Asymptotics of the Partition Function for Random Matrices

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Random Matrix Theory

- Consider the probability measure ("Deformed Unitary Ensemble") on $N \times N$ Hermitean matrices given by

$$\frac{1}{Z_N} \exp\left\{ -N \operatorname{Tr}\left[\frac{1}{2} M^2 + \sum_{k=1}^{v} t_k \, M^k \right] \right\} dM$$

where $\quad dM = \prod_{j<k} dM^R_{jk} \, dM^I_{jk} \prod_{j=1}^{N} dM_{jj}$

  - This induces a probability measure on eigenvalues.
  - The asymptotic expansion of Z_N gives asymptotic information on the statistics of eigenvalues of random matrices.

The Fundamental Problem

- To understand the detailed asymptotics for the following family of integrals:

$$Z_N(t_1,\ldots,t_\nu) = \int\cdots\int \exp\left\{-N^2\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{2}\lambda_j^2 + \sum_{k=1}^{\nu} t_k\,\lambda_j^k\right) - \frac{1}{N^2}\sum_{j\neq l}\log\left|\lambda_j - \lambda_l\right|\right]\right\} d^N\lambda$$

- We establish the following uniformly valid asymptotic expansion

$$\log\left(\frac{Z_N}{Z_N(\vec{0})}\right) = N^2 e_0(t_1,\ldots,t_\nu) + e_1(t_1,\ldots,t_\nu) + \frac{1}{N^2} e_2(t_1,\ldots,t_\nu) + \cdots$$

where $e_j(t_1,\ldots,t_\nu)$ is an analytic function of the complex vector $\boldsymbol{t}$ in a neighborhood of $\boldsymbol{0}$.

Brief background discussion

Basor, Chen, and Widom computed several terms in an asymptotic Expansion of the Partition function for weights of the form

$$w(x) = U(x)x^{\nu}e^{-x}, \qquad a_{i+j} = \int_0^{\infty} x^{i+j} w(x)\,dx$$

where $U(x)-1$ is a Schwartz function.

$$\det(a_{i+j})_{0\le i,j\le n-1} = \exp\left[c_1 n^2 \ln n + c_2 n^2 + c_3 n \ln n + c_4 n + c_5 n^{1/2} + c_6 \ln n + c_7 + o(1)\right]$$

$c_1 = 1$, $c_2 = -3/2$, $c_3 = \nu$, $c_4 = -\nu + \ln 2\pi$

$c_5 = c_5(U)$, $c_6 = \nu^2/2 - 1/6$, $c_7 = c_7(U)$.

Brief background discussion

- An expansion of this type was conjectured in 1980 by Bessis, Itzkson, and Zuber, with a very elegant heuristic explanation.

← Adv. Appl. Math 1. (1980) 109-157.

See also Di Francesco, Ginsparg, Zinn-Justin 2D Gravity and Random Matrices Phys. Reports 254 (1995).

- Their motivation: the Taylor coefficients of

$e_G(0,0,\ldots,\ t_j,0,\ldots,\ 0)$ should "count" the number of j–valent fat–graphs which fit on a Riemann surface of genus G.

- No rigorous proof of the necessary asymptotic expansion was ever given.

Gaussian Rules

$$d\mu_N = 2^{-\frac{N}{2}} \pi^{-\frac{N^2}{2}} e^{-\frac{1}{2}\mathrm{Tr} M^2} \, dM$$

$$\langle M_{ij} M_{ji} \rangle = 1 \, , \quad \langle M_{ij} M_{k\ell} \rangle = 0 \text{ if } (i,j) \neq (k,\ell).$$

Wick Rules.

$\ell_1, \dots, \ell_{2k}$ linear functions.

$$\langle \ell_1 \ell_2 \cdots \ell_{2k} \rangle = \sum \langle \ell_{r_1} \ell_{s_1} \rangle \langle \ell_{r_2} \ell_{s_2} \rangle \cdots \langle \ell_{r_k} \ell_{s_k} \rangle$$

Sum is over all $\{r_1, \dots, r_k, s_1, \dots, s_k\}$ such that

$1 \le r_j \le 2k-1$

$2 \le s_j \le 2k$

$r_1 < r_2 < \cdots < r_k$

$r_1 < s_1$ $\quad$ $s_1 \neq s_2$ etc.

$r_2 < s_2$

$\vdots$

$r_k < s_k$

$$\hat{Z}_N(t_1,\dots,t_\nu) := \frac{Z_N}{Z_N(0)} = \frac{1}{Z_N(0)} \int e^{-N\,\mathrm{Tr}\left(\frac{1}{2}M^2 + \sum_{k=1}^{\nu} t_k M^k\right)} dM$$

$$\hat{Z}_N = \left\langle e^{-N\,\mathrm{Tr} \sum_{k=1}^{\nu} \frac{t_k}{N^{k/2}} \tilde{M}^k} \right\rangle$$

$$\boxed{\frac{\partial^n}{\partial t_p^n} Z_N \Big|_{\vec{t}=0} = (-1)^n N^{n(1-p/2)} \left\langle \left(\mathrm{Tr}\, M^p\right)^n \right\rangle}$$

Use Wick "calculus" to assist in evaluating

$p=4$, $n=2$:

$$\left\langle \left(\mathrm{Tr}\, M^4\right)^2 \right\rangle = \left\langle \sum_{\text{configs}} M_{i_1 j_1} M_{j_1 k_1} M_{k_1 l_1} M_{l_1 i_1} M_{i_2 j_2} M_{j_2 k_2} M_{k_2 l_2} M_{l_2 i_2} \right\rangle$$

Sum over all $i_1, j_1, k_1, l_1, i_2, j_2, k_2, l_2$

$$= \sum_{\text{configs}} \left\langle f_1\, f_2\, f_3\, f_4\, f_5\, f_6\, f_7\, f_8 \right\rangle$$

Use Wick:

$$= \sum_{\text{configs}} \sum_{(u,v)\in W(8)} \langle f_{u_1} f_{v_1} \rangle \langle f_{u_2} f_{v_2} \rangle \langle f_{u_3} f_{v_3} \rangle \langle f_{u_4} f_{v_4} \rangle$$

$$= \sum_{W} \sum_{\text{configs}}$$

$$\left\langle (\mathrm{Tr}\, M^4)^2 \right\rangle = \sum_{W(8)} \sum_{\text{configs}} \langle f_{u_1} f_{v_1}\rangle \langle f_{u_2} f_{v_2}\rangle \langle f_{u_3} f_{v_3}\rangle \langle f_{u_4} f_{v_4}\rangle$$

$$= \sum_{\text{configs}} \langle f_1 f_2\rangle \langle f_3 f_4\rangle \langle f_5 f_6\rangle \langle f_7 f_8\rangle$$

$$+ \sum_{\text{configs}} \langle f_1 f_4\rangle \langle f_2 f_3\rangle \langle f_6 f_8\rangle \langle f_5 f_7\rangle .$$

$+ \ldots$

Take one of these terms:

$$\sum_{\substack{\text{configs} \\ i_1, j_1, k_1, l_1, i_2, j_2, k_2, l_2}} \langle M_{i_1 j_1} M_{l_1 i_1}\rangle \langle M_{j_1 k_1} M_{k_1 l_1}\rangle \langle M_{i_2 k_2} M \rangle \langle M\, M \rangle$$

etc.

$i_1 = i_1$, $j_1 = l_1$; $j_1 = l_1$, $k_1 = k_1$; $j_2 = i_2$, $i_2 = l_2$, $k_2 = l_2$, $j_2 = k_2$

N^4

Back to the general case:

$$\left\langle (\mathrm{Tr}\, M^p)^n \right\rangle = \sum_{\text{configs}} \Big\langle M_{i_1 j_1} M_{j_1 k_1} M_{k_1 l_1} \cdots \times M_{i_2 j_2} M_{j_2 k_2} \cdots \vdots \times M_{i_n j_n} M_{j_n k_n} \cdots \Big\rangle$$

$$= \sum_{\text{configs}} \langle f_1 f_2 \cdots f_{np} \rangle$$

$$= \sum_{W(np)} \sum_{\text{configs}} \langle f_{u_1} f_{v_1} \rangle \langle f_{u_2} f_{v_2} \rangle \cdots \langle f_{u_{\frac{np}{2}}} f_{v_{\frac{np}{2}}} \rangle$$

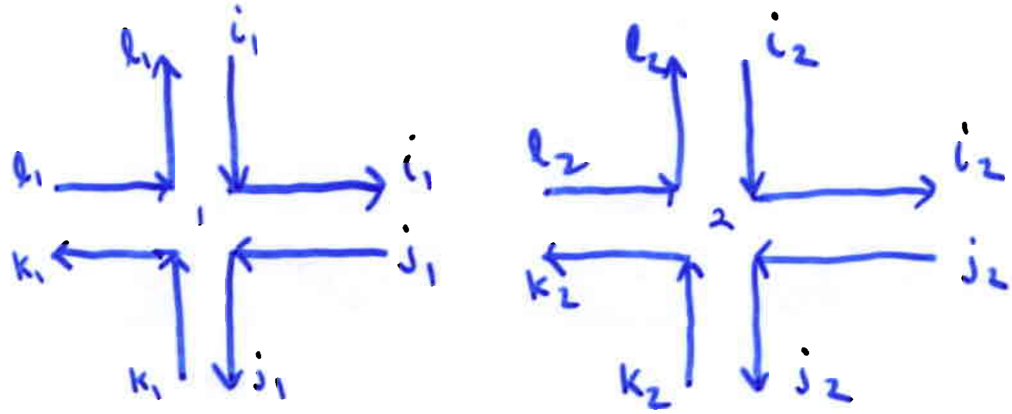
$$= \sum_{W=(u,v) \in W(np)} N^{F(W)} , \qquad W = \left((u_1, v_1), \ldots, (u_{\frac{np}{2}}, v_{\frac{np}{2}}) \right)$$

$F(W)$ = # of free indices among

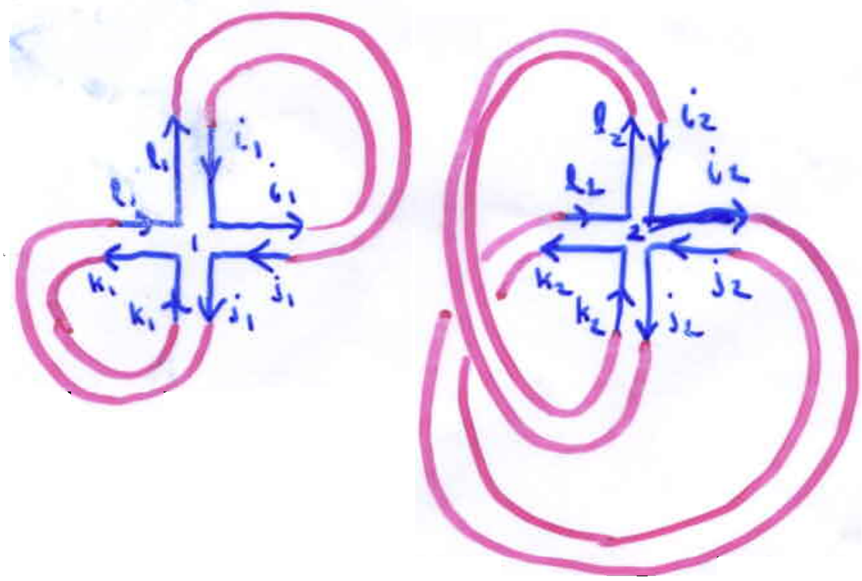
$i_1, j_1, k_1, \ldots$
$i_2, j_2, k_2, \ldots$
$\vdots$
$i_n, j_n, k_n, \ldots$

Connection to labelled maps.

① draw n vertices with valence p.



② form ribbons by connecting fattened edges, using Wick rules



These labelled "ribbon graphs" we call "diagrams."

If underlying graph is connected, the ribbon graph may be embedded in a Riemann surface, and we get a labelled map. Else embed each component separated

For each Wick coupling (diagram) we can associate a labelled map (or a union of labelled maps).

# of faces is $F(w)$.

# of vertices is n

# of edges is $np/2 = 2n$

Euler characteristic tells us the genus $2-2g = n-2n+F$

$$g = \frac{2+n-F}{2}.$$

So
$$\left\langle (\mathrm{Tr}\, M^4)^n \right\rangle = \sum_{w \in \mathcal{W}(\gamma)} N^{F(w)}$$
$$= \sum_g N^{2-2g+n} \# \{ w \in \mathcal{W}(\gamma) : F(w) = 2-2g+n \}$$

We can use this to count diagrams according to # vertices, and genus, if we could evaluate these integrals. (This can be done.)

$$\left\langle (\mathrm{Tr}\, M^p)^n \right\rangle = \sum_{w \in \mathcal{W}(np)} N^{F(w)} = \sum_g N^{2-2g+\frac{np}{2}-n} \# \{ w : F = 2-2g+\tfrac{np}{2}-n \}$$

Now $\frac{1}{N^2}\ln Z_N = e_0 + N^{-2}e_1 + N^{-4}e_2 + \dots$

We can differentiate this term-by term, and then evaluate at $\vec{t}=0$.

$$\frac{\partial^n}{\partial t_p^n}\frac{1}{N^2}\ln Z_N\Big|_{\vec{t}=0} = \frac{\partial^n}{\partial t_p^n}e_0\Big|_{\vec{t}=0} + N^{-2}\frac{\partial^n}{\partial t_p^n}e_1\Big|_{\vec{t}=0} + N^{-4}\frac{\partial^n}{\partial t_p^n}e_2\Big|_{\vec{t}=0} + \dots$$

① R.H.S. is a finite sum. (Zero from some point on.)

② $$\frac{\partial^n}{\partial t_p^n}e_g\Big|_{\vec{t}=0} = \#\left\{ W \in \mathcal{W}(np) : \begin{array}{l} F(W) = 2-2g+\frac{np}{2}-n \\ W \text{ is connected.} \end{array} \right\}$$

$$= \#\left\{ \text{diagrams with} \begin{array}{l} \bullet\ n \text{ vertices (valence } p) \\ \bullet\ \text{genus } g \\ \bullet\ \text{connected.} \end{array} \right\}$$

(diagram: graph with *labelled vertices* $(1,2,\dots,n)$ and a labelling of the edges incident to each vertex $(i,j,k,\dots)$)

Easy consequences $(\vec{t} \neq 0)$

$$\frac{\partial}{\partial t_l}\log Z_N = -NE_N(\operatorname{Tr} M^l)$$

$$\mathbb{E}_N\left(\frac{1}{N}\operatorname{Tr} M^\ell\right) \to e_{0,t_\ell}(\vec{t})$$

$$\frac{\partial^2}{\partial t_l \partial t_m}\log Z_N =$$

$$-N^2\left[E_N(\operatorname{Tr} M^l \operatorname{Tr} M^m) - E_N\left(\operatorname{Tr} M^l\right)E_N\left(\operatorname{Tr} M^m\right)\right]$$

Expansion implies that $\left\{\frac{1}{N}\operatorname{Tr}\left(M^\ell\right)\right\}$ Are asymptotically uncorrelated.

$$\mathbb{E}_N\left(\frac{1}{N}\operatorname{Tr} M^\ell \cdot \frac{1}{N}\operatorname{Tr} M^m\right) - \mathbb{E}_N\left(\frac{1}{N}\operatorname{Tr} M^\ell\right)\mathbb{E}_N\left(\frac{1}{N}\operatorname{Tr} M^m\right) \longrightarrow 0 = O(1/N^2)$$

The Approach

$$\frac{\partial}{\partial t_l}\log Z_N = -N^2\int_{-\infty}^{\infty}\lambda^l \rho_N^{(1)}(\lambda)\,d\lambda$$

Where $\rho_N^{(1)}(\lambda)$ is the mean density of eigenvalues. It has a very useful representation in terms of the associated orthogonal polynomials, defined via

We establish a complete asymptotic expansion for integrals:

$$\int_{-\infty}^{\infty} f(\lambda)\rho_N^{(1)}(\lambda)\,d\lambda = f_0 + \frac{1}{N^2} f_1 + \frac{1}{N^4} f_2 + \cdots$$

Provided f is C^∞ and doesn't grow too fast.

Eigenvalue Statistics via OP's

$$N\rho_N^{(1)}(\lambda) = \exp\left[-N\left\{\frac{1}{2}\lambda^2 + \sum_{k=1}^{v} t_k\ \lambda^k\right\}\right]K_N(\lambda,\lambda)$$

$$K_N(x,y) = \sum_{j=0}^{N-1} p_j(x)p_j(y)$$

$$= \frac{\kappa_{N-1}}{\kappa_N}\frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x-y}$$

$$K_N(x,x) = \frac{\kappa_{N-1}}{\kappa_N}\left[p_N{}'(x)p_{N-1}(x) - p_N(x)p_{N-1}{}'(x)\ \right]$$

Orthogonal Polynomials

$$N\rho_N^{(1)}(x) = e^{-NV(x)} \frac{\kappa_{N-1}}{\kappa_N}\left[p_N{}'(x)p_{N-1}(x) - p_N(x)p_{N-1}{}'(x)\right]$$

$$\frac{\partial}{\partial t_l}\log Z_N = -N^2\int_{-\infty}^{\infty}\lambda^l \rho_N^{(1)}(\lambda)d\lambda$$

One requires rigorous global asymptotics for , to evauate The asymptotics of $\frac{\partial}{\partial t_l}\log Z_N$

Riemann-Hilbert problem for O.P.'s
(Its, Kitaev, Fokas)

- Y analytic in $\mathbf{C} \setminus \mathbf{R}$
- $$Y_+(z) = Y_-(z)\begin{pmatrix} 1 & w_N(z) \\ 0 & 1 \end{pmatrix} \qquad z \in \mathbf{R},\ w_N(z) = e^{-NV(z)}$$

  $$Y(z) = \left(I + O\left(z^{-1}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$
- By Plemelj's formula, one finds

$$Y(z) = \begin{pmatrix} \dfrac{1}{\kappa_N^{(N)}} p_N(z) & \dfrac{1}{\kappa_N^{(N)}} \displaystyle\int_{\mathbf{R}} \frac{p_N(s)\, w_N(s)}{s-z} \frac{ds}{2\pi i} \\ -2\pi i \kappa_{N-1}^{(N)} p_{N-1}(z) & -\kappa_{N-1}^{(N)} \displaystyle\int_{\mathbf{R}} \frac{p_{N-1}(s)\, w_N(s)}{s-z} ds \end{pmatrix}$$

Mean density in terms of RHP

$$N\rho_N^{(1)}(x) = e^{-NV(x)} \frac{\kappa_{N-1}}{\kappa_N} \left[p_N{}'(x) p_{N-1}(x) - p_N(x) p_{N-1}{}'(x) \right]$$

$$= \frac{e^{-NV(x)}}{-2\pi i} \left[Y_{11}' Y_{21} - Y_{11} Y_{21}' \right]$$

Asymptotics for Riemann-Hilbert Problems

- Using work of Deift, Kriecherbauer, K.M., Venakides, and Zhou. Based on Deift-Zhou technique for asymptotic analysis of RHPs. The first application to OPs and random matrices: quartic potential considered by Bleher and Its.

$$Y \xrightarrow{1} M \xrightarrow{2} M_1 \xrightarrow{3} S:$$

$$Y = \begin{pmatrix} e^{\frac{N\ell}{2}} & 0 \\ 0 & e^{-\frac{N\ell}{2}} \end{pmatrix} S\, M_1^{(A)} \begin{pmatrix} 1 & 0 \\ e^{NG} & 1 \end{pmatrix} \begin{pmatrix} e^{N(g-\ell/2)} & 0 \\ 0 & e^{-N(g-\ell/2)} \end{pmatrix}$$

$S = S_N(z)$ possesses an explicit asymptotic expansion.

The other terms are all exact, and (relatively) explict in terms of the original parameters $\{t_j\}_1^\nu$

Equilibrium Measure and first transformation

- **Connection to approximation theory:** $Z_N(t)$ satisfies the following leading order asymptotic behavior

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_N(t) = \sup_{\mu\in\mathbf{A}} \left\{ - \int \left(\frac{1}{2}\lambda^2 + \sum_{k=1}^{\nu} t_k \lambda^k \right) d\mu(\lambda) + \iint \log|\lambda-\eta|\, d\mu(\eta)\, d\mu(\lambda) \right\}$$

where **A** is the set of all positive Borel measures on **R** with unit mass.

- The supremum is achieved at a unique measure, μ^*, called the *equilibrium measure,* DKM ('96),
  - μ^* is supported on finitely many disjoint intervals
  - On the interior of each interval, μ^* has analytic density $\psi(\lambda)$

Equilibrium Measure and first transformation

Theorem : $\exists T > 0 \ni$ if $|t| < T$, then $(t_\nu > 0)$

$d\mu^* = \psi d\lambda$ where

$$\psi(\lambda) = \frac{1}{2\pi i}\chi_{(\alpha,\beta)}(\lambda)\sqrt{(\lambda-\alpha)(\lambda-\beta)}\,h(\lambda) = \frac{1}{2\pi i}R_+(\lambda)h(\lambda)$$

with h a polynomial of deg V - 2, strictly positive on $[\alpha, \beta]$ defined by

$$h(z) = \frac{1}{2\pi i}\oint \frac{V'(s)}{\sqrt{(s-\alpha)(s-\beta)}}\frac{ds}{s-z}$$

Dynamics of the Equilibrium Measure

$$\int_{\alpha}^{\beta} \frac{V'(s)}{\sqrt{(s-\alpha)(\beta-s)}} = 0$$

$$\int_{\alpha}^{\beta} \frac{sV'(s)}{\sqrt{(s-\alpha)(\beta-s)}} = 2\pi$$

$\alpha(t)$ and $\beta(t)$ are analytic functions of t as are coefficients of of the polynomial $h(\lambda)$.

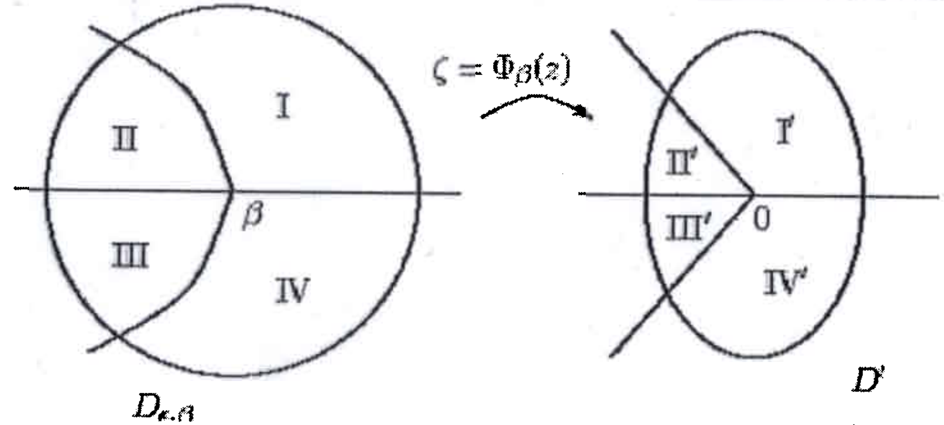
Expansion for mean density in the bulk

For $\lambda \in (\alpha, \beta)$

$$\rho_N(\lambda) = \psi(\lambda) + \frac{1}{4\pi N}\cos\left(N\int_{\lambda}^{\beta}\psi(s)ds\right)\left[\frac{1}{\lambda-\beta}-\frac{1}{\lambda-\alpha}\right]$$

$$+\frac{1}{N^2}\left[H(\lambda)+G(\lambda)\sin\left(N\int_{\lambda}^{\beta}\psi(s)ds\right)\right]+\cdots$$

Endpoint behavior: inner approx.



$$\Phi_\beta(z) = \left(\frac{3N}{4}\right)^{2/3} \left(\int_\beta^z \boldsymbol{Rh}\,\boldsymbol{ds} \right)^{2/3}$$

$$\Psi^\sigma : \mathbb{C} \setminus \gamma_\sigma \to \mathbb{C}^{2\times 2},$$

$$\Psi^\sigma(\zeta) = \begin{cases} \begin{pmatrix} Ai(\zeta) & Ai(\omega^2\zeta) \\ Ai'(\zeta) & \omega^2 Ai'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & \text{, for } \zeta \in I, \\ \begin{pmatrix} Ai(\zeta) & Ai(\omega^2\zeta) \\ Ai'(\zeta) & \omega^2 Ai'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{, for } \zeta \in II, \\ \begin{pmatrix} Ai(\zeta) & -\omega^2 Ai(\omega\zeta) \\ Ai'(\zeta) & -Ai'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{, for } \zeta \in III, \\ \begin{pmatrix} Ai(\zeta) & -\omega^2 Ai(\omega\zeta) \\ Ai'(\zeta) & -Ai'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & \text{, for } \zeta \in IV. \end{cases}$$

σ_3 is the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Uniformly Valid Representation of $\rho^{(1)}$

For $z \in [0, \beta + \delta]$ the following exact formula holds true:

$$
\begin{aligned}
N\rho_N^{(1)}(z) = &\left(\frac{\Phi_\beta'(z)}{4\Phi_\beta(z)} - \frac{\gamma'(z)}{\gamma(z)}\right)\left[2\,\Psi_{11}^\sigma(\Phi_\beta(z))\;\Psi_{21}^\sigma(\Phi_\beta(z))\right] \\
&+\Phi_\beta'(z)\left[\left(\Psi_{21}^\sigma(\Phi_\beta(z))\right)^2 - \Phi_\beta(z)\left(\Psi_{11}^\sigma(\Phi_\beta(z))\right)^2\right] \\
&+\frac{i}{2}\left[\left(S'\,B\,\Psi^\sigma(\Phi_\beta(z))\right)_{11}\left(S\,B\,\Psi^\sigma(\Phi_\beta(z))\right)_{21}\right. \\
&\qquad\left.-\left(S\,B\,\Psi^\sigma(\Phi_\beta(z))\right)_{11}\left(S'\,B\,\Psi^\sigma(\Phi_\beta(z))\right)_{21}\right].
\end{aligned}
$$

where

$$
\gamma(z) = \frac{(z-\beta)^{1/4}}{(z-\alpha)^{1/4}}, \quad \boldsymbol{B}(z) = \begin{pmatrix} \gamma^{-1}(z) & -\gamma(z) \\ -i\gamma^{-1}(z) & -i\gamma(z) \end{pmatrix} \Phi_\beta(z)^{\sigma_3/4}
$$

and $\boldsymbol{S}$ solves the RH problem for the error.

If |**t**| is sufficiently small, then there is a fixed size neighborhood of the interval [0,β] on which the following asymptotic expansion holds true.

$$
\begin{aligned}
\frac{1}{N}\frac{i}{2}\Big[&\left(S' \, B \, \Psi^{\sigma}\left(\Phi_{\beta}(z)\right)\right)_{11}\left(S \, B \, \Psi^{\sigma}\left(\Phi_{\beta}(z)\right)\right)_{21} \\
&- \left(S \, B \, \Psi^{\sigma}\left(\Phi_{\beta}(z)\right)\right)_{11}\left(S' \, B \, \Psi^{\sigma}\left(\Phi_{\beta}(z)\right)\right)_{21}\Big] \\
&= \sum_{j \text{ even}, \, j \geq 2} N^{-j}\bar{a}_j(z)\Psi_{11}^{2}(\Phi_{\beta})\frac{\sqrt{\Phi_{\beta}}}{\gamma(z)^2} \\
&+ \sum_{j \text{ even}, \, j \geq 2} N^{-j}\bar{b}_j(z)\frac{\Psi_{21}^{2}(\Phi_{\beta}(z))\gamma(z)^2}{\sqrt{\Phi_{\beta}(z)}} \\
&+ \sum_{j \text{ odd}, \, j \geq 3}^{\infty} N^{-j}\bar{c}_j(z)\Psi_{11}(\phi_{\beta}(z))\Psi_{21}(\phi_{\beta}(z)).
\end{aligned}
$$

Now we must evaluate $\int_{-\infty}^{\infty} f(\lambda)\, \rho_N^{(1)}(\lambda)\, d\lambda$.

First, the contribution from $(\beta+\delta,\infty)$ and $(-\infty,\alpha-\delta)$ is exponentially small, beyond all orders in N.

We are stuck with a series of integrals of the form

$$N^{-j} \int_0^{\beta+\delta} f(\lambda)\, \mathrm{Ai}(\Phi_\beta(\lambda))\, \mathrm{Ai}'(\Phi_\beta(\lambda))\, d\lambda, \quad j \text{ odd}$$

$$N^{-j} \int_0^{\beta+\delta} f(\lambda) \frac{(\mathrm{Ai}'(\Phi_\beta(\lambda)))^2}{\sqrt{\Phi_\beta(\lambda)}}\, d\lambda, \quad j \text{ even}$$

$$N^{-j} \int_0^{\beta+\delta} f(\lambda)\, (\mathrm{Ai}(\Phi_\beta(\lambda)))^2 \sqrt{\Phi_\beta(\lambda)}\, d\lambda, \quad j \text{ even}$$

And each of these possesses an asymptotic expansion In even powers of N.