n^{t-1}}{\Gamma(t)} \left(1 + O(n^{-1})\right).$$

Hence, the first part of the average is bounded by

$$\mathrm{const}_2 \cdot (1-\xi) \sum_{n:(1-\xi)n>C} e^{\mathrm{const}_3 \cdot n(1-\xi)}$$

where $\text{const}_3 = 2\theta k/K - \text{const}_1$. Choosing K large enough, we make const_3 negative, and then the expression in question is bounded by

$$\text{const}_5 \cdot \int_C^\infty e^{-\text{const}_4 \cdot s} ds, \qquad \text{const}_4 > 0,$$

which goes to 0 as $C \to \infty$.

The second part of the average has a limit as $\xi \nearrow 1$:

$$\sum_{\lambda:\,(1-\xi)|\lambda|<C} F(\lambda)\widetilde{M}_{z,z',\theta;\xi}(\lambda) \longrightarrow \int_{\omega:\,\delta(\omega)<C} \prod_{j=1}^k (E(\omega_\theta; -x_j))^2 \,\underline{\widetilde{M}}_{z,z',\theta}(d\omega).$$

Indeed,

$$F(\lambda) = \prod_{j=1}^k E\big((1-\xi)\iota(\lambda_\theta); -x_j + O(1-\xi)\big)\, E\big((1-\xi)\iota(\lambda_\theta); -x_j + O(1-\xi)\big)$$

is uniformly close to

$$\prod_{j=1}^k (E\big(\omega_\theta; -x_j\big))^2, \qquad \omega = (1-\xi)\iota(\lambda).$$

by Lemma 5.3(ii). On the other hand, by Theorem 1.6 and Lemma 4.3, the image of the measure $\widetilde{M}_{z,z',\theta;\xi}$ under the map $\lambda \mapsto \omega = (1-\xi)\iota(\lambda)$, viewed as a measure on $\widetilde{\Omega}$, weakly converges to $\underline{\widetilde{M}}_{z,z',\theta}$, as $\xi \nearrow 1$.

Since

$$\prod_{j=1}^k (E(\omega_\theta; -x_j))^2 = \prod_{j=1}^k (E_\theta(\omega; -x_j))^{2\theta},$$

by Proposition 5.1, in order to conclude the proof of Proposition 5.2 it remains to show that

$$\int_{\omega:\,\delta(\omega)>C} \prod_{j=1}^k (E(\omega_\theta; -x_j))^2 \,\underline{\widetilde{M}}_{z,z',\theta}(d\omega)$$

converges to 0 as $C \to \infty$ uniformly in ξ close to 1. This fact follows from Lemma 5.3(i) similarly to the argument in the beginning of the proof. Note that this estimate also justifies the convergence of the integral in the right–hand side of (5.1). Another way to estimate the integral over $\{\omega : \delta(\omega) > C\}$ is to directly use the integrability proved in Theorem 2.5. $\square$

Recall that the lifted correlation functions $\widetilde{\rho}_k(x_1, \ldots, x_k)$ (densities of the correlation measures $\widetilde{\rho}_k(dx)$) with positive arguments $x_1, \ldots, x_k$ were defined in §1.

Theorem 5.5. *For any* $k = 1, 2, \ldots,$ *and* $x_1, \ldots, x_k > 0$,

$$\widetilde{\rho}_k(x_1, \ldots, x_k) = \prod_{j=1}^k \frac{\Gamma(\theta)}{\Gamma(z - (j-1)\theta)\Gamma(z' - (j-1)\theta)}$$

$$\times (x_1 \cdots x_k)^{z+z'+\theta-1-2k\theta} e^{-(x_1+\cdots+x_k)} \prod_{1\leq i<j\leq k}(x_i - x_j)^{2\theta}$$

$$\times {}_2F_0^{(1/\theta)}\left(\frac{-z+k\theta}{\theta}, \frac{-z'+k\theta}{\theta}; \underbrace{-\frac{\theta}{x_1}, \ldots, -\frac{\theta}{x_1}}_{2\theta \text{ times}}, \ldots, \underbrace{-\frac{\theta}{x_k}, \ldots, -\frac{\theta}{x_k}}_{2\theta \text{ times}}\right).$$

Proof. The right–hand side is a real-analytic function in $x_1, \dots, x_n > 0$. Hence, by virtue of Proposition 1.9, it suffices to prove the claim for $x_1, \dots, x_k \gg 0$.

On the other hand, for large $x_1, \dots, x_k$, the equality directly follows from Proposition 4.2, Corollary 3.7, Proposition 5.2, and Theorem 2.5. Indeed, Proposition 4.2 shows that the correlation measures $\widetilde{\rho}_k(dx)$ of $\widetilde{\underline{M}}_{z,z',\theta}$ are weakly approximated by their discrete counterparts — the correlation measures $\widetilde{r}_k^{(\xi)}$ of $M_{z,z',\theta;\xi}$. Further, Corollary 3.7 expresses the values of the discrete correlation measures through averages of products of $E^*(\lambda; u)$. Proposition 5.2 then shows that the weak limit of $\widetilde{r}_k^{(\xi)}$, if it exists, must have the density equal to the integral

$$\int\limits_{\omega \in \widetilde{\Omega}} \prod_{j=1}^{k} (E_\theta(\omega; -x_j))^{2\theta} \, \widetilde{\underline{M}}_{z-k\theta, z'-k\theta, \theta}(d\omega)$$

(note the shift of z, z' due to Corollary 3.7) times the limit of $(1-\xi)^{-k} C'$ with C' from Corollary 3.7 (the factor $(1-\xi)^{-k}$ comes from the rescaling $\mathbb{Z} \to (1-\xi)\mathbb{Z}$). This limit is readily computed: for $a_i \sim x_i/(1-\xi)$ as $\xi \nearrow 1$ we have

$$\xi^{a_1 + \cdots + a_k + k(k+1)\theta/2} \sim e^{-x_1 - \cdots - x_k},$$

$$\prod_{j=1}^{k} \frac{\Gamma(z + a_j + \theta)\Gamma(z' + a_j + \theta)}{\Gamma(a_j + k\theta + 1)\Gamma(a_j + k\theta + \theta)}$$

$$\sim (1-\xi)^{-k(z+z'+\theta-1)+2k^2\theta}(x_1 \cdots x_k)^{z+z'-2k\theta+\theta-1},$$

$$\prod_{1 \le j < j' \le k} \prod_{\sigma=0}^{\theta-1} ((a_j - a_{j'})^2 - \sigma^2) \sim (1-\xi)^{k(k-1)\theta} \prod_{1 \le j < j' \le k} (x_j - x_{j'})^{2\theta}.$$

Gathering these pieces together and using Theorem 2.5 we obtain the result. $\square$

We can now invert the integral transform that relates the correlation functions $\widetilde{\rho}_k$ of the lifted measure $\widetilde{\underline{M}}_{z,z',\theta}$ and the correlation functions ρ_k of the initial measure $\underline{M}_{z,z',\theta}$, see §1.

It is convenient to introduce the notation, see [GS]

$$\frac{y_+^{c-1}}{\Gamma(c)} = \begin{cases} \dfrac{y^{c-1}}{\Gamma(c)}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

For $\Re c > 0$ this is a locally integrable function. As a distribution, it admits an analytic continuation in c to the whole complex plane. In particular, for $c = 0$, $\frac{y_+^{c-1}}{\Gamma(c)}$ is the delta-function at the origin.

Theorem 5.6. *For any $k = 1, 2, \ldots,$ and $x_1, \ldots, x_k > 0$*

$$\rho_k(x_1, \ldots, x_k) = \Gamma\left(\frac{zz'}{\theta}\right) \cdot \prod_{j=1}^{k} \frac{\Gamma(\theta)}{\Gamma(z - (j-1)\theta)\Gamma(z' - (j-1)\theta)}$$

$$\times (x_1 \cdots x_k)^{z+z'+\theta-1-2k\theta} \frac{(1-|x|)_+^{c-1}}{\Gamma(c)} \prod_{1 \le i < j \le k} (x_i - x_j)^{2\theta}$$

$$\times {}_2\widehat{F}_1^{(1/\theta)}\left(a, b; c; \underbrace{-\frac{\theta(1-|x|)}{x_1}, \ldots, -\frac{\theta(1-|x|)}{x_1}}_{2\theta \text{ times}}, \ldots, \underbrace{-\frac{\theta(1-|x|)}{x_k}, \ldots, -\frac{\theta(1-|x|)}{x_k}}_{2\theta \text{ times}}\right)$$

where $|x| = x_1 + \cdots + x_k$,

$$a = \frac{-z + k\theta}{\theta}, \qquad b = \frac{-z' + k\theta}{\theta}, \qquad c = ab\,\theta.$$

Note that the expression above vanishes unless $|x| \le 1$. This agrees with the fact that the correlation measure ρ_k is supported by the set where $|x| \le 1$ as was mentioned in §1.

Proof of Theorem 5.6. As was pointed out in §1, the lifting (1.4) is invertible. Therefore, it suffices to check that (1.4) holds with ρ_k given by the formula above and $\widetilde{\rho}_k$ given by Theorem 5.5. We have (recall that $t = zz'/\theta$)

$$\int_0^\infty \frac{s^{t-1}e^{-s}}{\Gamma(t)} \rho_k\left(\frac{x_1}{s}, \ldots, \frac{x_k}{s}\right) \frac{ds}{s^k} = \prod_{j=1}^{k} \frac{\Gamma(\theta)}{\Gamma(z - (j-1)\theta)\Gamma(z' - (j-1)\theta)}$$

$$\times \int_0^\infty \left(\frac{x_1 \cdots x_k}{s^k}\right)^{z+z'+\theta-1-2k\theta} \cdot \frac{(s-|x|)_+^{c-1}}{s^{c-1}\Gamma(c)} \cdot \frac{\prod_{1 \le i < j \le k}(x_i - x_j)^{2\theta}}{s^{k(k-1)\theta}}$$

$$\times {}_2\widehat{F}_1^{(1/\theta)}\left(a, b; c; \underbrace{-\frac{\theta(s-|x|)}{x_1}, \ldots, -\frac{\theta(s-|x|)}{x_1}}_{2\theta \text{ times}}, \ldots, \underbrace{-\frac{\theta(s-|x|)}{x_k}, \ldots, -\frac{\theta(s-|x|)}{x_k}}_{2\theta \text{ times}}\right)$$

$$\times s^{t-1-k}e^{-s}ds.$$

Making the change of variable $s - |x| \to s$ and using (2.5), we obtain the result. $\square$

Remark 5.7. Assume, as in Remarks 1.10 and 2.6, that $z = m\theta$, $m = 1, 2, \ldots,$ and $z' > (m-1)\theta$. Then Theorems 5.5 and 5.6 show that ρ_k and $\widetilde{\rho}_k$ vanish identically for $k \ge m + 1$, which agrees with the fact that the measures $\underline{M}_{z,z',\theta}$ and $\widetilde{\underline{M}}_{z,z',\theta}$ live on the subsets of Ω and $\widetilde{\Omega}$ with no more than m nonzero alpha-coordinates. (The vanishing is caused by the gamma–prefactors.)

The mth correlation function gives the distribution function for $\alpha_1, \ldots, \alpha_m$ given by (1.5) and (1.6). Further, the formulas of Theorems 5.6 and 5.5 with $k < m$ provide the correlation functions for the m-particle Laguerre ensemble (1.6) and its simplex analog (1.5).

Remark 5.8. Theorems 5.5, 5.6, and Remark 2.7 provide integral representations for the density functions $\widetilde{\rho}_1$ and ρ_1 which involve only elementary functions. A similar integral representations has been used in [BF, §5.3] for (saddle point) asymptotic analysis of the density function in the Hermite ensemble when the number of particles goes to infinity.

6. Asymptotics of the correlation functions at the origin

In this section we compute the asymptotics of the correlation functions $\rho_k(x)$ and $\widetilde{\rho}_k(x)$ when $x_1, \ldots, x_k \to +0$. In the variables $y_i = -\ln x_i$ the answer is translation invariant and is the same for both lifted and non-lifted correlation functions. This limit transition is similar to the bulk scaling limit in random matrix models.

We will need certain multivariate special functions $\varphi_s^{(\nu)}(x_1, \ldots, x_l)$, $s \in \mathbb{C}^l$, $x \in (\mathbb{R}_{>0})^l$. These functions are symmetric with respect to permutations of $\{x_i\}$ and generalize the normalized Jack polynomials $P_\lambda^{(\nu)}(x_1, \ldots, x_l)/P_\lambda^{(\nu)}(1, \ldots, 1)$: if $s = \lambda + \rho$, where

$$\rho = \nu \left(\frac{l-1}{2}, \frac{l-3}{2}, \ldots, -\frac{l-3}{2}, -\frac{l-1}{2} \right),$$

then these two functions coincide.

The functions $\varphi_s^{(\nu)}$ can be defined as symmetric, normalized at $(1, \ldots, 1)$ eigenfunctions of the Sekiguchi system of differential operators with appropriate eigenvalues depending on s, see [Sek] and also [Ma2]. The functions $\varphi_s^{(\nu)}$ are symmetric with respect to the permutations of $\{s_i\}$.

When $\nu = 1/2, 1, 2$, the functions $\varphi_s^{(\nu)}$ are spherical functions for the symmetric space $GL(l, \mathbb{F})/U(l, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively, and they admit a matrix integral representation, see [FK, chapter XIV, §3].

In the case $\theta = 1$ the spherical function is given by the explicit formula

$$\varphi_{s_1, \ldots, s_l}^{(1)}(x_1, \ldots, x_l) = 0!1! \cdots (l-1)! \cdot \frac{(x_1 \ldots x_l)^{\frac{l-1}{2}} \det[x_i^{s_j}]}{\prod_{i<j}(x_i - x_j)(s_i - s_j)}.$$

Theorem 6.1. *For any $k = 1, 2, \ldots$, the image of the correlation measure $\rho_k(dx)$ or $\widetilde{\rho}_k(dx)$ under the change of variables*

$$x_i = e^{-y_i - T}, \qquad i = 1, \ldots, k,$$

converges, as $T \to +\infty$, to

$$C \cdot \prod_{1 \le i < j \le k} (e^{-y_i} - e^{-y_j})^{2\theta} \cdot \varphi_s^{(1/\theta)}(\underbrace{e^{-y_1}, \ldots, e^{-y_1}}_{2\theta \text{ times}}, \ldots, \underbrace{e^{-y_k}, \ldots, e^{-y_k}}_{2\theta \text{ times}}) \, dy, \quad (6.1)$$

where

$$C = \prod_{j=0}^{k-1} \frac{\Gamma(j\theta + 1)\Gamma(\theta)\Gamma(j\theta + z - z' + 1)\Gamma(j\theta + z' - z + 1)}{\Gamma(j\theta + k + 1)\Gamma(j\theta - z + 1)\Gamma(j\theta - z' + 1)\Gamma(z - j\theta)\Gamma(z' - j\theta)},$$

$$s = (s'_1, \ldots, s'_{k\theta}, s''_1, \ldots, s''_{k\theta}),$$

$$s'_j = \frac{z' - z - 2j + \theta + 1}{2\theta}, \quad s''_j = \frac{z - z' - 2j + \theta + 1}{2\theta}, \qquad j = 1, \ldots, k\theta.$$

Note that the measure (6.1) is translation invariant. Indeed, this follows from the fact that

$$\varphi_s^{(\nu)}(a \cdot x_1, \ldots, a \cdot x_l) = a^{|s|}\varphi_s^{(\nu)}(x_1, \ldots, x_l), \qquad |s| = s_1 + \cdots + s_l,$$

for any $a > 0$ and $l = 1, 2, \ldots$.

The result for $\theta = 1$ was proved in [P.III]. A stronger result involving joint correlation functions of $\{\alpha_i\}$ and $\{\beta_i\}$ (also for $\theta = 1$) was proved in [P.V].

The proof of Theorem 6.1 is based on multivariate Mellin-Barnes integral representations of ${}_2F_0^{(\nu)}$ and ${}_2\widehat{F}_1^{(\nu)}$. The details will appear elsewhere.

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