Duality, Biorthogonal Polynomials and Multi–Matrix Models

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(Joint work with Marco Bertola and Bertrand Eynard)

1 Random matrices

2-Matrix Measure:

$$\frac{1}{\tau_N} d\mu(M_1, M_2) := \frac{1}{\tau_N} \exp \mathrm{tr}\, (-V_1(M_1) - V_2(M_2) + M_1 M_2) dM_1 dM_2$$

$$V_1(x) = u_0 + \sum_{K=1}^{d_1+1} \frac{u_K}{K} x^K , \qquad V_2(y) = v_0 + \sum_{J=1}^{d_2+1} \frac{v_J}{J} y^J .$$

Partition function is:

$$\tau_N = \int_{M_1} \int_{M_2} d\mu . \qquad \text{(2 Toda } \tau \text{ function)}$$

Relation to **bi-orthogonal polynomials**:

<det(x1 - M1)> <det(y1 - M2)>

$$\pi_n(x) = x^n + \cdots , \qquad \sigma_n(y) = y^n + \cdots , \qquad n = 0, 1, \ldots$$

$$\int \int \mathrm{d}x\, \mathrm{d}y\ \pi_n(x) \sigma_m(y) \mathrm{e}^{-V_1(x) - V_2(y) + xy} = h_n \delta_{mn},$$

Fredholm Kernels:

$$\overset{N}{K}_{12}(x, y) = \sum_{n=0}^{N-1} \frac{1}{h_n} \pi_n(x) \sigma_n(y) \mathrm{e}^{-V_1(x)} \mathrm{e}^{-V_2(y)} , \qquad \overset{N}{K}_{11}(x, x') = \int \mathrm{d}y\ \underset{12}{\overset{N}{K}}(x, y)\, \mathrm{e}^{x'y},$$

$$\overset{N}{K}_{22}(y', y) = \int \mathrm{d}x\ \underset{12}{\overset{N}{K}}(x, y)\, \mathrm{e}^{xy'} , \qquad \overset{N}{K}_{21}(y', x') = \int \int \mathrm{d}x\, \mathrm{d}y \overset{N}{K}_{12}(x, y)\, \mathrm{e}^{xy'} \mathrm{e}^{x'y} .$$

3 Recursion relations and generalized Darboux-Christoffel formulae

Bi-orthogonal Quasi-polynomials:

$$\psi_n(x) := \frac{1}{\sqrt{h_n}}\pi_n(x)\mathrm{e}^{-V_1(x)} , \qquad \phi_n(y) := \frac{1}{\sqrt{h_n}}\sigma_n(y)\mathrm{e}^{-V_2(y)} ,$$

Their **Fourier–Laplace transforms:**

$$\underline{\psi}_n(y) = \int_{\Gamma_x} \mathrm{d}x\, \mathrm{e}^{xy}\psi_n(x) , \qquad \underline{\phi}_n(x) = \int_{\Gamma_y} \mathrm{d}y\, \mathrm{e}^{xy}\phi_n(y) .$$

Orthogonality relations:

$$\int \mathrm{d}x\, \psi_n(x)\underline{\phi}_m(x) = \int \mathrm{d}y\, \underline{\psi}_n(y)\phi_m(y) = \int\int \mathrm{d}x\mathrm{d}y\, \psi_n(x)\phi_m(y)\mathrm{e}^{xy} = \delta_{mn} .$$

Infinite vectors and finite-band recursion matrices:

$$\underset{\infty}{\Psi} := [\psi_0, ..., \psi_n, ...]^t , \qquad \underset{\infty}{\Phi} := [\phi_0, ..., \phi_n, ...]^t$$

$$x \underset{\infty}{\Psi} := Q \underset{\infty}{\Psi} ; \qquad \frac{\mathrm{d}}{\mathrm{d}x} \underset{\infty}{\Psi} := P \underset{\infty}{\Psi}$$

$$y \underset{\infty}{\Phi} = -P^t \underset{\infty}{\Phi}; \qquad \frac{\mathrm{d}}{\mathrm{d}y} \underset{\infty}{\Phi} = -Q^t \underset{\infty}{\Phi} ,$$

where

$$Q := \begin{bmatrix} \alpha_0(0) & \gamma(0) & 0 & 0 & \cdots \\ \alpha_1(1) & \alpha_0(1) & \gamma(1) & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \alpha_{d_2}(d_2) & \alpha_{d_2-1}(d_2) & \cdots & \alpha_0(d_2) & \gamma(d_2) \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix} ,$$

$$-P := \begin{bmatrix} \beta_0(0) & \beta_1(1) & \cdots & \beta_{d_1}(d_1) & \cdots \\ \gamma(0) & \beta_0(1) & \beta_1(2) & \ddots & \beta_{d_1}(d_1+1) \\ 0 & \gamma(1) & \beta_0(2) & \ddots & \ddots \\ 0 & 0 & \gamma(2) & \beta_0(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} ,$$

satisfy the string equation.

$$[P, Q] = \mathbf{1} .$$

Density of eigenvalues of the first matrix:

$$\overset{N}{\rho}_1(x) = \frac{K^N{}_{11}(x,x)}{N}.$$

Correlation functions:

$$\overset{N}{\rho}_{11}(x,x') = \frac{1}{N^2}\left(\overset{N}{K}_{11}(x,x)\overset{N}{K}_{11}(x',x') - \overset{N}{K}_{11}(x,x')\overset{N}{K}_{11}(x',x)\right).$$

$$\overset{N}{\rho}_{12}(x,y) = \frac{1}{N^2}\left(\overset{N}{K}_{11}(x,x)\overset{N}{K}_{22}(y,y) - \overset{N}{K}_{12}(x,y)(\overset{N}{K}_{21}(y,x) - e^{xy}\right) .$$

Spacing distributions:

$$\overset{N^1}{p}_J = \det\left(\mathbf{1} - \overset{N}{\hat{\mathbf{K}}}_{11} \circ \chi_J\right),$$

(where χ_J is the characteristic function of the set J.)

2 Dual Isomonodromic Deformations

Dual isomonodromic pair:

$$\mathcal{D}_1 := \frac{\partial}{\partial x} + \mathbf{L}(x,\mathbf{u}) , \qquad \mathcal{D}_2 := \frac{\partial}{\partial y} + \mathbf{M}(y,\mathbf{u}) ,$$

Spectral duality:

$$\det(\mathbf{L}(x,\mathbf{u}) - y\mathbf{1}) = 0 , \qquad \det(\mathbf{M}(y,\mathbf{u}) - x\mathbf{1}) = 0$$

are biholomorphically equivalent.

4 Finite "windows" and Generalized Christoffel-Darboux fomulae

Finite dual "windows"

$$\underset{N}{\Psi} := [\psi_{N-d_2}, ..., \psi_N]^t\ , \quad N \geq d_2, \qquad \underset{N}{\Phi} := [\phi_{N-d_1}, ..., \phi_N]^t\ , \quad N \geq d_1$$

$$\overset{N}{\Psi} := [\psi_N, ..., \psi_{N+d_1}]^t\ , \quad N \geq 0, \qquad \overset{N}{\Phi} := [\phi_N, ..., \phi_{N+d_2}]^t\ , \qquad N \geq 0$$

Proposition 4.1 Generalized Darboux-Christoffel relations:

$$\overset{N}{K}_{11}(x, x') = -\frac{\left(\underline{\Phi}^{N-1}(x'), \mathbb{A}^N\, \Psi_N(x)\right)}{(x - x')}\ , \qquad \overset{N}{K}_{22}(y', y) = \frac{\left(\underline{\Psi}^{N-1}(y'), \mathbb{B}^N\, \Phi_N(y)\right)}{(y' - y)}\ .$$

"Differential" generalized Darboux–Christoffel relations:

$$(\partial_{x'} + \partial_x)\, \overset{N}{K}_{11}(x', x) = -\left(\underset{N}{\Phi}(x'), (\overset{N}{\mathbb{B}})^t\, \overset{N-1}{\Psi}(x)\right)\ ,$$

$$(\partial_{y'} + \partial_y)\, \overset{N}{K}_{22}(y', y) = -\left(\underline{\Psi}_N(y'), (\overset{N}{\mathbb{A}})^t\, \overset{N-1}{\Phi}(y)\right)\ .$$

where

$$\overset{N}{\mathbb{A}} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \hline \alpha_{d_2}(N) & \cdots & \alpha_2(N) & \alpha_1(N) & 0 \\ 0 & \alpha_{d_2}(N+1) & \cdots & \alpha_1(N+1) & 0 \\ 0 & 0 & \alpha_{d_2}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \alpha_{d_2}(N+d_2-1) & 0 \end{array}\right],$$

$$\overset{N}{\mathbb{B}} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \hline \beta_{d_1}(N) & \cdots & \beta_2(N) & \beta_1(N) & 0 \\ 0 & \beta_{d_1}(N+1) & \cdots & \beta_1(N+1) & 0 \\ 0 & 0 & \beta_{d_1}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \beta_{d_1}(N+d_1-1) & 0 \end{array}\right]$$

5 Folding

Lemma 5.1 *The sequence of matrices*

$$\underset{N}{\mathbf{a}}(x) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N)}{\gamma(N)} & \cdots & \frac{-\alpha_1(N)}{\gamma(N)} & \frac{(x-\alpha_0(N))}{\gamma(N)} \end{bmatrix}, \quad N \geq d_2,$$

$$\underset{N}{\mathbf{b}}(y) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N)}{\gamma(N)} & \cdots & \frac{-\beta_1(N)}{\gamma(N)} & \frac{(y-\beta_0(N))}{\gamma(N)} \end{bmatrix}, \quad N \geq d_1 \ .$$

implement the shift $N \mapsto N+1$*:*

$$\underset{N}{\mathbf{a}}\,\underset{N}{\Psi}(x) = \underset{N+1}{\Psi}(x)\ , \qquad \underset{N}{\mathbf{b}}\,\underset{N}{\Phi}(y) = \underset{N+1}{\Phi}(y)\ .$$

Lemma 5.2 *The windows of quasi-polynomials* $\underset{N}{\Psi}$, $\underset{N}{\Phi}$ *satisfy the differential systems*

$$\frac{\partial}{\partial x}\underset{N}{\Psi} = -\overset{N}{D_1}(x)\,\underset{N}{\Psi}\ ,\ , \qquad \frac{\partial}{\partial y}\underset{N}{\Phi} = -\overset{N}{D_2}(y)\,\underset{N}{\Phi}\ ,$$

where

$$\overset{N}{D_1}(x) := \overset{N}{\gamma}\,\overset{N-1}{\mathbf{a}} + \overset{N}{\beta_0} + \sum_{j=1}^{d_1}\overset{N}{\beta_j}\,\underset{N+j-1}{\mathbf{a}}\,\underset{N+j-2}{\mathbf{a}}\cdots\underset{N}{\mathbf{a}} \in gl_{d_2+1}[x]$$

$$\overset{N}{D_2}(y) := \overset{N}{\gamma}\,\overset{N-1}{\mathbf{b}} + \overset{N}{\alpha_0} + \sum_{j=1}^{d_2}\overset{N}{\alpha_j}\,\underset{N+j-1}{\mathbf{b}}\,\underset{N+j-2}{\mathbf{b}}\cdots\underset{N}{\mathbf{b}} \in gl_{d_1+1}[y]\ .$$

$$\overset{N}{\alpha_j} := \mathrm{diag}\left[\alpha_j(N+j-d_1), \alpha_j(N+j-d_1+1), \ldots, \alpha_j(N+j)\right]\ ,\ j = -1, \ldots d_2\ ,$$

$$\overset{N}{\beta_j} := \mathrm{diag}\left[\beta_j(N+j-d_2), \beta_j(N+j-d_2+1), \ldots, \beta_j(N+j)\right]\ ,\ j = -1, \ldots d_1\ .$$

$$\overset{N}{\alpha}_{-1} = \overset{N}{\gamma} := \mathrm{diag}\left[\gamma(N-d_1-1), \ldots, \gamma(N-1)\right]$$

or

$$\overset{N}{\beta}_{-1} = \overset{N}{\gamma} := \mathrm{diag}\left[\gamma(N-d_2-1), \gamma(N-d_2), \ldots, \gamma(N-1)\right]\ .$$

Deformation equations

Lemma 5.3 *The deformation equations can be written in the folded windows (and dual windows) as*

$$\frac{d}{du_K}\underset{N}{\Psi} = \overset{N,\Psi}{\mathbf{U}}_K \underset{N}{\Psi}\,, \qquad \frac{d}{du_K}\underline{\overset{N-1}{\Psi}} = \underline{\overset{N-1}{\Psi}}\;\underline{\overset{N,\Psi}{\mathbf{U}}}_K\,,$$

$$\frac{d}{dv_J}\underset{N}{\Psi} = -\overset{N,\Psi}{\mathbf{V}}_J \underset{N}{\Psi}\,, \qquad \frac{d}{dv_J}\underline{\overset{N-1}{\Psi}} = -\underline{\overset{N-1}{\Psi}}\;\underline{\overset{N,\Psi}{\mathbf{V}}}_J\,,$$

$$\frac{d}{du_K}\underset{N}{\Phi} = -\overset{N,\Phi}{\mathbf{U}}_K \underset{N}{\Phi}\,, \qquad \frac{d}{du_K}\underline{\overset{N-1}{\Phi}} = -\underline{\overset{N-1}{\Phi}}\;\underline{\overset{N,\Phi}{\mathbf{U}}}_K\,,$$

$$\frac{d}{dv_J}\underset{N}{\Phi} = \overset{N,\Phi}{\mathbf{V}}_J \underset{N}{\Phi}\,, \qquad \frac{d}{dv_J}\underline{\overset{N-1}{\Phi}} = \underline{\overset{N-1}{\Phi}}\;\underline{\overset{N,\Phi}{\mathbf{V}}}_J\,,$$

where

$$\overset{N,\Psi}{\mathbf{U}}_K := \sum_{j=0}^{K} \overset{N,d_2}{U}_{j,K}\, \underset{N+j-1}{\mathbf{a}} \cdots \underset{N}{\mathbf{a}}\,, \qquad \underline{\overset{N,\Psi}{\mathbf{U}}}_K := \sum_{j=0}^{K} \underset{N}{\underline{\mathbf{b}}} \cdots \underset{N+j-1}{\underline{\mathbf{b}}}\, \overset{N+d_1-1,d_1}{U}_{j,K}\,,$$

$$\overset{N,\Psi}{\mathbf{V}}_J := \sum_{j=0}^{J} \overset{N,d_2}{V}_{j,J}\, \overset{N-j}{\mathbf{a}} \cdots \overset{N-1}{\mathbf{a}}\,, \qquad \underline{\overset{N,\Psi}{\mathbf{V}}}_J := \sum_{j=0}^{J} \overset{N-1}{\underline{\mathbf{b}}} \cdots \overset{N-j}{\underline{\mathbf{b}}}\, \overset{N+d_1-1,d_1}{V}_{j,J}\,,$$

$$\mathbf{U}_K^{N,\Phi} := \sum_{j=0}^{K} \overset{N-j,d_1}{U}_{j,K}\, \overset{N-j}{\mathbf{b}} \cdots \overset{N-1}{\mathbf{b}}\,, \qquad \underline{\overset{N,\Phi}{\mathbf{U}}}_K := \sum_{j=0}^{K} \overset{N-1}{\underline{\mathbf{a}}} \cdots \overset{N-j}{\underline{\mathbf{a}}}\, \overset{N+d_2-1-j,d_2}{U}_{j,K}$$

$$\overset{N,\Phi}{\mathbf{V}}_J := \sum_{j=0}^{J} \overset{N+j,d_1}{V}_{j,J}\, \underset{N+j-1}{\mathbf{b}} \cdots \underset{N}{\mathbf{b}}\,, \qquad \underline{\overset{N,\Phi}{\mathbf{V}}}_J := \sum_{j=0}^{J} \underset{N}{\underline{\mathbf{a}}} \cdots \underset{N-1+j}{\underline{\mathbf{a}}}\, \overset{N+j+d_2-1,d_2}{V}_{j,J}\,.$$

$$\overset{N,d}{U}_{j,K} := \mathrm{diag}(U_j^K(N-d),\ldots,U_j^K(N)) \qquad \overset{N,d}{V}_{j,J} := \mathrm{diag}(V_j^J(N-d-j),\ldots,V_j^J(N-j))\,.$$

$$U_j^K(n) := U_{n,j+n}^K\,, \quad V_j^J(n) := V_{n,j+n}^J\,.$$

$$U^K := -\frac{1}{K}\left\{[Q^K]_{>0} + \frac{1}{2}[Q^K]_0\right\}\,, \quad V^J := -\frac{1}{J}\left\{[(-P^t)^J]_{>0} + \frac{1}{2}[(-P^t)^J]_0\right\}\,.$$

6 Compatibility

Theorem 6.1 The system of equations

$$\left(\partial_x + \overset{N}{D}_1\right) \underset{N}{\Psi}(x) = 0\ ,$$

$$\left(\partial_{u_K} - \overset{N,\Psi}{\mathbf{U}}_K\right) \underset{N}{\Psi}(x) = 0\ ,$$

$$\left(\partial_{v_J} + \overset{N,\Psi}{\mathbf{V}}_{(J)}\right) \underset{N}{\Psi}(x) = 0\ ,$$

$$\underset{N+1}{\Psi}(x) = \underset{N}{\mathbf{a}}(x)\, \underset{N}{\Psi}(x)$$

is compatible and hence sequences of fundamental systems of solutions $\left\{ \underset{N}{\Psi}(x) \right\}_{N>d_2}$ to all equations exist.

Similar statements hold for the barred quantities and those related to the Φ_N sequences.

7 Spectral Duality

Theorem 7.1 The spectral curves associated to the characteristic polynomials of $\overset{N}{D}_1$, $\overset{N}{\underline{D}}_2$, $\overset{N}{D}_2$, $\overset{N}{\underline{D}}_1$ are pairwise equal. More precisely, we have:

$$u_{d_1+1} \det\left[x\mathbf{1}_{d_1+1} - \overset{N}{D}_2(y)\right] = v_{d_2+1} \det\left[y\mathbf{1}_{d_2+1} - \overset{N}{\underline{D}}_1(x)\right]$$

$$u_{d_1+1} \det\left[x\mathbf{1}_{d_1+1} - \overset{N}{\underline{D}}_2(y)\right] = v_{d_2+1} \det\left[y\mathbf{1}_{d_2+1} - \overset{N}{D}_1(x)\right]\ ,$$

which connect the spectral curves of the differential operators of different dimensions operating on the two pairs of dual windows.

The proof is based on the linear algebra lemma.

Lemma 7.1 *Let T be a square matrix having the block form*

$$T = \left[\begin{array}{c|c|c|c|c} 0 & F_1 & 0 & 0 & 0 \\ \hline 0 & 0 & F_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & F_d \\ \hline G_0 & G_1 & G_2 & \cdots & G_d \end{array}\right] ,$$

where the $d+1$ blocks have compatible sizes and the diagonal blocks are square. Then

$$\det\left[\mathbf{1} - T\right] = \det\left[\mathbf{1} - D\right]$$

where

$$D := G_d + \sum_{k=0}^{d-1} G_k \cdot F_{k+1} \cdots F_d \ ,$$

and $\mathbf{1}$ denotes, according to the context, the unit matrix of appropriate size.

Theorem 7.2 If $\{\underline{\tilde{\psi}}_n(y')\}_{n\in\mathbb{N}}$ and $\{\tilde{\phi}_n(y)\}_{n\in\mathbb{N}}$ (or $\{\underline{\tilde{\phi}}_n(x')\}_{n\in\mathbb{N}}$ and $\{\tilde{\psi}_n(x)\}_{n\in\mathbb{N}}$) are two arbitrary pairs of sequences of functions satisfying the recurrence relations of Lemma 5.1, the differential relations, Lemma 5.2, and the deformation equations of Lemma 5.3 then the bilinear expressions

$$\tilde{f}_N(y) := \left(\overset{N-1}{\underline{\tilde{\Psi}}}(y), \overset{N}{\mathbb{B}} \underset{N}{\tilde{\Phi}}(y) \right) ,$$

$$\tilde{g}_N(x) := \left(\overset{N-1}{\underline{\tilde{\Phi}}}(x), \overset{N}{\mathbf{A}} \underset{N}{\tilde{\Psi}}(x) \right) ,$$

are independent of y (resp. x) and N, and also are constant in the deformation parameters $\{u_K, v_J\}$.

Corollary 7.1 There exist two pairs of sequences of fundamental matrix solutions to the difference–differential–deformation equations $(\underset{N}{\boldsymbol{\Psi}}, \overset{N-1}{\underline{\boldsymbol{\Phi}}})$, $(\underset{N}{\boldsymbol{\Phi}}, \overset{N-1}{\underline{\boldsymbol{\Psi}}})$ such that

$$\left(\overset{N-1}{\underline{\boldsymbol{\Phi}}}, \overset{N}{\mathbf{A}}\, \underset{N}{\boldsymbol{\Psi}}\right) \equiv \mathbf{1}\,, \qquad \left(\overset{N-1}{\underline{\boldsymbol{\Psi}}}, \overset{N}{\mathbb{B}}\, \underset{N}{\boldsymbol{\Phi}}\right) \equiv \mathbf{1}\,. \tag{7-1}$$

Theorem 7.3 The differential-deformation systems

$$\left\{\partial_y \underset{N}{\boldsymbol{\Phi}} = -\overset{N}{D_2}(y)\, \underset{N}{\boldsymbol{\Phi}},\ \partial_{u_K} \underset{N}{\boldsymbol{\Phi}} = -\overset{N,\Phi}{\mathbf{U}_K}\, \underset{N}{\boldsymbol{\Phi}},\ \partial_{v_J} \underset{N}{\boldsymbol{\Phi}} = \overset{N,\Phi}{\mathbf{V}_J}\, \underset{N}{\boldsymbol{\Phi}},\right\}$$

$$\left\{\partial_y \overset{N-1}{\underline{\boldsymbol{\Psi}}} = \overset{N-1}{\underline{\boldsymbol{\Psi}}}\, \overset{N}{\underline{D}_2}(y),\ \partial_{u_K} \overset{N-1}{\underline{\boldsymbol{\Psi}}} = \overset{N-1}{\underline{\boldsymbol{\Psi}}}\, \overset{N,\Psi}{\underline{\mathbf{U}}_K},\ \partial_{v_J} \overset{N-1}{\underline{\boldsymbol{\Psi}}} = -\overset{N-1}{\underline{\boldsymbol{\Psi}}}\, \overset{N,\Psi}{\underline{\mathbf{V}}_J},\right\}.$$

are put in duality by the matrix $\overset{N}{\mathbb{B}}$,

$$\overset{N}{\mathbb{B}}\, \overset{N}{D_2}(y) = \overset{N}{\underline{D}_2}(y)\, \overset{N}{\mathbb{B}}$$

$$\partial_{u_K} \overset{N}{\mathbb{B}} = \overset{N}{\mathbb{B}}\, \overset{N,\Phi}{\mathbf{U}_K}(y) - \overset{N,\Psi}{\underline{\mathbf{U}}_K}(y)\, \overset{N}{\mathbb{B}} \qquad \partial_{v_J} \overset{N}{\mathbb{B}} = \overset{N,\Psi}{\underline{\mathbf{V}}_J}(y)\, \overset{N}{\mathbb{B}} - \overset{N}{\mathbb{B}}\, \overset{N,\Phi}{\mathbf{V}_J}(y)\,.$$

In particular since the matrices $\overset{N}{\underline{D}_2}(y)$ and $\overset{N}{D_2}(y)$ are conjugate to each other, their spectral curves are the same.

Similarly, the differential-deformation systems

$$\left\{\partial_x \underset{N}{\boldsymbol{\Psi}} = -\overset{N}{D_1}(x)\, \underset{N}{\boldsymbol{\Psi}},\ \partial_{u_K} \underset{N}{\boldsymbol{\Psi}} = \overset{N,\Psi}{\mathbf{U}_K}\, \underset{N}{\boldsymbol{\Psi}},\ \partial_{v_J} \underset{N}{\boldsymbol{\Psi}} = -\overset{N,\Psi}{\mathbf{V}_J}\, \underset{N}{\boldsymbol{\Psi}},\right\}$$

$$\left\{\partial_x \overset{N-1}{\underline{\boldsymbol{\Phi}}} = \overset{N-1}{\underline{\boldsymbol{\Phi}}}\, \overset{N}{\underline{D}_1}(x),\ \partial_{u_K} \overset{N-1}{\underline{\boldsymbol{\Phi}}} = -\overset{N-1}{\underline{\boldsymbol{\Phi}}}\, \overset{N,\Phi}{\underline{\mathbf{U}}_K},\ \partial_{v_J} \overset{N-1}{\underline{\boldsymbol{\Phi}}} = \overset{N-1}{\underline{\boldsymbol{\Phi}}}\, \overset{N,\Phi}{\underline{\mathbf{V}}_J},\right\}.$$

are put in duality by the matrix $\overset{N}{\mathbf{A}}$,

$$\overset{N}{\mathbf{A}}\, \overset{N}{D_1}(x) = \overset{N}{\underline{D}_1}(x)\, \overset{N}{\mathbf{A}}$$

$$\partial_{v_J} \overset{N}{\mathbf{A}} = \overset{N}{\mathbf{A}}\, \overset{N,\Psi}{\mathbf{V}_J}(x) - \overset{N,\Phi}{\underline{\mathbf{V}}_J}(x)\, \overset{N}{\mathbf{A}}\,, \qquad \partial_{u_K} \overset{N}{\mathbf{A}} = \overset{N,\Phi}{\underline{\mathbf{U}}_K}(x)\, \overset{N}{\mathbf{A}} - \overset{N}{\mathbf{A}}\, \overset{N,\Psi}{\mathbf{U}_K}(x)\,,$$

and hence the spectral curves of $\overset{N}{\underline{D}_1}(x)$ and $\overset{N}{D_1}(x)$ are also the same.

Conclusion The four spectral curves

$$
\begin{array}{ccc}
\det\left[y\mathbf{1} - \overset{N}{\underline{D}}_2(x)\right] = 0 & \overset{\text{Prop. 7.1}}{\longleftrightarrow} & \det\left[x\mathbf{1} - \overset{N}{D}_1(y)\right] = 0 \\
\updownarrow \text{Thm. 7.2} & & \updownarrow \text{Thm. 7.2} \\
\det\left[y\mathbf{1} - \overset{N}{D}_2(x)\right] = 0 & \overset{\text{Prop. 7.1}}{\longleftrightarrow} & \det\left[x\mathbf{1} - \overset{N}{\underline{D}}_1(y)\right] = 0
\end{array}
$$

all coincide.

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Stokes Matrices and Riemann–Hilbert problem for 2-matrix models

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1) "Duality, Biorthogonal Polynomials and Multi–Matrix Models", *Comm. Math. Phys.* ~~(in press, 2002)~~ 229, 73-120 (2002)

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2) "Differential systems for biorthogonal polynomials appearing in 2-Matrix Models and the associated Riemann-Hilbert problem
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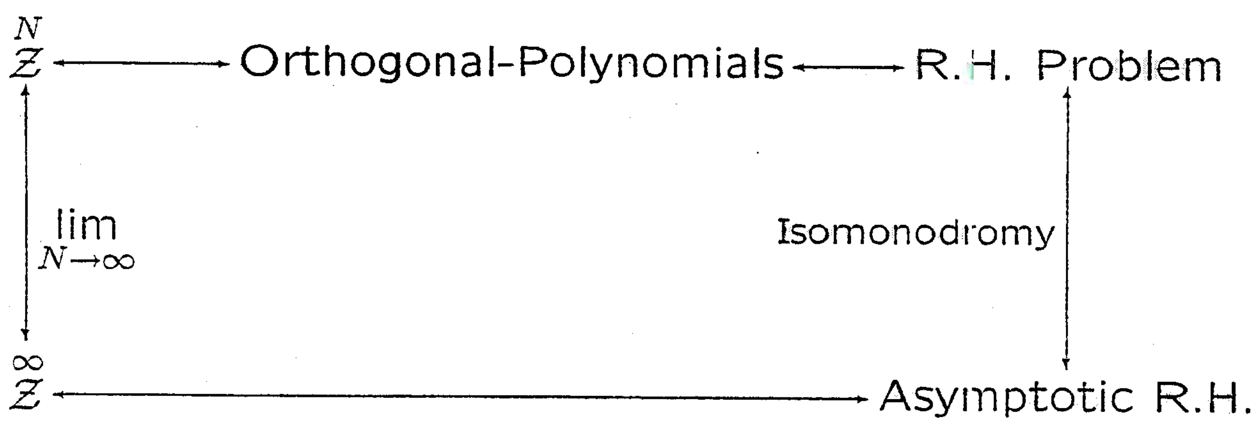
•) Aim: rigorous large N asymptotics for partition function of 2-matrix models using bi–orthogonal polynomials.

$$\overset{N}{\mathcal{Z}}(V_1, V_2) := \int\int \mathrm{d}M_1 \mathrm{d}M_2\, \mathrm{e}^{-\mathrm{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)} . \tag{1}$$

•) Idea: apply the same ideology already successful in 1-matrix models (Its, Bleher, Fokas, Kitaev, Moore), Deift, Kriecherbauer, McLaughlin, Venakides, Zhou, ...

$$\overset{N}{\mathcal{Z}} := \int \mathrm{d}M\, \mathrm{e}^{-\mathrm{Tr}\, V(M)} \tag{2}$$



Technical difficulty: for the 2-matrix model, which R.H. problem?

Bi-orthogonal (quasi)-polynomials

$$\psi_n(x) := \frac{\pi_n(x)}{\sqrt{h_n}} \mathrm{e}^{-V_1(x)} \ ; \ \ \varphi_n(y) := \frac{\sigma_n(y)}{\sqrt{h_n}} \mathrm{e}^{-V_2(y)}$$

$$\pi_n(x) := x^n + \dots \ ; \qquad \sigma_n(y) := y^n + \dots$$

$$\delta_{nm} = \sum_{ij} \varkappa_{ij} \int_{\Gamma_x^{(i)}} \int_{\Gamma_y^{(j)}} \mathrm{d}x \mathrm{d}y\, \psi_n(x) \mathrm{e}^{xy} \varphi_n(y) =:$$

$$=: \quad \int_{\varkappa\Gamma} \mathrm{d}x \wedge \mathrm{d}y\, \psi_n(x) \mathrm{e}^{xy} \varphi_n(y)$$

↑

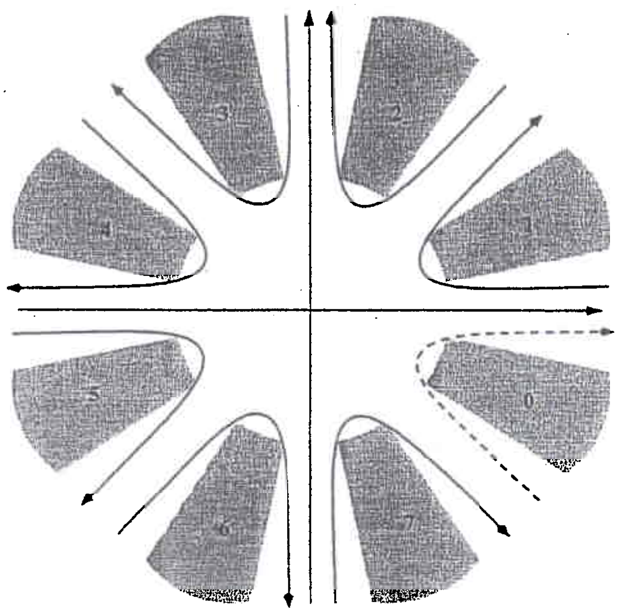
generalized orthogonality

The contours $\Gamma_x^{(i)}, \Gamma_y^{(j)}$ depend on the degree of the potentials

$$V_1(x) = \frac{u_{d_1+1}}{d_1+1} x^{d_1+1} + \sum_{J=1}^{d_1} \frac{u_J}{J} x^J$$

$$V_2(y) = \frac{v_{d_2+1}}{d_2+1} y^{d_2+1} + \sum_{J=1}^{d_1} \frac{v_K}{K} y^K$$

An example of contours for V_1 of degree $d_1 + 1 = 8$ (and $u_{d_1+1} \in \mathbb{R}_+$)



Degree can be odd.

Dual pairs of Isomonodromic DDD equations

$$\underset{N}{\Psi}(x) := \begin{bmatrix} \psi_{N-d_2}(x) \\ \vdots \\ \psi_N(x) \end{bmatrix} \in \mathbb{C}^{d_2+1} \qquad (3)$$

$$\partial_x \underset{N}{\Psi}(x) = -\overset{N}{D}_1(x)\, \underset{N}{\Psi}(x) \quad \text{Differential (RH)}$$

$$\partial_u \underset{N}{\Psi}(x) = \overset{N}{\mathcal{U}}_1(x)\, \underset{N}{\Psi}(x) \quad \text{Deformation}$$

$$\underset{N+1}{\Psi}(x) = \underset{N}{\mathrm{a}}(x)\, \underset{N}{\Psi}(x) \quad \text{Difference.}$$

Frobenius compatibility $\Rightarrow$ Isomonodromic deformations of $\partial_x + D_1(x)$ (discrete and continuous). Size d_2+1: degree (Poincaré rank) d_1+1;

$$\overset{N}{D}_1(x) = x^{d_1(+1)} \left[\begin{array}{c|c} 0_{d_2\times d_2} & 0 \\ \hline 0 & 1 \end{array}\right] + \mathcal{O}(x^{d_1}) \ . \qquad (4)$$

Stokes Phenomenon is complicated by the degeneracy of the eigenvalues (of the leading term).

Dual (barred) system

Define the Fourier–Laplace transforms

$$\underline{\varphi}_n(x) := \int \mathrm{d}y \, \mathrm{e}^{xy} \varphi_n(y)$$

They satisfy similar DDD isomonodromic system:

$$\overset{N}{\underline{\phi}}(x) := \left[\underline{\varphi}_{N-1}(x), \dots, \underline{\varphi}_{N+d_2-1}(x)\right] \in \mathbb{C}^{d_2+1^\vee}$$

$$\partial_x \overset{N}{\underline{\phi}}(x) = \overset{N}{\underline{\phi}}(x) \, \overset{N}{\underline{D}}_1(x)$$

$$\partial_u \overset{N}{\underline{\phi}}(x) = \overset{N}{\underline{\phi}}(x) \overset{N}{\underline{\mathcal{U}}}_1(x)$$

$$\overset{N+1}{\underline{\phi}}(x) = \overset{N}{\underline{\phi}}(x) \, \underset{N}{\underline{a}}(x)$$

Christoffel–Darboux pairing

The (generalized) CD formula for the kernel

$$\overset{N}{K}_{11}(x, x') := \frac{1}{N} \sum_{n=0}^{N-1} \psi_n(x) \underline{\varphi}_n(x') = \frac{1}{x - x'} \underset{N}{\Psi}(x) \overset{N}{\mathbb{A}} \overset{N}{\Phi}(x')$$

$$\overset{N}{\mathbb{A}} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \hline \alpha_{d_2}(N) & \cdots & \alpha_2(N) & \alpha_1(N) & 0 \\ 0 & \alpha_{d_2}(N+1) & \cdots & \alpha_2(N+1) & 0 \\ 0 & 0 & \alpha_{d_2}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \alpha_{d_2}(N+d_2-1) & 0 \end{array}\right]$$

The Christoffel–Darboux pairing $\overset{N}{\mathbb{A}}$ establishes duality barred $\leftrightarrow$ unbarred DDD systems.

Duality $\longleftrightarrow$ String Equation
$$\overset{N}{\underline{D}_1}(x)\,\overset{N}{\mathbb{A}} = \overset{N}{\mathbb{A}}\,\overset{N}{D_1}(x)$$

By duality, we can formulate the RH problem for one or the other system: for the barred system is much easier.

Fundamental system of the barred system of DDD equations

Two types of solutions:

- Laplace Transform type: if $\overset{N}{\phi}(y) := \left[\varphi_{N-1}(y), \ldots, \varphi_{n-1+d_2}(y)\right]$ then:

$$\underline{\overset{N}{\phi}}^{(j)}(x) := \int_{\Gamma_y^{(j)}} \mathrm{d}y\, \mathrm{e}^{xy} \overset{N}{\phi}(y) \tag{5}$$

are solution of the DDD system for any contour Γ for which the integral converges. If $\deg(V_2) = d_2 + 1$ there are d_2 "homologically" independent contours. We thus have d_2 (lin. indep.) entire solutions.

- Hilbert–Transform type:

$$\underline{\overset{N}{\phi}}^{(0)}(x) := \mathrm{e}^{V_1(x)} \int_{\varkappa\Gamma} \mathrm{d}s \wedge \mathrm{d}y \frac{\mathrm{e}^{-V_1(s)+sy}\,\phi(y)}{x-s} \tag{6}$$

"Discontinuities" along the contours $\Gamma_x^{(i)}$ in the x–plane. (R.H.)

$$\underline{\phi}_+^{(0)}(x) = \underline{\phi}_-^{(0)}(x) + 2i\pi \sum_{j=1}^{d_2} \varkappa_{kj} \underline{\phi}^{(j)}(x) \ , \quad x \in \Gamma_x^{(k)} \ . \tag{7}$$

Fundamental System

$$\underset{\sim}{\overset{N}{\Phi}}(x) := \begin{bmatrix} \underset{\sim}{\overset{N}{\phi}}{}^{(0)}(x) \\ \hline \underset{\sim}{\overset{N}{\phi}}{}^{(1)}(x) \\ \vdots \\ \underset{\sim}{\overset{N}{\phi}}{}^{(d_2)}(x) \end{bmatrix} \begin{matrix} \leftarrow \text{Hilbert–like: jumps} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Laplace–like: Stokes matrices} \end{matrix} \tag{8}$$

The R.H. problem needs to be complemented with the formal asymptotics at $x = \infty$ and Stokes matrices.

For the Hilbert–like solution, in each conn. comp. of $\mathbb{C}_x \setminus \bigcup_{j=1}^{d_1} \Gamma_x^{(j)}$ we have

$$\underset{\sim}{\varphi}{}_n^{(0)}(x) = \mathrm{e}^{V_1(x)} \int_{\varkappa\Gamma} \mathrm{d}s \wedge \mathrm{d}y \frac{\mathrm{e}^{-V_1(s)+sy}\varphi_n(y)}{x-s}$$

$$\sim \sqrt{h_n}\,\mathrm{e}^{V_1(x)} x^{-n-1} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right)$$

Solutions of the Laplace-type

Each of them is of the form

$$\int_\Gamma \mathrm{d}y\, \mathrm{e}^{xy - V_2(y)} P(y) \tag{9}$$

↑ polynomial

Asymptotics is obtained by the **Steepest Descent** method

$$\int_\gamma \mathrm{d}y\, \mathrm{e}^{xy - V_2(y)} P(y) \sim C \frac{\mathrm{e}^{-V_2(y_k(x)) - xy_k(x)}}{\sqrt{V_2''(y_k(x))}} P(y_k(x)) \tag{10}$$

↑ S.D. Contour

where $y_k(x)$ is the saddle point solution of

$$V_2'(y) - x = 0 \ , \qquad y(x) \sim x^{1/d_2} \sum_{j=0}^{\infty} t_j\, x^{-j/d_2} \tag{11}$$

$$\Im\Big(xy - V_2(y)\Big)\Big|_{y\in\gamma} = \text{const} = \Im(xy_k(x) - V_2(y_k(x)))$$

Choice of γ determines the choice of the root of x; it is integral lin. comb. of the contours $\Gamma_1, \dots, \Gamma_{d_2}$. As arg$(x)$ crosses Stokes' lines the homology of γ changes discontinously.

Formal asymptotics

In each Stokes' sector in the x-plane one solution behaves

$$\Phi \sim e^{T(x)} \cdot W \cdot x^G \cdot Y\left(x^{1/d_2}\right)$$

$$T(x) := \text{diag}\left(-V_1(x), [-V_2(y_1(x)) + x y_1(x)]_+, \dots \right.$$
$$\left. \dots, [-V_2(y_{d_2}(x)) + x y_{d_2}(x)]_+\right)$$

$$W := \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & [\omega^{ij}]_{ij=1\dots d_2} \end{array}\right], \quad \omega := e^{2i\pi/d_2}$$

$$G := \text{diag}\left(-N, \frac{N - \frac{d_2-1}{2}}{d_2}, \dots \frac{N + \frac{d_2-1}{2}}{d_2}\right)$$

$$Y(\lambda) := Y_0 + \mathcal{O}\left(\frac{1}{\lambda}\right) \;;\; Y_0 = \text{diag. invertible}$$

Note: $T(x)$ and logarithmic part do not commute.

Stokes Matrices for Laplace solutions

We have to express the change in homology basis of the SDC as $\alpha := \arg(x)$ increases. No loss of generality in assuming

$$V_2(y) := \frac{y^{d+1}}{d+1} \qquad (12)$$

Change variable

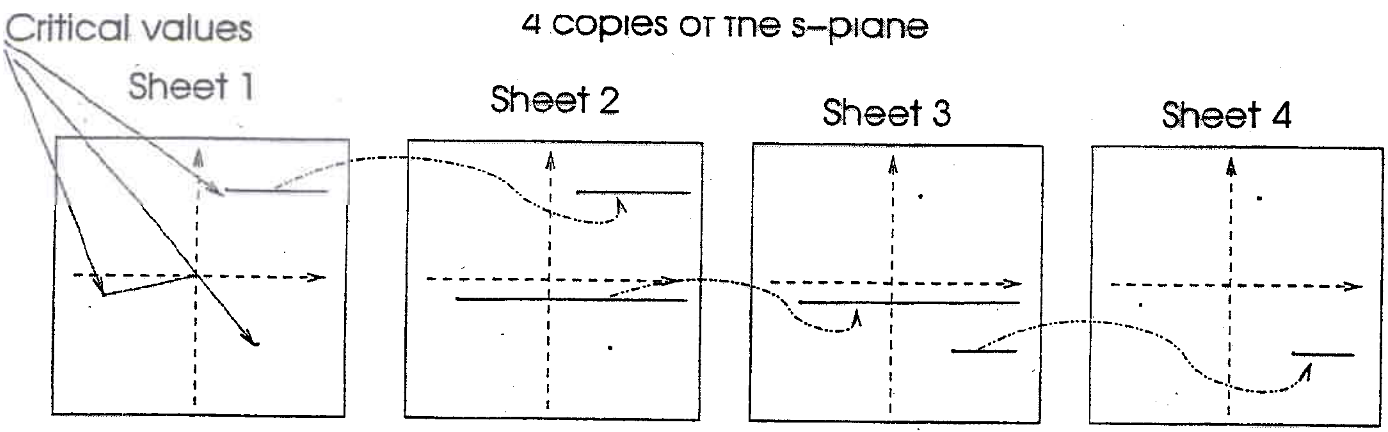
$$xy - V_2(y) = |x|^{\frac{d+1}{d}} \left[\mathrm{e}^{i\alpha} z - \frac{z^{d+1}}{d+1} \right] \; ; \quad z = |x|^{-\frac{1}{d}} y$$

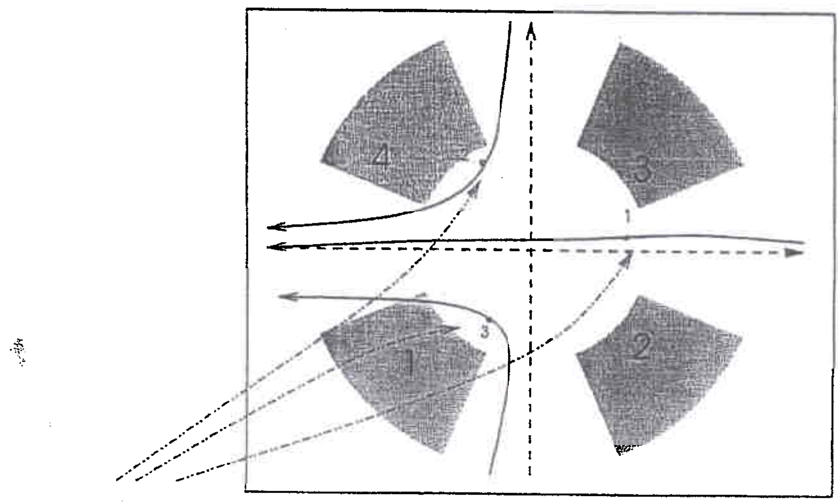
$$s(z) := \frac{z^{d+1}}{d+1} - \mathrm{e}^{i\alpha} z \, , \quad \Lambda := |x|^{\frac{d+1}{d}}$$

$$|x|^{-\frac{1}{d}} \int_\gamma \mathrm{d}y \, \mathrm{e}^{xy - V_2(y)} = \int_\gamma \mathrm{d}z \, \mathrm{e}^{-\Lambda\left(\frac{z^{d+1}}{d+1} - \mathrm{e}^{i\alpha} z\right)} = \int_{\mathrm{cut}} \mathrm{d}s \, \frac{\mathrm{d}z(s)}{\mathrm{d}s} \, \mathrm{e}^{-\Lambda s}$$

↑ Square-root branchcuts

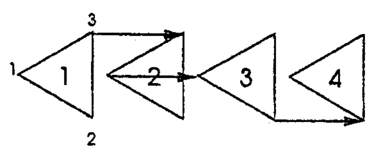
Structure of homology ⇔ sheet structure z(s)
SDC are z–images of the two rims of the cuts in the s–plane.



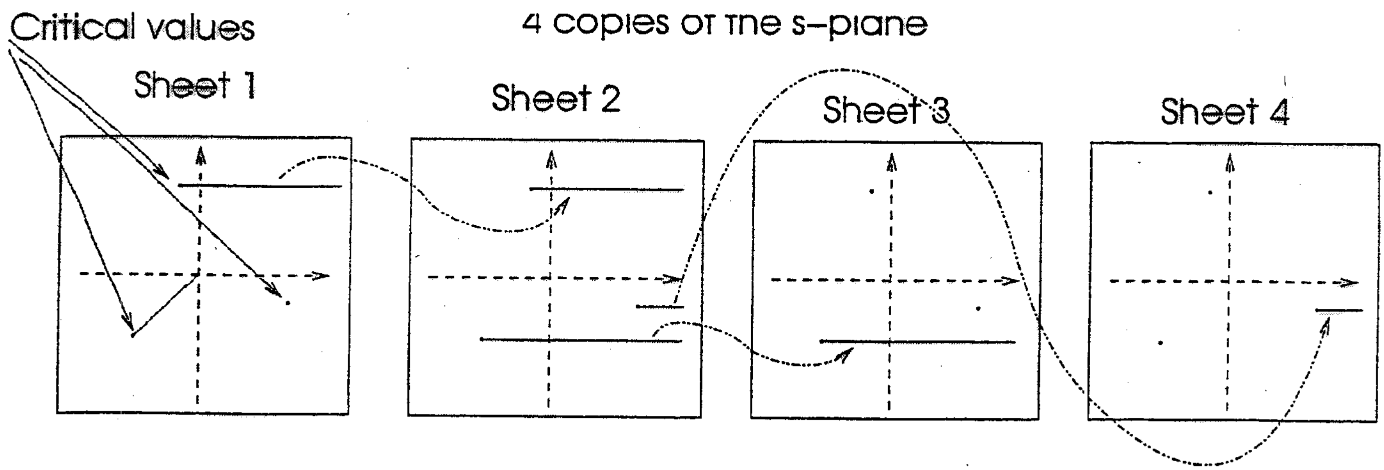


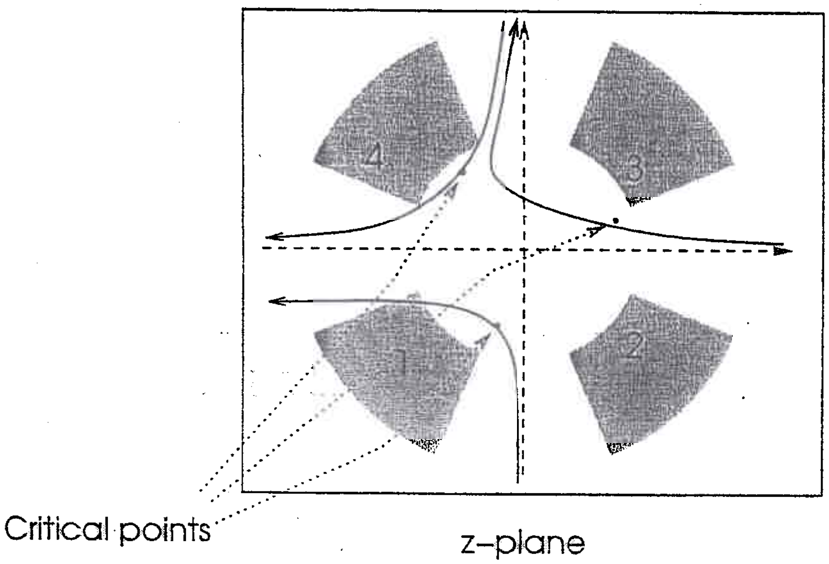
z-plane

Extended Hurwitz diagram

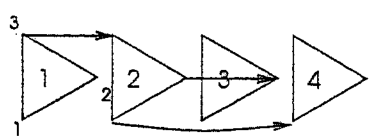


$$Q_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$





Extended Hurwitz diagram



$$Q_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The Stokes matrices

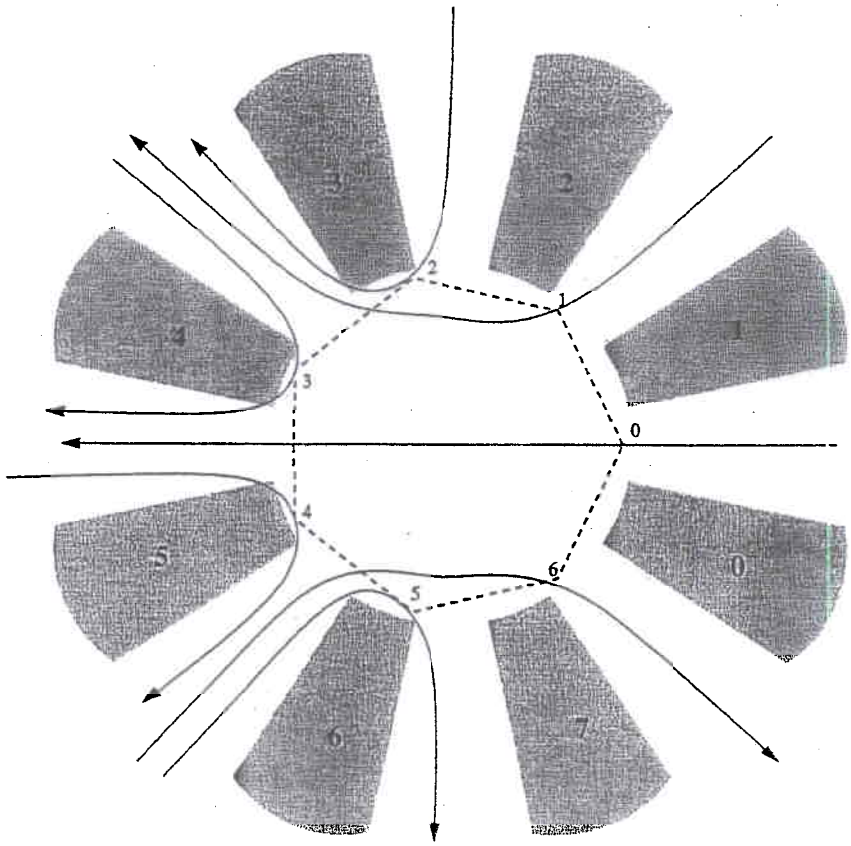
$$M_0 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \; ; \; d = 3$$

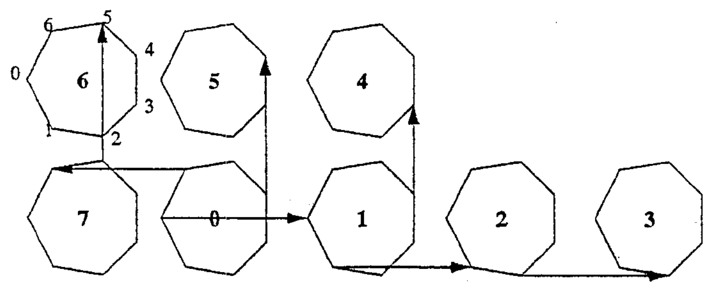
$$M_{k+2} := p M_k p^{-1}$$

$$p := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$S_k := \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & M_k \end{array}\right]$$

Another example with $d = 7$





$$Q_0 := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Conclusions

• Well defined R. H. problem; use it! (ansatz needed: see Bertrand's lecture)

• Define isomonodromic Tau function for the system (technical hurdle: degenerate spectrum) *á la* Miwa, Jimbo, Sato, Mori.

• Compare it with partition function $\mathcal{Z}$.

E.g. for one–matrix models $D(x)$ is 2×2 and spectrum is nondegenerate ⇒ Isom. tau function is defined. Then

$$\mathcal{Z}_n(V) := \int \mathrm{d}M_n\, \mathrm{e}^{-\mathrm{Tr}\, V(M)}$$

$$V(x) := \frac{u_{d+1}}{d+1}x^{d+1} + \sum_{J=1}^{d} \frac{u_J}{J}x^J$$

$$\mathcal{Z}_n(V) := u_{d+1}^{-\frac{n^2}{d+1}}\, \tau_{IM}(V)$$

[M.B.,B.E.,J.H. , "Partition functions for Matrix Models and Isomonodromic Tau functions", nlin.SI/0204054]

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