Randomly perturbed Toeplitz matrices and structured pseudospectra

A. Böttcher

Mark Embree (Oxford)

Slava Sokolov (Chemnitz)

Sergei Grudsky (Mexico City)

Anderson model (1958)

Discrete Laplacian

$$C_n = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & 0 \end{pmatrix}$$

$\mathrm{sp}\, C_n = \underset{-2}{\cdot}\cdots\cdots\cdots\underset{2}{\cdot}$

eigenvectors  delocalized extended

Discrete Hamiltonian with random potential

$$C_n + X = \begin{pmatrix} x_1 & 1 & & & 1 \\ 1 & x_2 & 1 & & \\ & 1 & x_3 & 1 \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & x_n \end{pmatrix}$$

$x_1, \dots, x_n$ randomly chosen real numbers

$\mathrm{sp}\,(C_n + X) = \underset{-2-\delta}{\cdot}\cdots\cdots\cdots\underset{2+\varepsilon}{\cdot}$

eigenvectors  localized with probability 1

This explains some basic phenomena in physics (supraconduction, conductor-isolator-transition, ...)

1977 Nobel Prize for Physics $\longrightarrow$ P.W. Anderson

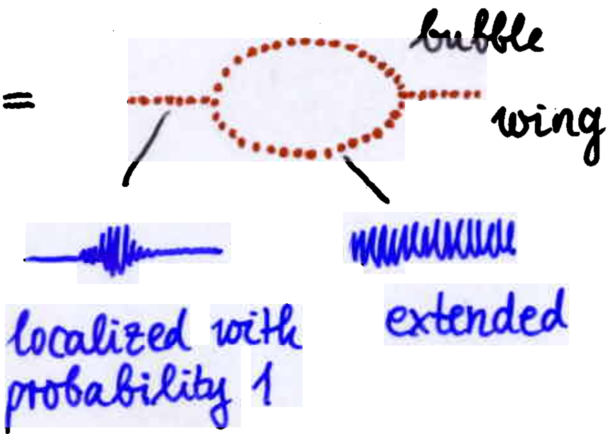
Hatano-Nelson model (1996)

$$C_n = \begin{pmatrix} 0 & \alpha^2 & & & 1 \\ 1 & 0 & \alpha^2 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & \alpha^2 \\ \alpha^2 & & & 1 & 0 \end{pmatrix}$$

$\mathrm{sp}\, C_n =$ 

$$C_n + X = \begin{pmatrix} x_1 & \alpha^2 & & & 1 \\ 1 & x_2 & \alpha^2 & & \\ & 1 & x_3 & \ddots & \\ & & \ddots & \ddots & \alpha^2 \\ \alpha^2 & & & 1 & x_n \end{pmatrix}$$

$x_1, \dots, x_n$ randomly chosen real numbers

$\mathrm{sp}\,(C_n + X) =$ 

Applications in non-Hermitian quantum mechanics, population biology, small world networks, eigenvalue perturbation theory for general matrices, ...

Population Dynamics and Non-Hermitian Localization

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delocalization is accompanie ... ıvalues into the complex plane. This important signature of delocalization is preserved ... lattice approximation to the continuum Eq. (1.1), useful for treating both square well ... s described in Section III. Exact results and various similarity transformations for the ... . [11] are summarized in Section IV. The one dimensional lattice model is used to explo ... ɩions in Section V. In the appendix we give analytical expressions for the spectrum of ... ıs system for two different lattices and the continuum model.

SQUARE WELL GROWTH PROFILES

In the follo ... ation Eq. (1.1) with a simple "square well" growth profile $U(\mathbf{x})$, imposing a positive gr ... ı patch ("oasis"), and a negative growth rate $-\epsilon a$ outside ("desert") [10]:

$$U(\mathbf{x}) = \begin{cases} a, & \text{for } |\mathbf{x}| < \frac{W}{2}, \\ -\epsilon a, & \text{for } |\mathbf{x}| \geq \frac{W}{2}, \end{cases} \tag{2.1}$$

wh ... oasis. Experimentally this situation can be realized using a very simple setup, w ... of localization and delocalization and leads to interesting further questions. A one ... in figure 1, where a solution with photosynthetic bacteria in a thin circular pipe, or ... .ted by a fixed uniform light source through a mask, leading to a "square well" intensity ... oved at a small, constant velocity around the sample to simulate convective flow. (Moving ... .troducing convective flow in the system, up to a change of reference frame [9].) The bacteria ... the brightly illuminated area ("oasis") at a certain rate, but division ceases or proceeds at a ... .he darker region ("desert") outside. As a result, the growth rate in this continuum population ... sitive in the oasis and small (positive or negative) in the surrounding desert region. Using this ... wth model, one can study the total number of bacteria expected to survive in the steady state, ... distribution in space and other quantities, as a function of the "convection velocity" of the light

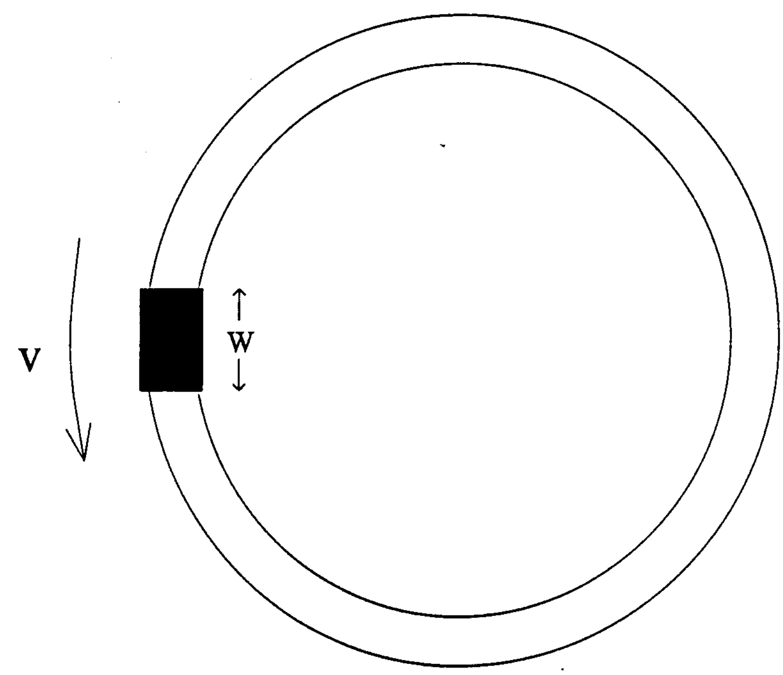

FIG. 1. Experimental setup: a solution with photosynthetic bacteria in a circular pipe or a thin annular track in a petri dish is illuminated only in a small area, while the rest of the sample is either kept dark or illuminated with reduced intensity. The light source (shaded rectangle) is moved slowly around the sample to model convective flow. The bacteria are assumed to divide in the illuminated area ("oasis") at a certain growth rate $a > 0$, and die (or grow modestly) in the remaining area ("desert") with growth rate $-\epsilon a$.

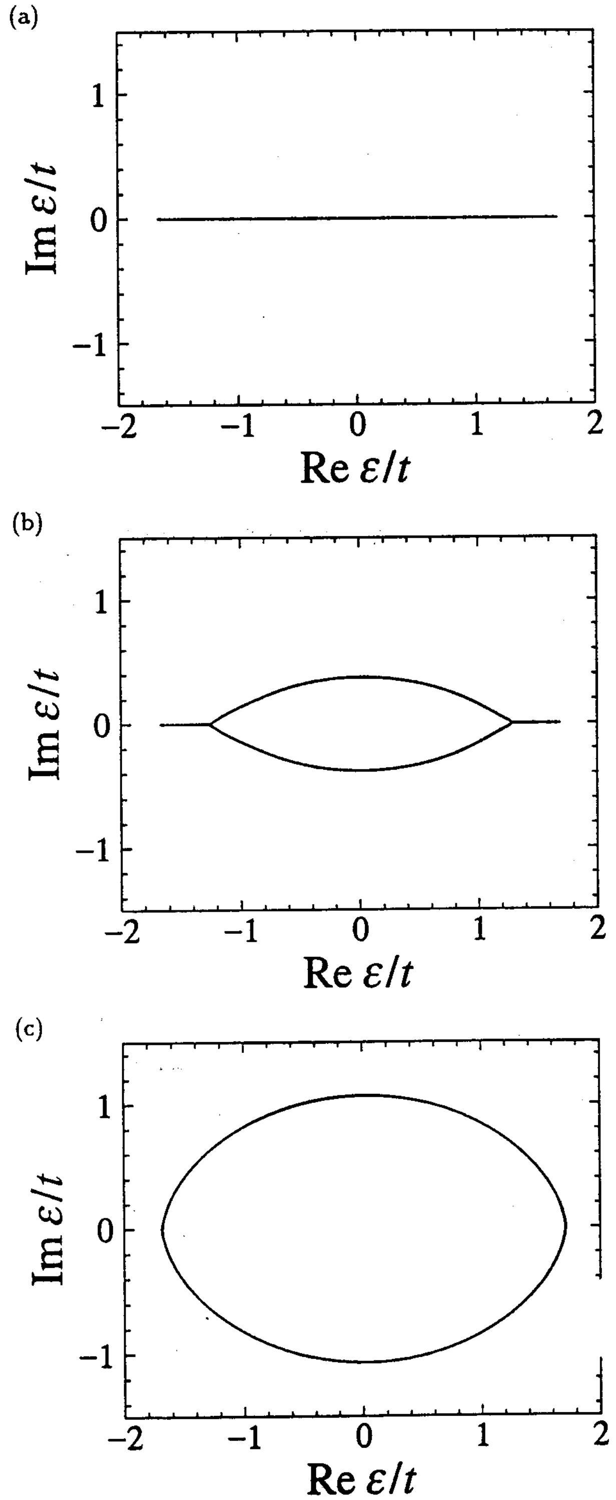

FIG. 2. Energy spectra of one-dimensional 1000-site lattice model with randomness $\Delta/w = 1$. The resulting spectrum for the same realization of the random potential $U(x) \in [-\Delta, \Delta]$ is plotted here for three different values of g. (a) Case $g < g_1$; all eigenstates are localized; (b) $g_1 < g < g_2$; bubble of complex eigenvalues indicating extended states appears near the center of the band; (c) $g_2 < g$; all the eigenstates are extended. (After Ref. [25]).

For g less than a critical value g_1, all eigenmodes are localized, and the eigenvalues remain real and locked to their values for $g = 0$. For $g_1 < g < g_2$, extended states with complex eigenvalues appear near the center of the band. Localized states still appear near the band edges. For $g > g_2$, every localized state is destroyed by the non-Hermitian perturbation, and all states are extended. In this limit, eigenfunctions are slightly perturbed Bloch states—the lattice version of plane waves. The spectrum is well approximated by the disorder-free limit, i.e., the lattice analogue of Eq. (1.15)

$$\Gamma(k) = 2w\cos(k\ell_0 + ig\ell_0), \tag{1.21}$$

where ℓ_0 is the lattice constant.

With our definition of $\tilde{\mathcal{L}}$, states near the *top* of the band should give a reasonable approximatior ʰe spectrum of the continuous operator (1.14); E' ʰhen describes the upper edge of the ellipse ' Fig. 2c. The states at the bottom of t' hand, have spatial characteristi' the lattice discretization.

Figure 3 shows typical sp ator in *two* dimensions [2' s near the top of the ban' ıa- tion to growth modes i' ı, all eigenvalues remain re' ain localized when $g < g_1$ xtended and localized stat the band [25]. When $g >$ ng states at the top of the ' y large g (not shown) the ıe disorder-free limit of th' one-dimensional case. Hc [25], this apparent simplif e artifact in $d = 2$: level ification of the Bloch wa' ven for weak disorder in Thus, chaotic spectra sim- xpected for large v due to namic limit" of large system we argue that the growth for n fact described by the $(d-1)$- ger-like equation with space- and domness. The anomalous critical ex- scribe this situation lead to universal arities in the density of states near the ıe complex plane.

I a gradient and set $\mathbf{u}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t)$, the dy cal growth model (1.11) which results from a Cole-Hopf transformation reads

$$\begin{aligned} \partial_t \mathbf{u}(\mathbf{x}, t) + (\mathbf{v} \cdot \nabla)\mathbf{u}(\mathbf{x}, t) + 2\lambda D(\mathbf{u}(\mathbf{x}, t) \cdot \nabla)\mathbf{u}(\mathbf{x}, t) \\ = D\nabla^2 \mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}), \end{aligned} \tag{1.22}$$

with $\mathbf{f}(\mathbf{x}) = \nabla U(\mathbf{x})$ and subject to the constraint $\nabla \times \mathbf{u} = 0$. This is a variant of the d-dimensional generalization of Burger's equation with noise studied 20 years ago by Forster *et al.* [26]. In the form (1.11), such problems are sometimes referred to as "KPZ equations".

Non-Hermitian Localization and Population Biology

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(November 8, 1999)

Toeplitz matrices: boundedness

Infinite Toeplitz matrix

$$A = (a_{j-k})_{j,k=1}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Toeplitz 1911

$$\boxed{\text{A bounded} \iff \text{the numbers $\{a_n\}$ are the Fourier coefficients of some } a \in L^\infty := L^\infty(\mathbb{T}).}$$

a is unique, is called the symbol

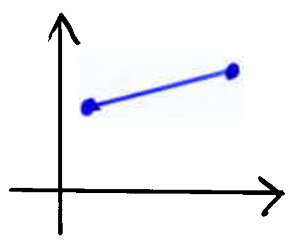
$$a(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$$

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \qquad T_n(a) = \begin{pmatrix} a_0 & \dots & a_{-(n-1)} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \dots & a_0 \end{pmatrix}$$

$T(a)$ compact $\iff$ $a = 0$ (Gohberg 1952)

$T(a)$ selfadjoint $\iff$ a real-valued (trivial)

$T(a)$ normal $\iff$ conv $\mathcal{R}(a)$ is a line segment



(Brown/Halmos 1963)

Toeplitz matrices: spectra

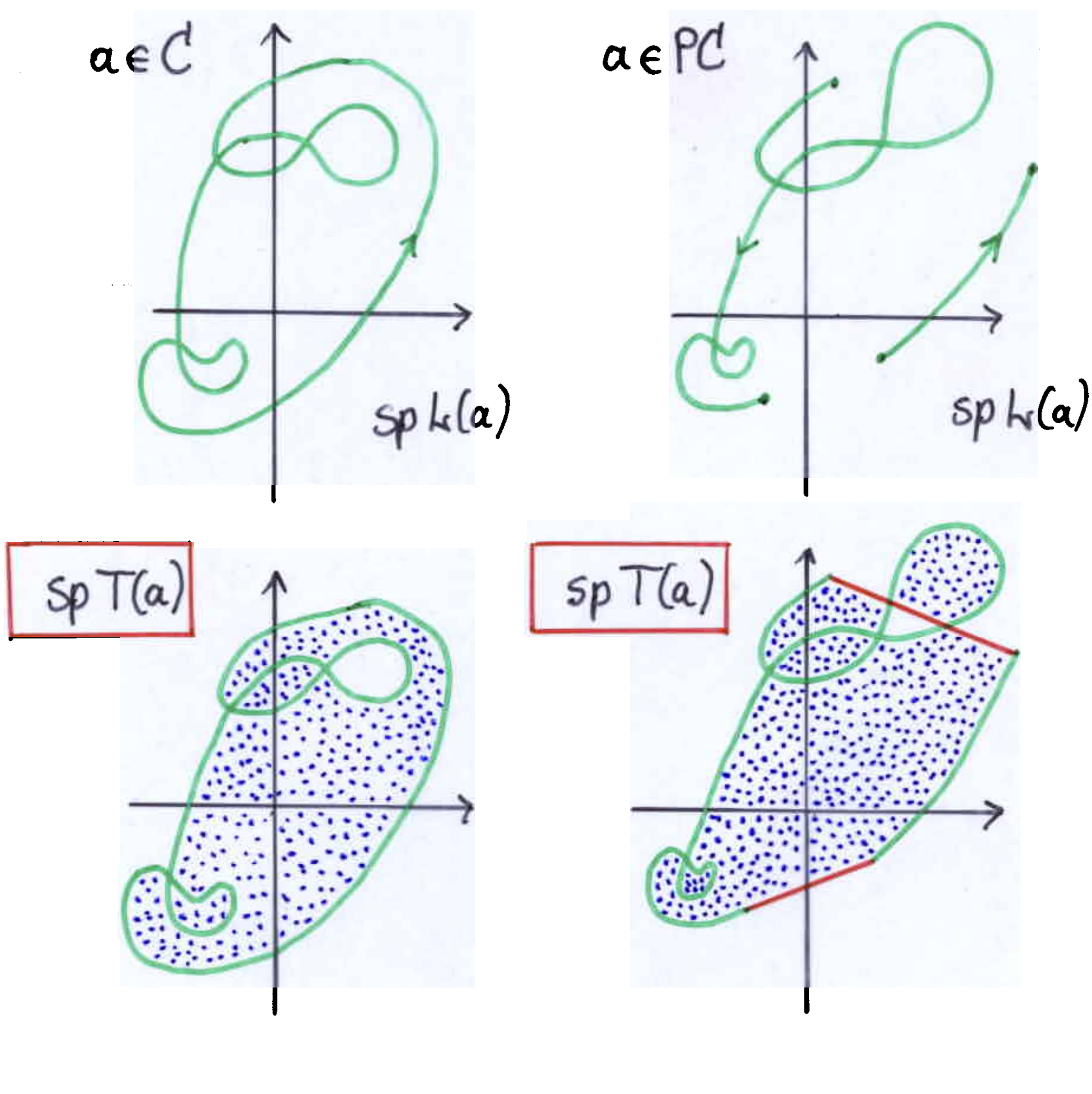
$$L(a) = \left(\begin{array}{cccc|cccc} \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & & a_{-2} & a_{-3} & \cdots \\ \cdots & a_1 & a_0 & & a_{-1} & a_{-2} & \cdots \\ \hline \cdots & a_2 & a_1 & & a_0 & a_{-1} & \cdots \\ \cdots & a_3 & a_2 & & a_1 & a_0 & \cdots \\ \cdots & a_4 & a_3 & & a_2 & a_1 & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \end{array}\right)$$

Laurent matrix

unitarily equivalent to the operator of multiplication by a on $L^2(\mathbb{T})$

$$\Rightarrow \boxed{\operatorname{sp} L(a) = \mathcal{R}(a)} \quad \text{(essential range of } a\text{)}$$

Spectra of infinite Toeplitz matrices

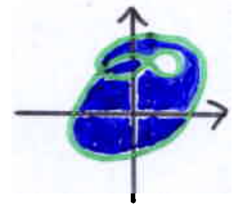


Finite Toeplitz matrices

$T_n(a)$ as a submatrix of $T(a)$

$$\left(\begin{array}{cccc|cc}
a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\
a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\
a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\
\hline
a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right)$$

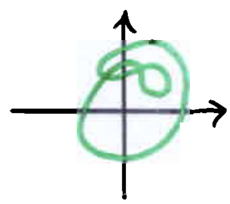
$$\boxed{\operatorname{sp} T_n(a) \xrightarrow{?} \operatorname{sp} T(a)}$$



$T_n(a)$ as a submatrix of $L(a)$

$$\left(\begin{array}{cccc|cccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & a_{-5} & \cdots \\
\cdots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\
\cdots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\
\hline
\cdots & a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
\cdots & a_4 & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\
\cdots & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right)$$

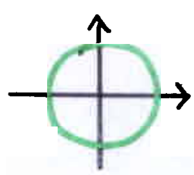
$$\boxed{\operatorname{sp} T_n(a) \xrightarrow{?} \operatorname{sp} L(a)}$$

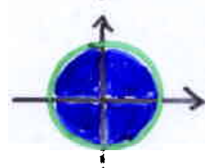


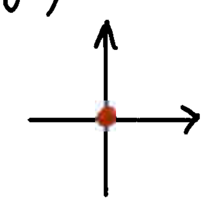
Example 1

$a(e^{i\theta}) = e^{i\theta}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\operatorname{sp} L(a) =$ 

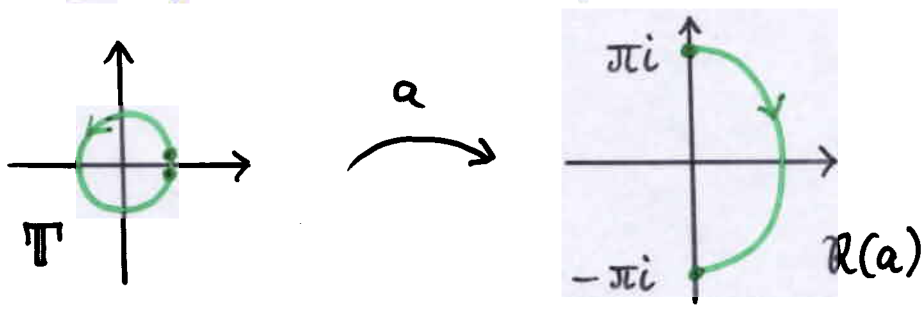
$\operatorname{sp} T(a) =$ 

$\operatorname{sp} T_n(a) = \{0\} =$ 

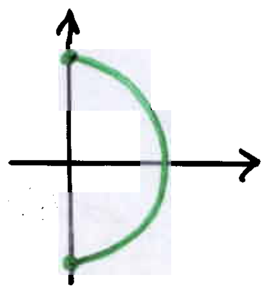
Example 2 Cauchy/Hilbert Toeplitz matrices

$$T(a) = \left(\frac{1}{j-k+\frac{1}{2}} \right)_{j,k=1}^{\infty}$$

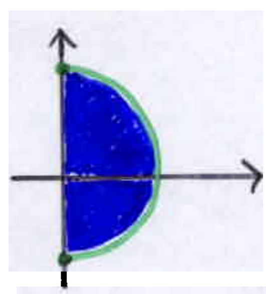
$$a(e^{i\theta}) = \pi i \, e^{-i\theta/2}, \quad \theta \in [0, 2\pi)$$



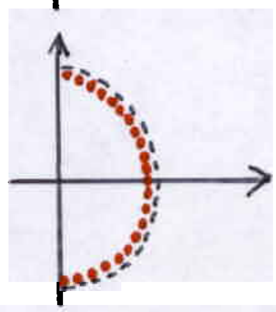
$\operatorname{sp} L(a) =$



$\operatorname{sp} T(a) =$



$\operatorname{sp} T_n(a) =$



$$\boxed{\liminf \operatorname{sp} T_n(a) = \limsup \operatorname{sp} T_n(a) = \operatorname{sp} L(a)}$$

Bö/Silbermann 1986, Widom 1990

Banded Toeplitz matrices

$$a(t) = a_{-q}t^{-q} + \dots + a_0 + \dots + a_p t^p \qquad (t = e^{i\theta})$$

trigonometric polynomial

$$P(z) = a_{-q} + a_{-q+1} z + \dots + a_p z^{p+q}$$

Label the zeros $z_1(\lambda), \dots, z_{p+q}(\lambda)$ of

$$P(z) - \lambda z^q = 0$$

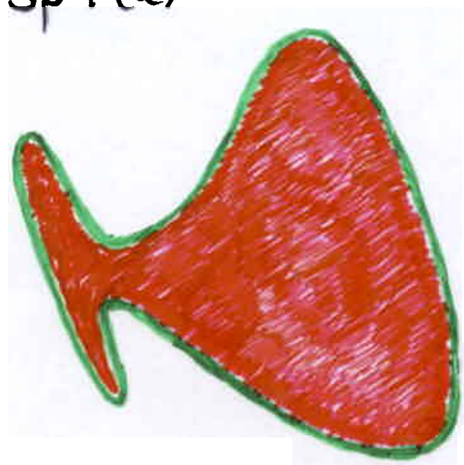
so that

$$|z_1(\lambda)| \leqq |z_2(\lambda)| \leqq \dots \leqq |z_{p+q}(\lambda)|.$$

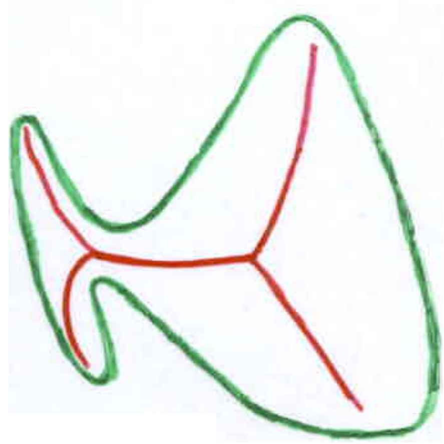
$$\Lambda(a) := \{\lambda \in \mathbb{C} : |z_q(\lambda)| = |z_{q+1}(\lambda)|\}$$

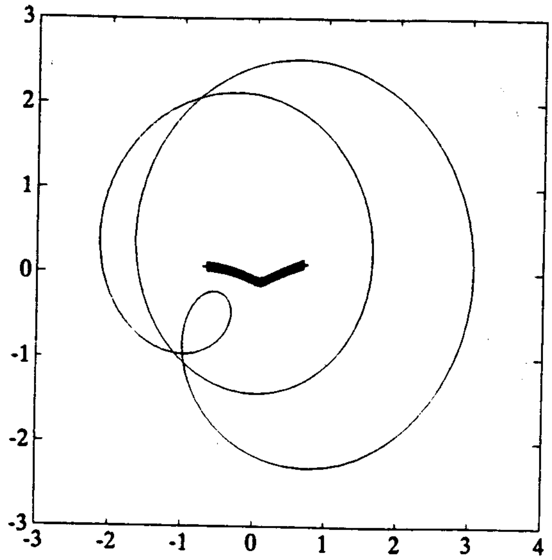
$$\boxed{\begin{array}{l} \text{Schmidt/Spitzer 1960} \\ \displaystyle\lim_{n\to\infty} \operatorname{sp} T_n(a) = \Lambda(a) \end{array}}$$

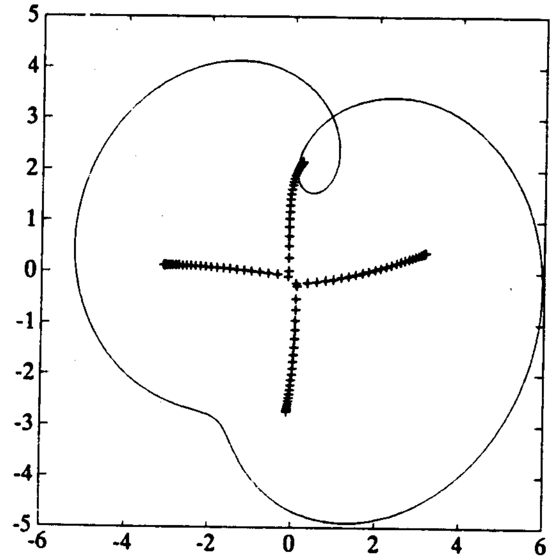
$\operatorname{sp} T(a)$

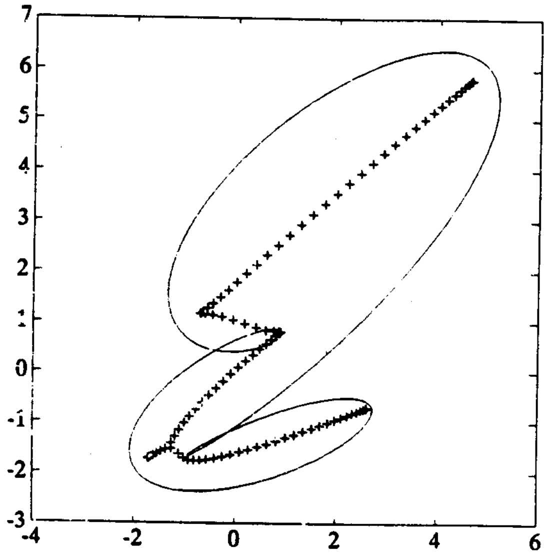


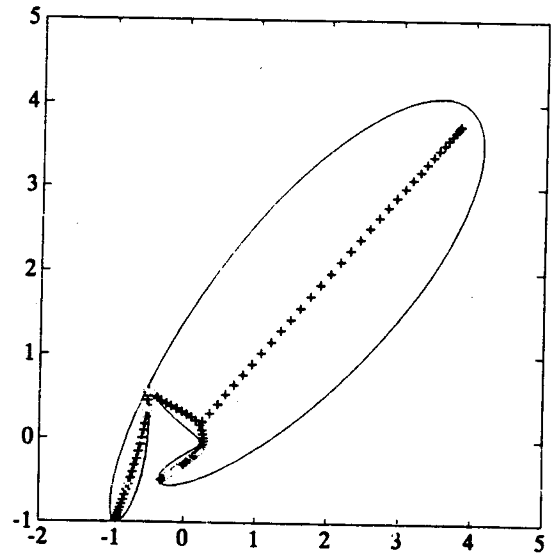
$\Lambda(a)$

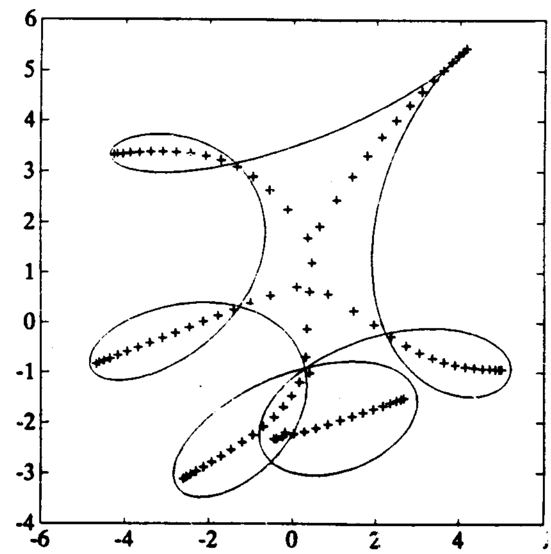


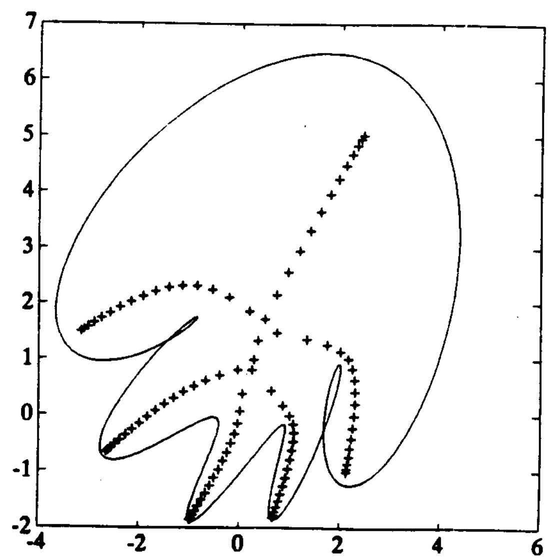

3 2 1 0 -1 -2 -3
-3 -2 -1 0 1 2 3 4


5 4 3 2 1 0 -1 -2 -3 -4 -5
-6 -4 -2 0 2 4 6


7 6 5 4 3 2 1 0 -1 -2 -3
-4 -2 0 2 4 6


5 4 3 2 1 0 -1
-2 -1 0 1 2 3 4 5


6 5 4 3 2 1 0 -1 -2 -3 -4
-6 -4 -2 0 2 4


7 6 5 4 3 2 1 0 -1 -2
-4 -2 0 2 4 6

Pseudospectra

$\varepsilon > 0$, ε-pseudospectrum of $A \in \mathcal{B}(H)$

$$\mathrm{sp}_\varepsilon A = \bigcup_{\|E\| \leq \varepsilon} \mathrm{sp}\,(A+E)$$

Alternatively,

$$\mathrm{sp}_\varepsilon A = \left\{ \lambda \in \mathbb{C} : \|(A-\lambda I)^{-1}\| \geq \frac{1}{\varepsilon} \right\}$$

In a sense, pseudospectra are of even greater import than spectra

Question: Is A invertible?

$\downarrow$

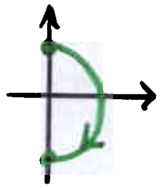
Question: What is $\|A^{-1}\|$?

In contrast to spectra, pseudospectra of Toeplitz matrices behave as nicely as we could ever expect.

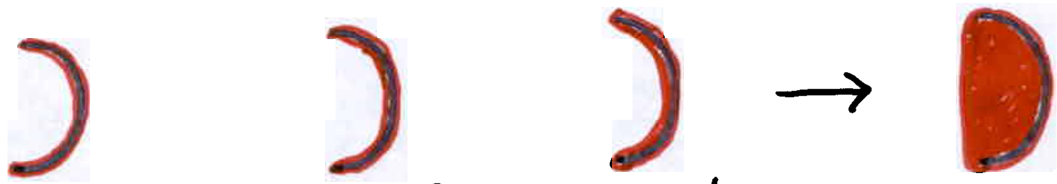
$$\boxed{\begin{array}{l} \text{Landau 1975} \\ \text{Reichel and Trefethen 1992} \\ a \text{ smooth, } \varepsilon > 0 \\ \Rightarrow \quad \mathrm{sp}_\varepsilon T_n(a) \to \mathrm{sp}_\varepsilon T(a) \end{array}}$$

Bö 1994

Slow convergence

$a(e^{i\theta}) = \pi i e^{-i\theta/2}$  $T_n(a) = \left(\frac{1}{j-k+\frac{1}{2}}\right)_{j,k=1}^{n}$

$sp_\varepsilon T_n(a)$ $sp_\varepsilon T(a)$



$\varepsilon = 10^{-5}$

$n = 10^2$ $n = 10^3$ $n = 10^4$

$$sp_\varepsilon T_n(a) = \left\{\lambda : \|T_n^{-1}(a-\lambda)\| \geq \frac{1}{\varepsilon}\right\} \to sp_\varepsilon T(a)$$

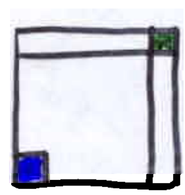
Reason: $\|T_n^{-1}(a-\lambda)\|$ grows very slowly

$a(\mathbb{T})$

$$T_n(a-\lambda) = T_n(c\varphi_\beta)$$

Bö/Silbermann 1986

$$\|T_n^{-1}(c\varphi_\beta)\| \simeq n^{2\beta-1}$$



Lower estimate:

$$\|T_n^{-1}(c\varphi_\beta)\| \geq \left|[T_n^{-1}(c\varphi_\beta)]_{n,1}\right|$$

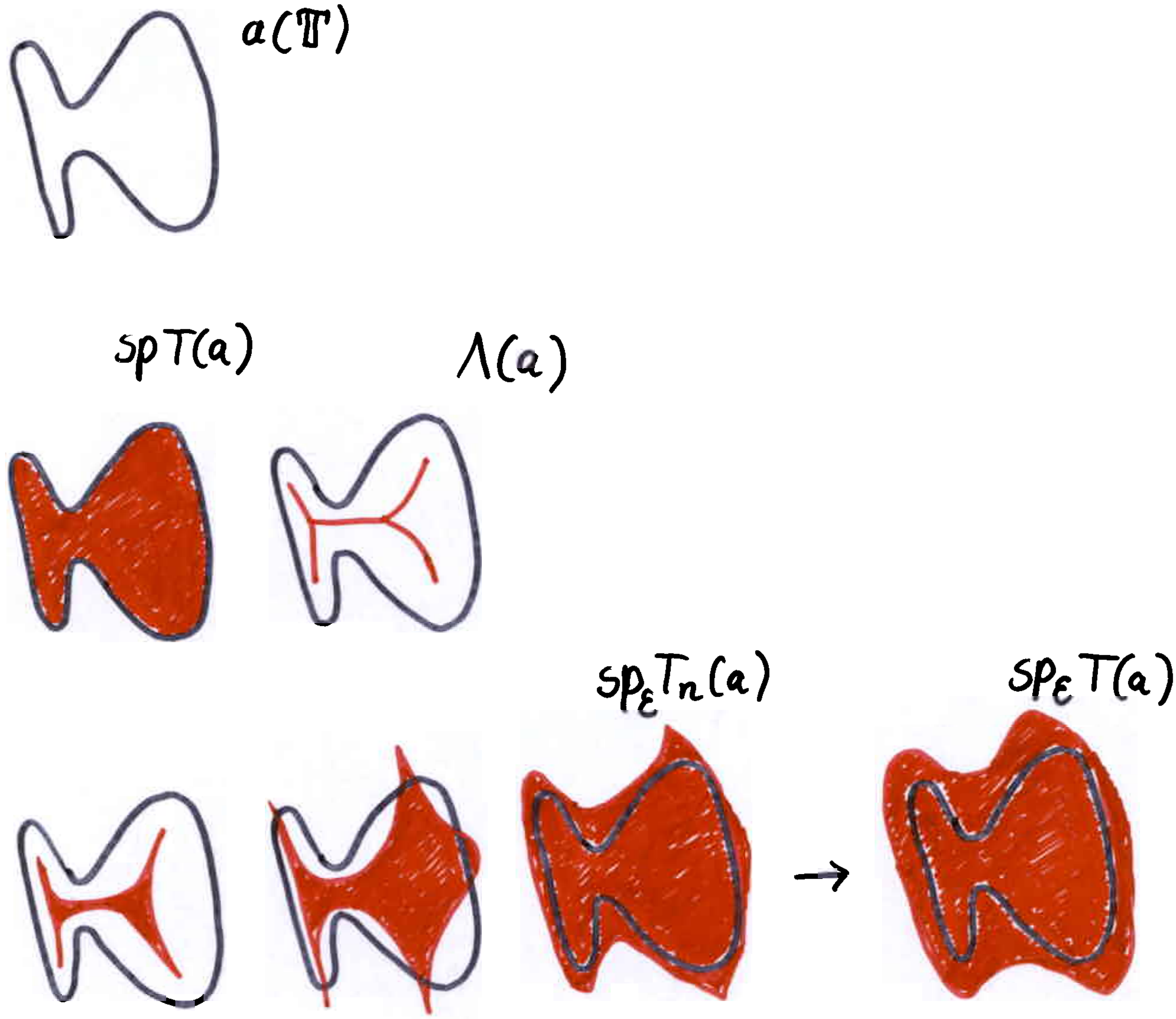
$$= \left|\frac{D_{n-1}(c\varphi_{\beta-1})}{D_n(c\varphi_\beta)}\right| \simeq \frac{n^{-(\beta-1)^2}}{n^{-\beta^2}} = n^{2\beta-1}$$

Example: $\|T_n^{-1}(a-\frac{1}{2})\| \approx 3.8\, n^{0.30} \approx 10^5$

for $n \approx 10^{15}$

(macroscopic systems have $10^8 = \sqrt[3]{10^{24}}$ particles)

$n = 10^{15}$


$a(\mathbb{T})$
$\operatorname{sp} T(a)$
$\Lambda(a)$
$\operatorname{sp}_\varepsilon T_n(a)$
$\operatorname{sp}_\varepsilon T(a)$

Structured pseudospectra

$$sp_\varepsilon A = \bigcup sp(A+K)$$

K arbitrary, $\|K\| \leqq \varepsilon$

$$sp^{diag}_{\varepsilon\overline{\mathbb{D}}} A = \bigcup sp(A+K)$$

K diagonal matrix, $\|K\| \leqq \varepsilon$

$\Leftrightarrow$ K diagonal matrix, $K_{ii} \in \varepsilon\overline{\mathbb{D}}$

$$sp^{diag}_{[-\varepsilon,\varepsilon]} A = \bigcup sp(A+K)$$

K diagonal matrix, $K_{ii} \in [-\varepsilon, \varepsilon]$

$$sp^{(i,j)}_{[-\varepsilon,\varepsilon]} A = \bigcup sp(A+K)$$

the only nonzero entry of K is K_{ij}
and $K_{ij} \in [-\varepsilon, \varepsilon]$

Linear systems theory

(Hinrichsen, Kelb, Pritchard, Gallestey, ...)

$$\left.\begin{array}{l} \dot{x} = Ax + v \\ \text{feed-back} \quad v = Kx \end{array}\right\} \Rightarrow \quad \dot{x} = (A+K)x$$

$$\left.\begin{array}{l} \dot{x} = Ax + P_i v \\ \text{feed-back} \quad v = KP_j x \end{array}\right\} \Rightarrow \quad \dot{x} = (A + P_i K P_j)x$$

Toeplitz matrices

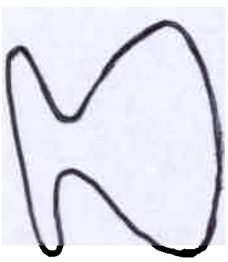
$a \in \mathcal{P}$

$\boxed{\operatorname{sp} T_n(a) \to \Lambda(a)}$ Schmidt/Spitzer 1960

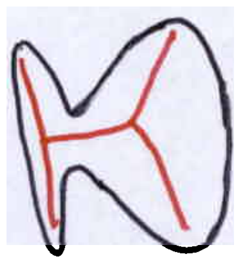
(unstructured) pseudospectra $\operatorname{sp}_\varepsilon A = \bigcup_{\|K\| \leq \varepsilon} \operatorname{sp}(A+K)$

$\boxed{\operatorname{sp}_\varepsilon T_n(a) \longrightarrow \operatorname{sp}_\varepsilon T(a)}$ H. Landau 1975

L. Reichel / L.N. Trefethen 1992



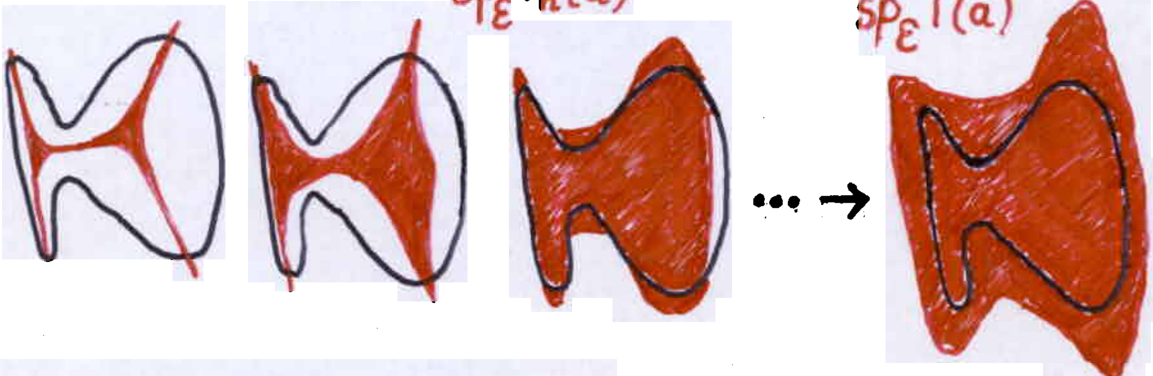
$a(\mathbb{T})$



$\Lambda(a)$

$\operatorname{sp}_\varepsilon T_n(a)$

$\operatorname{sp}_\varepsilon T(a)$



$\boxed{\operatorname{sp}^{(j,k)}_{\Omega} T_n(a) \overset{?}{\longrightarrow} \operatorname{sp}^{(j,k)}_{\Omega} T(a)}$

Theorem (Bö/Embree/Sokolov 2001)

$a \in \mathcal{P}$

under certain assumptions,

$$\mathrm{sp}_{\Omega}^{(j,k)}\, T(a) = \mathrm{sp}\, T(a) \cup H_{\Omega}^{jk}(a)$$

$$\lim_{n\to\infty} \mathrm{sp}_{\Omega}^{(j,k)}\, T_n(a) = \Lambda(a) \cup H_{\Omega}^{jk}(a)$$

finite volume case $\xrightarrow[\text{discontinuous}]{}$ infinite volume case

True if one of the following is satisfied:

- $(j,k) = (1,1)$
- $\mathbb{C} \setminus \Lambda(a)$ is connected
- $T(a)$ is Hessenberg
- $p+q \leqq 5$ or $p+q = 7$
  (no result for $p+q=6$)
- $p+q$ is a prime number and p or q equals 2

$$a(t) = \sum_{k=-p}^{q} a_k t^k$$

$a_q \neq 0$

$a_{-p} \neq 0$

Tridiagonal Toeplitz matrices

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & & \\ a_1 & a_0 & a_{-1} & \ddots \\ & a_1 & a_0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad a(t) = a_{-1}t^{-1} + a_0 + a_1 t$$

without loss of generality

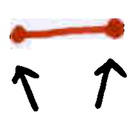
$$T(a) = \begin{pmatrix} 0 & \alpha^2 & & \\ 1 & 0 & \alpha^2 & \ddots \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad a(t) = t + \alpha^2 t^{-1}$$

$$\alpha \in [0,1]$$

$$\begin{aligned} a(e^{i\theta}) &= e^{i\theta} + \alpha^2 e^{-i\theta} \\ &= \cos\theta + i\sin\theta + \alpha^2\cos\theta - \alpha^2 i \sin\theta \\ &= (1+\alpha^2)\cos\theta + i\,(1-\alpha^2)\sin\theta \end{aligned}$$

$$a(\mathbb{T}) = \text{ellipse} \quad \frac{x^2}{(1+\alpha^2)^2} + \frac{y^2}{(1-\alpha^2)^2} = 1$$

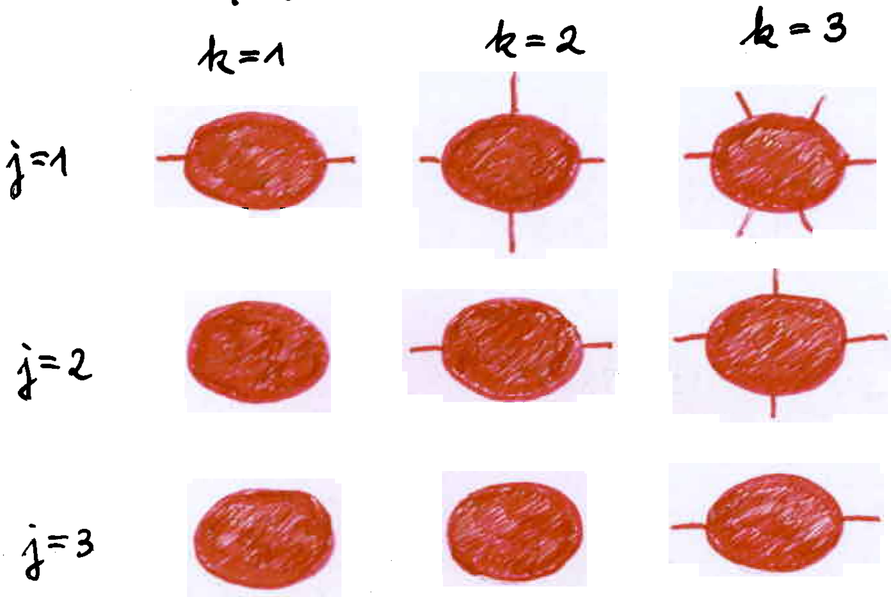
$\mathrm{sp}\, T(a) =$ 

$\Lambda(a) =$ 

foci of the ellipse

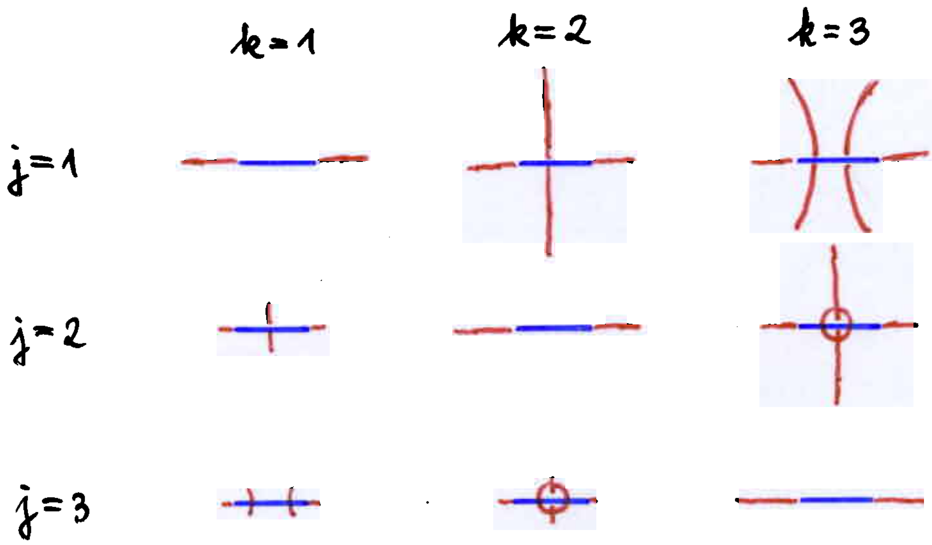
Tridiagonal Toeplitz matrices

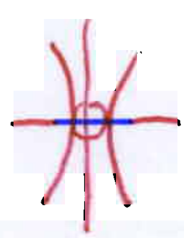
$$\mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]}\, T(a)$$



$\mathrm{sp}\, T(a) \cup H_{jk}$

$$\lim_{n\to\infty} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]}\, T_n(a)$$

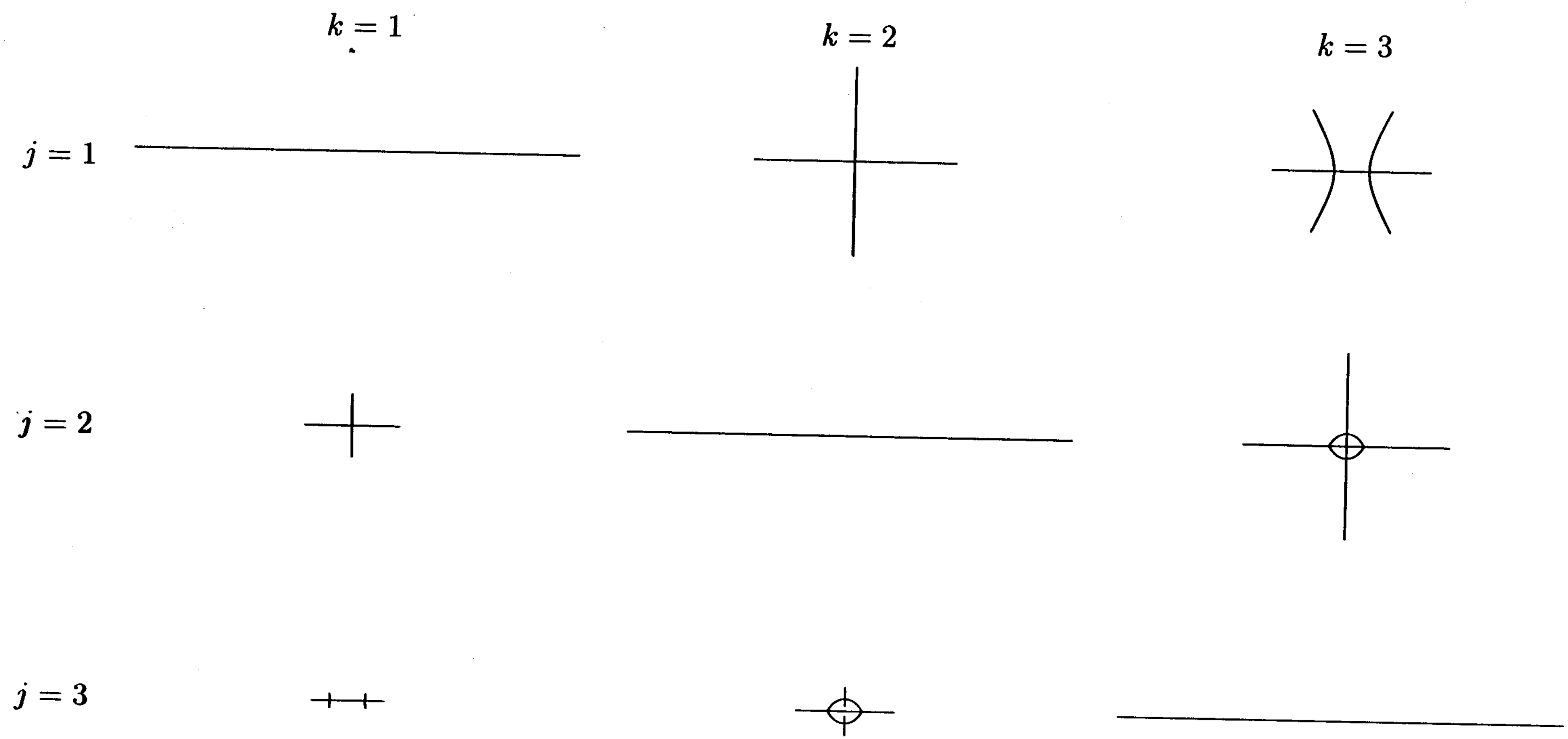




$\Lambda(a) \cup H_{jk}$

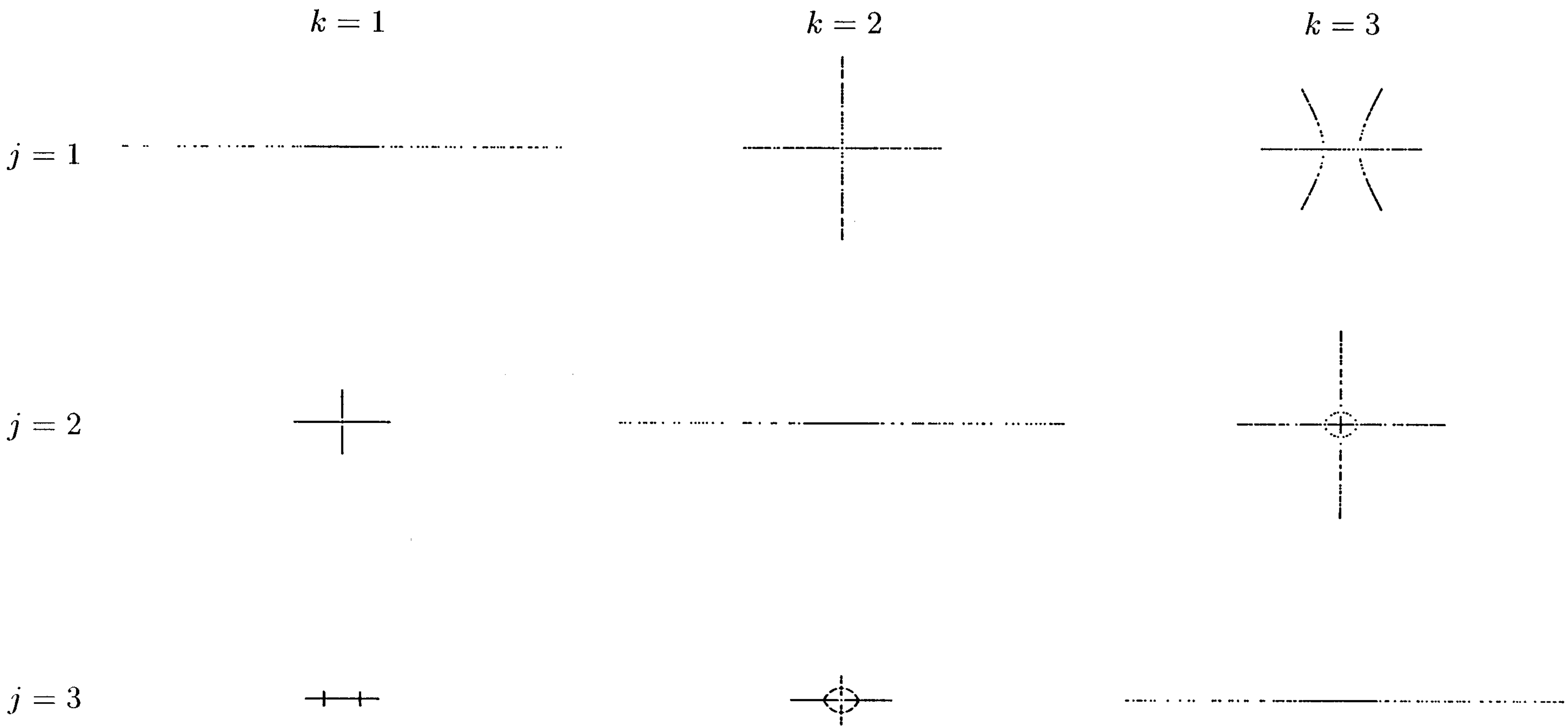
REAL SINGLE-ENTRY PERTURBATIONS OF $T_n(a)$ FOR $a(t) = t + \alpha^2 t^{-1}$

$$\lim_{n\to\infty} \mathrm{sp}_\Omega^{(j,k)} T_n(a) \text{ for } \alpha = \tfrac{4}{10} \text{ and } \Omega = [-5,5]$$



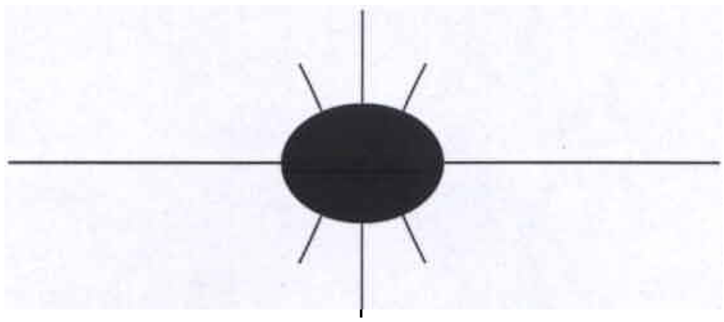
REAL SINGLE-ENTRY PERTURBATIONS OF $T_n(a)$ FOR $a(t) = t + \alpha^2 t^{-1}$

Eigenvalues of $T_{25}(a)$ with $\alpha = \frac{1}{4}$, perturbed in the (j, k) entry by a random number from $[-5, 5]$. Each plot is the union of 100 samples.

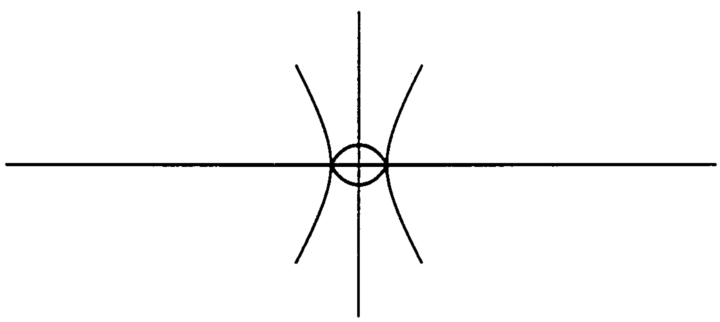


REAL SINGLE-ENTRY PERTURBATIONS OF $T(a)$ AND $T_n(a)$ FOR $a(t) = t + \alpha^2 t^{-1}$

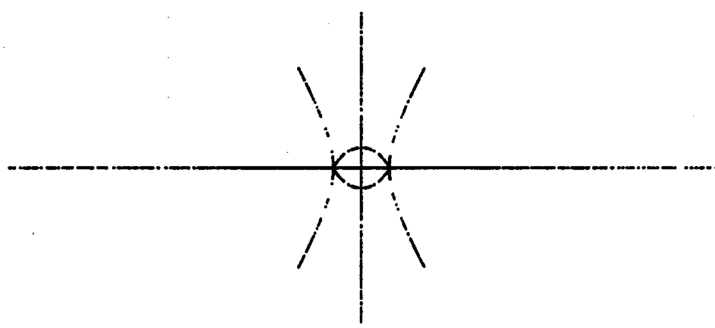
$$\alpha = \frac{4}{10} \qquad \Omega = [-5, 5]$$



$\mathrm{sp}^{(j,k)}_{[-5,5]}\, T(a)$, union over all (j, k) in the upper 3×3 block.



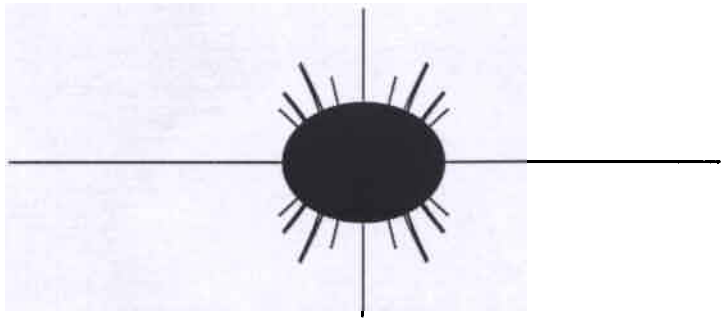
$\lim_{n\to\infty} \mathrm{sp}^{(j,k)}_{[-5,5]}\, T_n(a)$, union over all (j, k) in the upper 3×3 block.



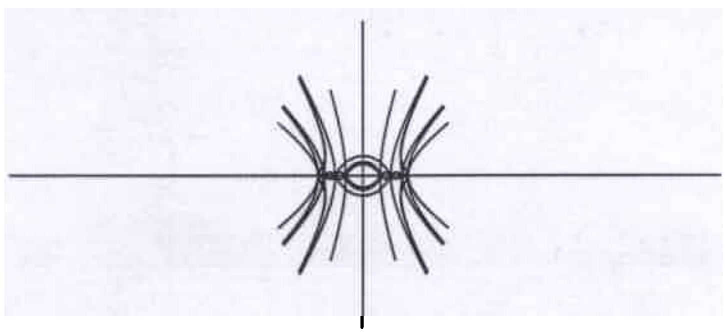
Computed eigenvalues of $T_{25}(a)$, randomly perturbed in one entry of the upper 3×3 block by a random number in $[-5, 5]$, union of 1000 samples.

REAL SINGLE-ENTRY PERTURBATIONS OF $T(a)$ AND $T_n(a)$ FOR $a(t) = t + \alpha^2 t^{-1}$

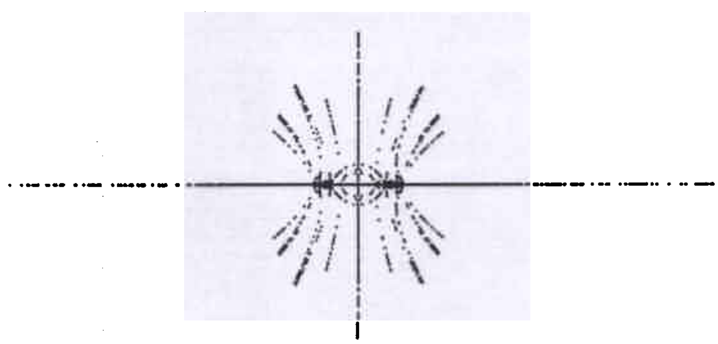
$$\alpha = \frac{4}{10} \qquad \Omega = [-5, 5]$$



$\mathrm{sp}_{[-5,5]}^{(j,k)}\, T(a)$, union over all (j,k) in the upper 5×5 block.

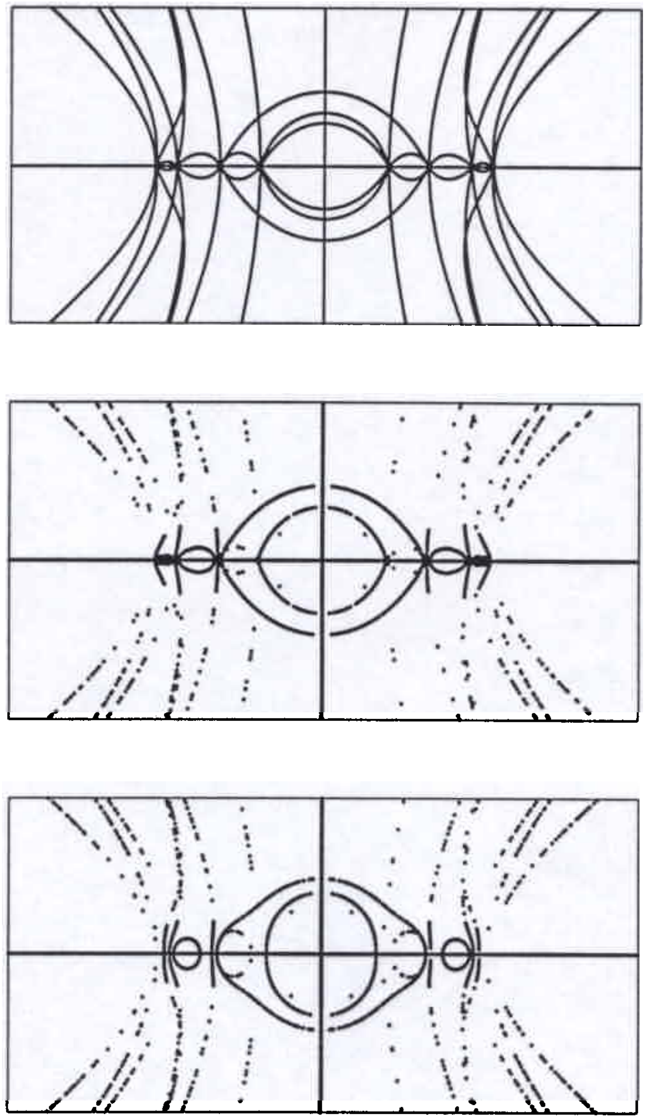


$\lim_{n\to\infty} \mathrm{sp}_{[-5,5]}^{(j,k)}\, T_n(a)$, union over all (j,k) in the upper 5×5 block.



Computed eigenvalues of $T_{25}(a)$, randomly perturbed in one entry of the upper 5×5 block by a random number in $[-5, 5]$, union of 1000 samples.

Figures are believed to be correct to the printed accuracy.



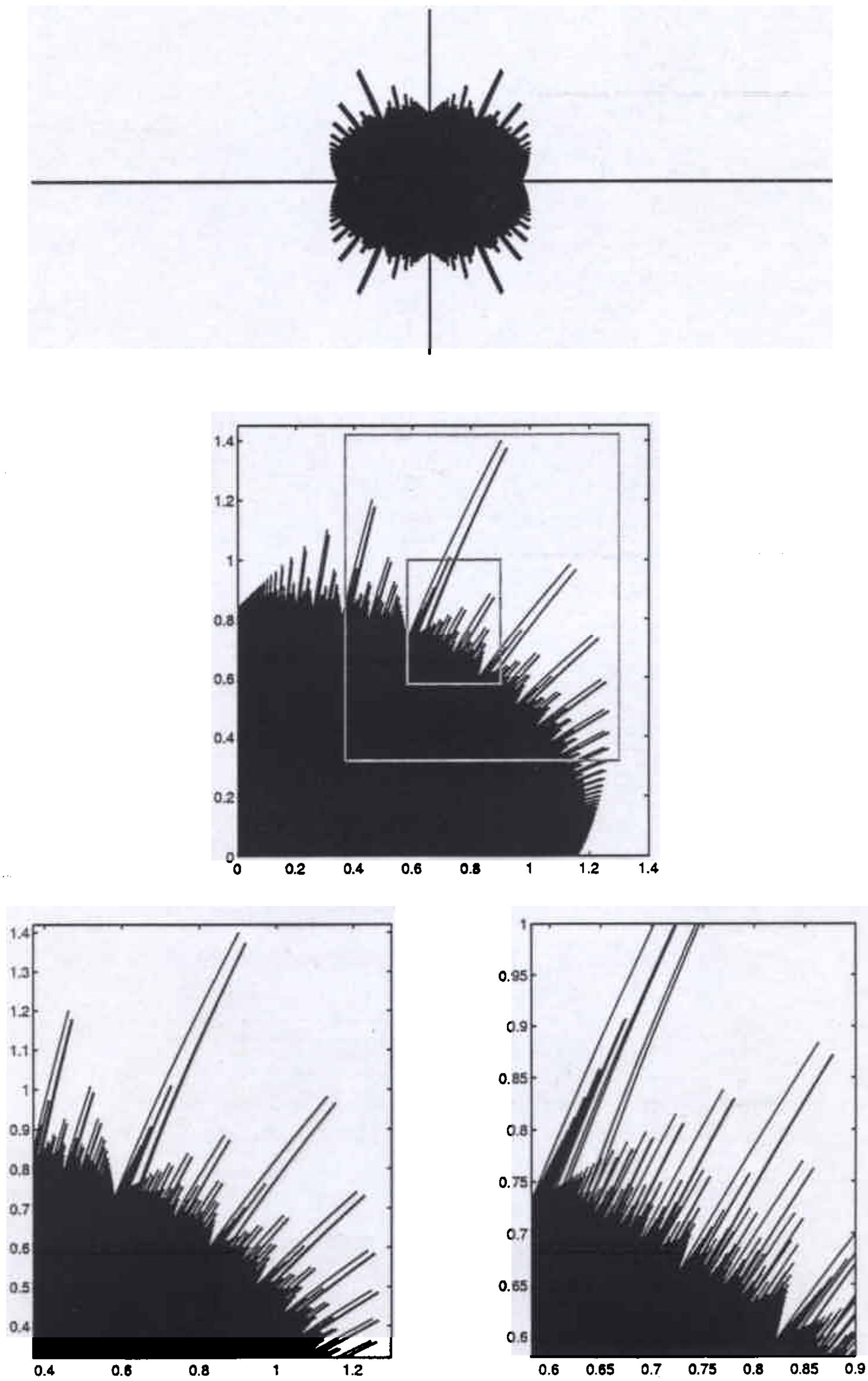
$N \to \infty$

$N = 50$

$N = 10$

FIGURE 5. Closer inspection of Figure 3. The top plot shows a portion of $\lim_{n\to\infty} \mathrm{sp}_{\Omega}^{(j,k)} T_n(a)$ over all (j,k) in the top 5×5 corner for $\Omega = [-5,5]$. The middle image shows eigenvalues of 10,000 random perturbations to a single entry in the top corner of $T_{50}(a)$; the bottom image shows the same for $T_{10}(a)$. (The accurate eigenvalues for $N = 50$ were obtained by reducing the non-normality in the problem via a similarity transformation.)

REAL SINGLE-ENTRY PERTURBATIONS OF $T(a)$ FOR $a(t) = t + \alpha^2 t^{-1}$ WITH $\alpha = 4/10$ AND $\varepsilon = 5$



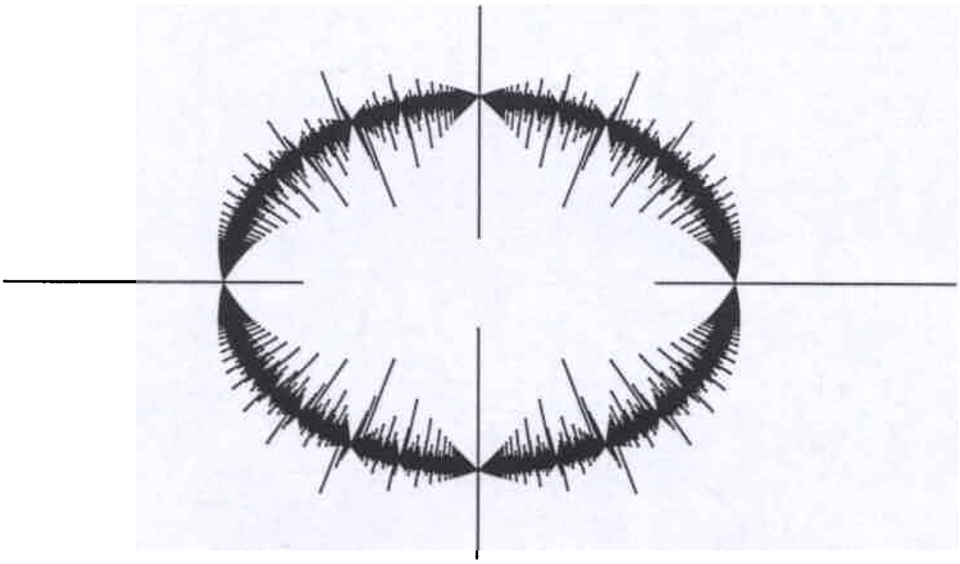


Fig. 1. $\bigcup_{(j,k)\in S} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} L(a)$ with $S = \{(j,k) \in \mathbf{Z}\times\mathbf{Z} : j-k \neq 1\}$ for $a(t) = t+\alpha^2 t^{-1}$ with $\alpha = \frac{2}{5}$ and $\varepsilon = 5$.

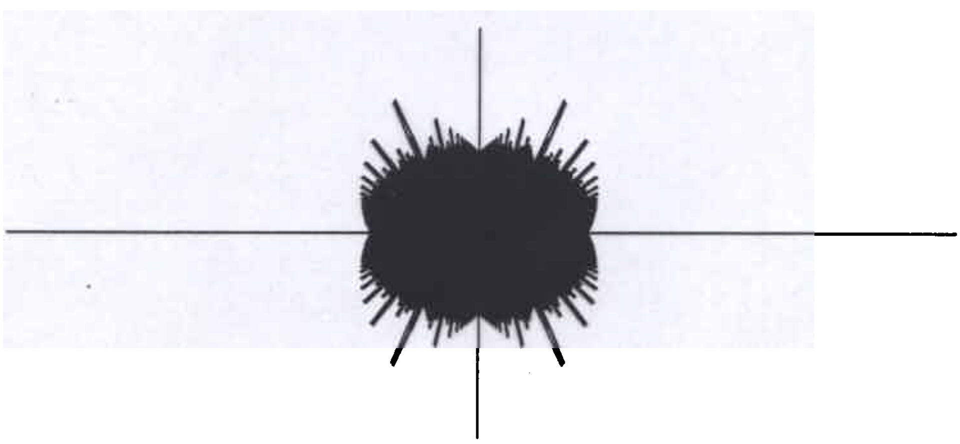


Fig. 2. $\bigcup_{(j,k)\in \mathbf{N}\times\mathbf{N}} \mathrm{sp}^{(j,k)}_{[-\varepsilon,\varepsilon]} T(a)$ for $a(t) = t+\alpha^2 t^{-1}$ with $\alpha = \frac{2}{5}$ and $\varepsilon = 5$. (Note the different scale from Figure 1.)

Date sent: Fri, 5 May 2000 15:13:28 +0100 (BST)
From: Mark Embree <Mark.Embree@comlab.ox.ac.uk>
To: <albrecht.boettcher@mathematik.tu-chemnitz.de>
Organization: Oxford University Computing Lab

Albrecht --

> we aren't in a hurry. Our goal is beauty - and beauty requires time
> (among other things).

If you don't mind, I'll take you at your word and wait until Monday to pass the TeX file, with figures, back to you. I'm beginning to see the antennas, and, indeed, they are beautiful!

Take care,
Mark

Date sent: Tue, 9 May 2000 14:41:18 +0100 (BST)
From: Mark Embree <Mark.Embree@comlab.ox.ac.uk>
To: <albrecht.boettcher@mathematik.tu-chemnitz.de>
Organization: Oxford University Computing Lab

Albrecht --

Finally, well over-due, I have Version 3 of our paper.

I apologize for the continued delay. Figures 1 & 2 were pretty challenging, among the toughest plots I've ever made. Very rewarding stuff!

I hope you enjoy this work,

Mark

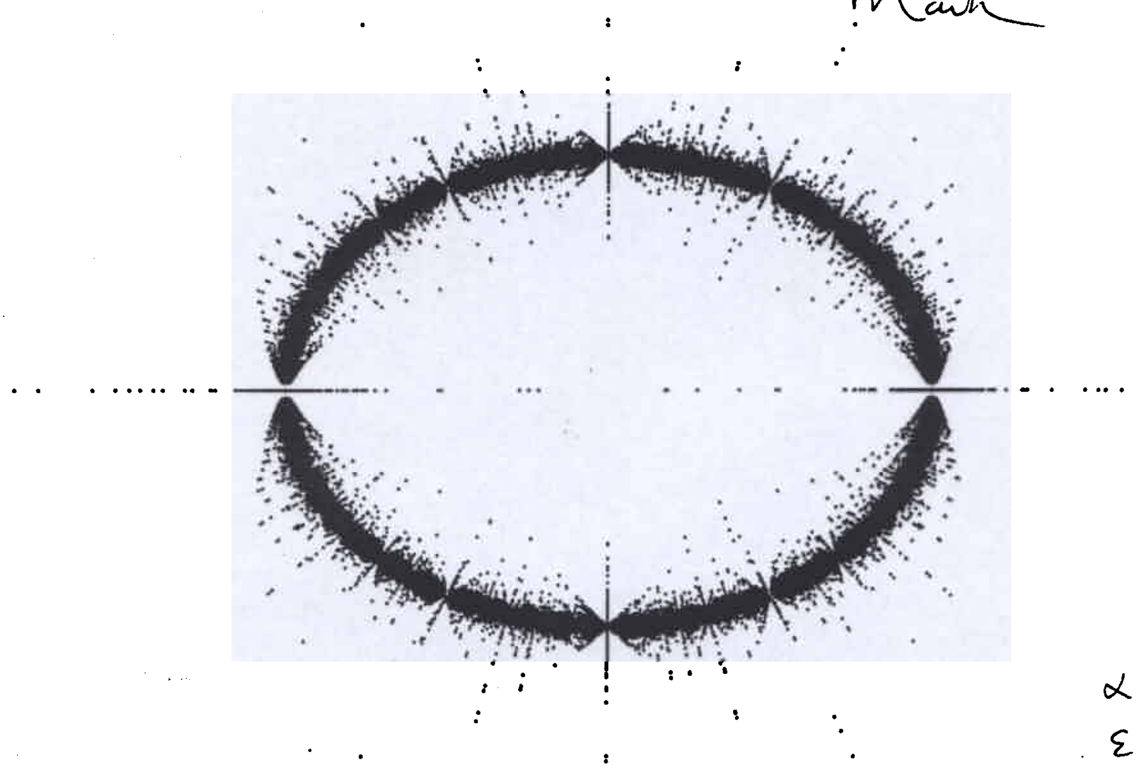
18 May 2000

Albrecht –

Please find enclosed the two Kreiss stability articles. As you may know, these are very famous in the world of numerical PDEs.

I hope Italy was grand!

— Mark



$\alpha = .4$

$\varepsilon = 5$

1000 trials of N = 100 dimension matrices

1 entry perturbed at a time.

No perturbations on the first subdiagonal

Several eigenvalues off the scale of these axes.

Can structured pseudospectra jump?

pseudospectrum

$$sp_{\varepsilon\overline{\mathbb{D}}}A = spA \cup \{\lambda \notin spA : \|(A-\lambda I)^{-1}\| \geq \tfrac{1}{\varepsilon}\}$$

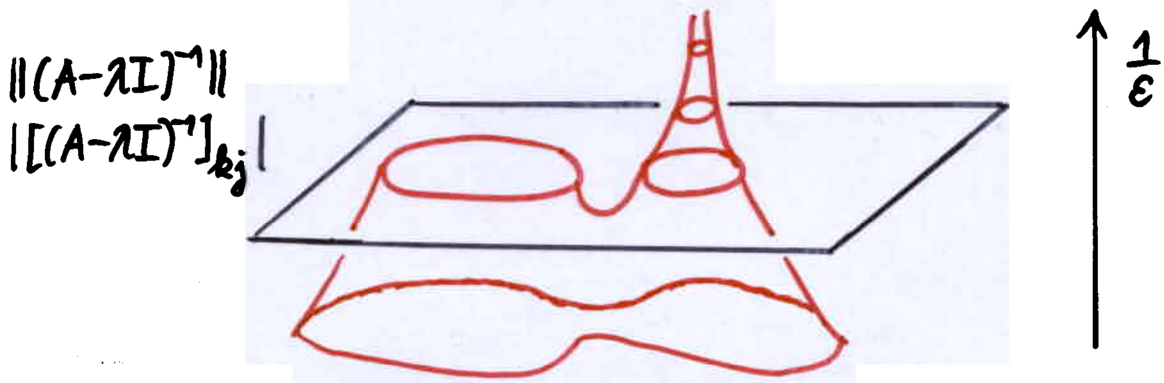
structured pseudospectrum

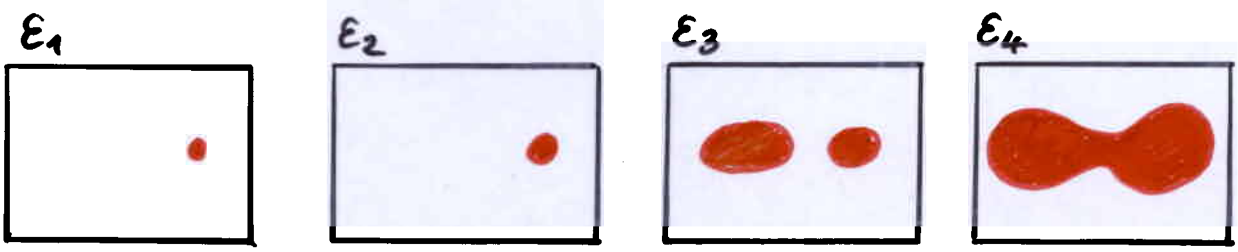
$$sp^{(j,k)}_{\varepsilon\overline{\mathbb{D}}}A = spA \cup \{\lambda \notin spA : |[(A-\lambda I)^{-1}]_{kj}| \geq \tfrac{1}{\varepsilon}\}$$

Do (structured) pseudospectra change continuously if ε changes continuously?

$$\Updownarrow$$

Can $\|(A-\lambda I)^{-1}\|$ or $|[(A-\lambda I)^{-1}]_{kj}|$ be locally constant?





A. Daniluk 1994:

$\|(A-\lambda I)^{-1}\|$ is nowhere locally constant (Hilbert space case)

$\Rightarrow$ $\boxed{sp_{\varepsilon\overline{\mathbb{D}}}A \text{ cannot jump}}$

Laurent matrices

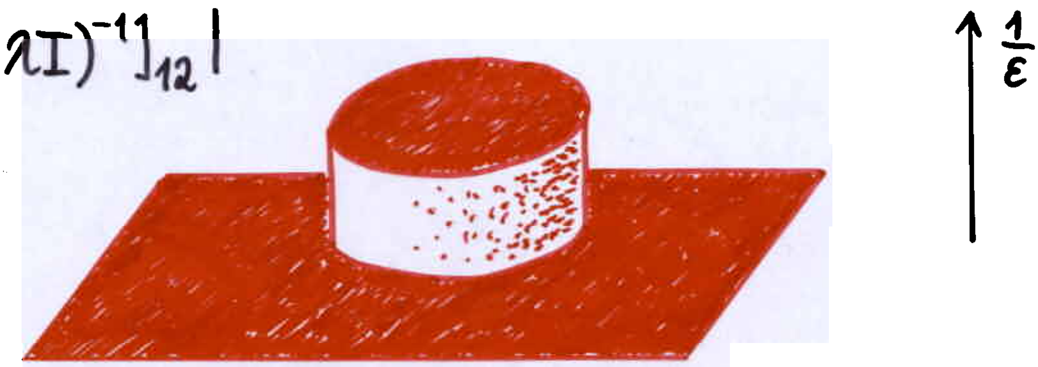
$$U = L(a) = \left(\begin{array}{cc|cc} 1 & 0 & & \\ & 1 & 0 & \\ \hline & & 1 & 0 \\ & & & \color{red}{1} \;\; 0 \\ & & & \quad 1 \;\; 0 \end{array}\right), \quad \operatorname{sp} U = \mathbb{T}$$

$$(U - \lambda I)^{-1} =$$

$$\left(\begin{array}{c|ccc} 1 & \lambda & \lambda^2 & \lambda^3 \\ 0 & 1 & \lambda & \lambda^2 \\ \hline 0 & 0 & \color{red}{1} & \lambda \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{c|cc} -1/\lambda & 0 & 0 \\ \hline -1/\lambda^2 & -1/\lambda & \color{red}{0} \\ -1/\lambda^3 & -1/\lambda^2 & -1/\lambda \\ -1/\lambda^4 & -1/\lambda^3 & -1/\lambda^2 \end{array}\right)$$

$|\lambda| < 1$ $\qquad\qquad$ $|\lambda| > 1$

$|[(U-\lambda I)^{-1}]_{12}|$



$\operatorname{sp}^{(2,1)}_{\varepsilon \mathbb{D}} U$

$0 < \varepsilon < 1$

$\varepsilon = 1$

$1 < \varepsilon < \infty$

Thus, structured pseudospectra can jump.

Large Finite Toeplitz Matrices

$a \in \mathcal{P}$

$\operatorname{sp} T(a) =$  $\qquad \Lambda(a) = \lim_{n\to\infty} \operatorname{sp} T_n(a) =$ 

$$a_\varrho(e^{i\theta}) = \sum_k a_k \varrho^k e^{ik\theta}$$

$$\boxed{\Lambda(a) = \bigcap_{\varrho > 0} \operatorname{sp} T(a_\varrho)}$$

Schmidt/Spitzer 1960

$\lambda \in \mathbb{C} \setminus \Lambda(a) \qquad \exists \varrho > 0 : T(a_\varrho - \lambda)$ is invertible

$$\boxed{T^{-1}(a_\varrho - \lambda) = \left(\varrho^{j-k} d_{jk}(\lambda)\right)_{j,k=0}^{\infty}}$$

we obtain analytic functions

$$d_{jk} : \mathbb{C} \setminus \Lambda(a) \to \mathbb{C}$$

put

$$H_\Omega^{jk}(a) = \left\{ \lambda \in \mathbb{C} \setminus \Lambda(a) : d_{kj}(\lambda) \in -\frac{1}{\Omega} \right\}$$

Toeplitz band matrices

Conjecture: $sp^{(j,k)}_{\varepsilon\bar{\mathbb{D}}} T(a)$ does not jump

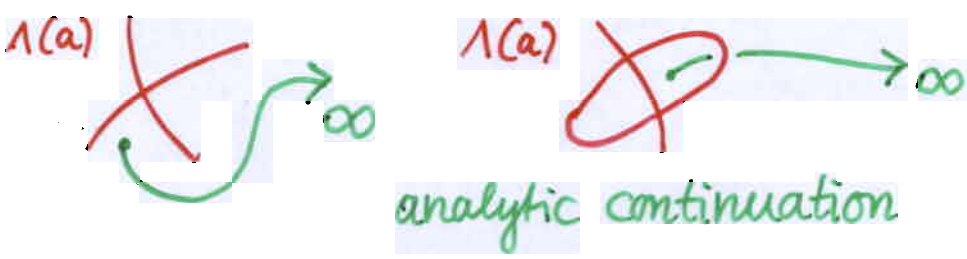
Equivalent conjecture:

$[T^{-1}(a-\lambda)]_{kj}$ is nowhere locally constant

Equivalent conjecture:

$d_{kj} : \mathbb{C} \setminus \Lambda(a) \to \mathbb{C}$ is nowhere locally constant

$$\boxed{\begin{array}{l} \mathbb{C} \setminus \Lambda(a) \text{ connected} \\ \Rightarrow d_{kj} \text{ nowhere locally constant} \end{array}}$$



$\mathbb{C} \setminus \Lambda(a)$ is connected in many, many cases.
Examples of symbols with disconnected $\mathbb{C} \setminus \Lambda(a)$ were first discovered numerically by

Beam / Warming 1993

and "exactly" computed by

Bö / Widom 1994 (Wiener-Hopf)

Bö / Grudsky 2001 (Toeplitz)

And what if $\mathbb{C}\setminus\Lambda(a)$ is disconnected?

$$a(z) = a_{-r}z^{-r} + \dots + a_s z^s$$

$$z^r(a(z)-\lambda) = a_{-r} + \dots + (a_0-\lambda)z^r + \dots + a_s z^{r+s}$$

zeros $z_1(\lambda), \dots, z_{r+s}(\lambda)$ labeled so that

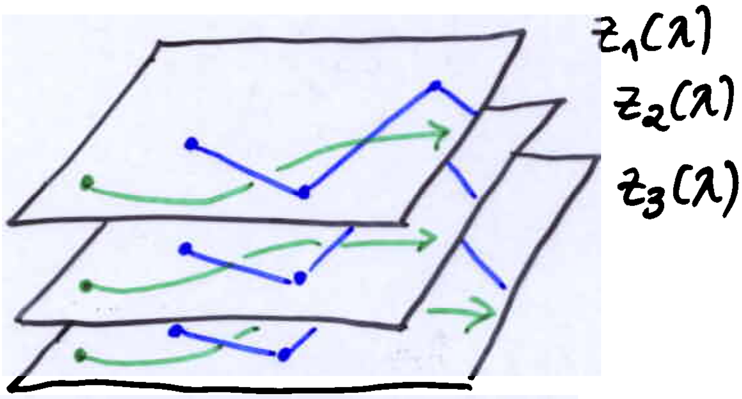
$$|z_1(\lambda)| \leqq \dots \leqq |z_r(\lambda)| \leqq |z_{r+1}(\lambda)| \leqq \dots \leqq |z_{r+s}(\lambda)|$$

Schmidt / Spitzer 1960

$$\Lambda(a) = \{\lambda \in \mathbb{C} : |z_r(\lambda)| = |z_{r+1}(\lambda)|\}$$

$\lambda \notin \Lambda(a)$: we have r small zeros and s large zeros

Riemann surface of $a_{-r} + \dots + (a_0-\lambda)z^r + \dots + a_s z^{r+s} = 0$



at each $\lambda \notin \Lambda(a)$ we have
r small branches
s large branches

λ plane

Each path from λ to ∞ induces a permutation of the branches ($\to$ monodromy group).

Is there a path from λ to ∞ such that the r small branches at λ are permuted into the r small branches at infinity?

If yes, then d_{kj} is nowhere locally constant.

Bö/Grudsky 2001

$$a(t) = a_{-r}t^{-r} + \dots + a_s t^s, \quad a_{-r} \neq 0, \; a_s \neq 0$$

d_{kj} is nowhere locally constant if one of the following conditions is satisfied:

- $k = j = 1$
- $r = 1$ or $s = 1$ (that is, $T(a)$ is Hessenberg)
- $r+s$ is a prime number and $r=2$ or $s=2$ (examples: $(2,3), (2,5), (2,9), (2,11), \dots$)

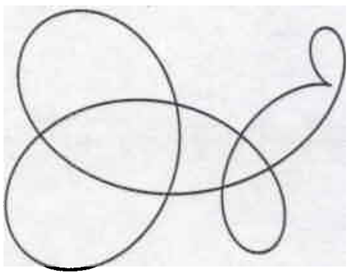
Thus, so far we have ($r \leq s$)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
		(3,3)	(3,4)	(3,5)	(3,6)

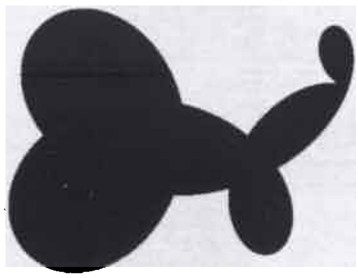
- extra proofs

$$\boxed{\text{Overall, if } r+s \leq 5 \text{ or } r+s = 7, \text{ then } \operatorname{sp}^{(j,k)}_{\varepsilon \overline{\mathbb{D}}} T(a) \text{ does not jump}}$$

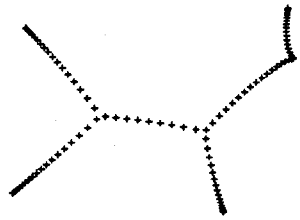
REAL SINGLE-ENTRY PERTURBATIONS OF $T_n(a)$



$a(\mathbf{T})$



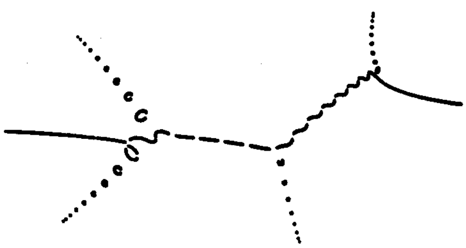
$\operatorname{sp} T(a)$



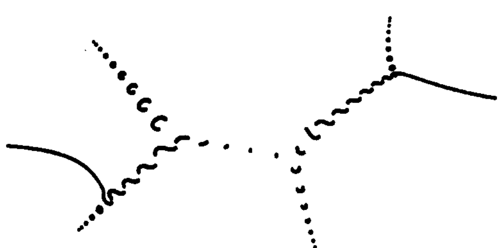
$\operatorname{sp} T_{100}(a)$

Eigenvalues of $T_{50}(a)$ perturbed in entry (j, k)
by a random number uniformly distributed in $[-4, 4]$
as computed by MATLAB; union of 500 samples.

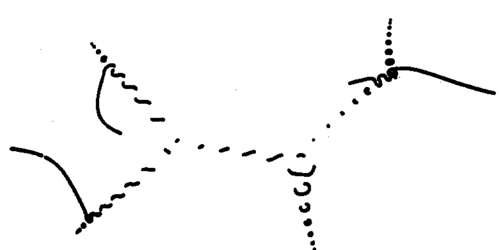
$(1, 1)$



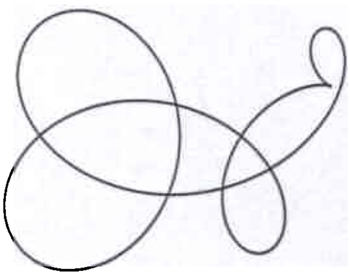
$(2, 2)$



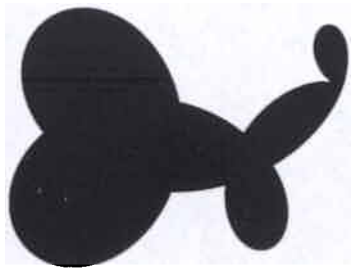
$(3, 3)$



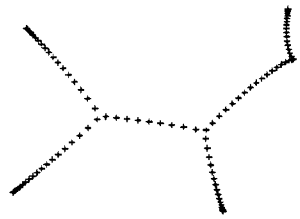
REAL SINGLE-ENTRY PERTURBATIONS OF $T_n(a)$



$a(\mathbf{T})$



$\operatorname{sp} T(a)$



$\operatorname{sp} T_{100}(a)$

Eigenvalues of $T_{50}(a)$ perturbed in entry (j, j)
by a random number uniformly distributed in $[-4, 4]$
for $j = 1$, 2, or 3 (randomly), as computed by MATLAB;
union of 1000 samples.