Robust Portfolio Selection

Garud Iyengar and Don Goldfarb

IEOR Department, Columbia University

Talk outline

- Portfolio Selection Problem
- Robust Portfolio Selection
- Linear regression and *uncertainty sets*
- *Second-order cone programming*
- Theoretical implications of the robust approach
- Preliminary computational results
- Conclustion

Market and portfolios

- Market

Market and portfolios

- Market
  - Discrete time market with n assets
  - Described by a sequence of return vectors: $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3, \ldots \in \mathbf{R}^n$

$$r_i^t = \frac{p_i^t - p_i^{(t-1)}}{p_i^{(t-1)}}, \quad i = 1, \ldots, n$$

where $\mathbf{p}^t$ is the price vector in period t

Market and portfolios

- Market
  - Discrete time market with n assets
  - Described by a sequence of return vectors: $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3, \ldots \in \mathbf{R}^n$
- Portfolio $\boldsymbol{\phi} \in \mathbf{R}^n$: fraction of wealth in assets, i.e. $\mathbf{1}^T \boldsymbol{\phi} = 1$
  - portfolio return r_ϕ^t in period t: $r_\phi^t = \sum_{i=1}^n \phi_i r_i^t = (\mathbf{r}^t)^T \boldsymbol{\phi}$

Market and portfolios

- Market
  - Discrete time market with n assets
  - Described by a sequence of return vectors: $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3, \ldots \in \mathbf{R}^n$
- Portfolio $\boldsymbol{\phi} \in \mathbf{R}^n$: fraction of wealth in assets, i.e. $\mathbf{1}^T \boldsymbol{\phi} = 1$
  - portfolio return r_ϕ^t in period t: $r_\phi^t = \sum_{i=1}^n \phi_i r_i^t = (\mathbf{r}^t)^T \boldsymbol{\phi}$
- Portfolio selection problem:
  - Choose a model M from a model class $\mathcal{M}$
  - Given a model M, choose a risk-return optimal $\boldsymbol{\phi}^*$

Markowitz portfolio selection

- Formulated by Markowitz ... extended by Sharpe and others.
- Model class $\mathcal{M}$: Return sequence $\{\mathbf{r}^t : t \geq 1\}$ IID Normal $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Markowitz portfolio selection

- Formulated by Markowitz ... extended by Sharpe and others.
- Model class $\mathcal{M}$: Return sequence $\{\mathbf{r}^t : t \geq 1\}$ IID Normal $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - model selection: Maximum likelihood estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

Markowitz portfolio selection

- Formulated by Markowitz ... extended by Sharpe and others.
- Model class $\mathcal{M}$: Return sequence $\{\mathbf{r}^t : t \geq 1\}$ IID Normal $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Risk-return optimality criterion
  - return: $\mathbf{E}[r_\phi] = \boldsymbol{\mu}^T \boldsymbol{\phi}$
  - risk: $\mathbf{Var}[r_\phi] = \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}$
  - Objective: Pareto optimal $\boldsymbol{\phi}$

Markowitz portfolio selection

- Formulated by Markowitz ... extended by Sharpe and others.
- Model class $\mathcal{M}$: Return sequence $\{\mathbf{r}^t : t \geq 1\}$ IID Normal $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Risk-return optimality criterion
  - return: $\mathbf{E}[r_\phi] = \boldsymbol{\mu}^T \boldsymbol{\phi}$
  - risk: $\mathbf{Var}[r_\phi] = \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}$
  - Objective: Pareto optimal $\boldsymbol{\phi}$
- Versions:
  - Minimum variance portfolio selection:

$$\begin{array}{rl} \text{minimize} & \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi} \\ \text{subject to} & \boldsymbol{\mu}^T \boldsymbol{\phi} \geq \alpha, \\ & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Markowitz portfolio selection

- Formulated by Markowitz ... extended by Sharpe and others.
- Model class $\mathcal{M}$: Return sequence $\{\mathbf{r}^t : t \geq 1\}$ IID Normal $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Risk-return optimality criterion
  - return: $\mathbf{E}[r_\phi] = \boldsymbol{\mu}^T \boldsymbol{\phi}$
  - risk: $\mathbf{Var}[r_\phi] = \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}$
  - Objective: Pareto optimal $\boldsymbol{\phi}$
- Versions:
  - Maximum Sharpe ratio portfolio selection:

$$\begin{array}{ll} \text{maximize} & \frac{\boldsymbol{\mu}^T \boldsymbol{\phi} - r_f}{\sqrt{\boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}}} \\ \text{subject to} & \mathbf{1}^T \boldsymbol{\phi} = 1, \end{array}$$

where r_f is the risk-free rate of return

Markowitz portfolio selection

- Formulated by Markowitz ... extended by Sharpe and others.
- Model class $\mathcal{M}$: Return sequence $\{\mathbf{r}^t : t \geq 1\}$ IID Normal $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Risk-return optimality criterion
  - return: $\mathbf{E}[r_\phi] = \boldsymbol{\mu}^T \boldsymbol{\phi}$
  - risk: $\mathbf{Var}[r_\phi] = \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}$
  - Objective: Pareto optimal $\boldsymbol{\phi}$
- Versions:
  - Value-at-risk (VaR) portfolio selection:

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\mu}^T \boldsymbol{\phi} \\ \text{subject to} & \mathbf{P}(r_\phi \leq \alpha) \leq \beta, \\ & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Comments on Markowitz formulation

- Great theoretical success ... 1990 Nobel prize to Markowitz and Sharpe

Comments on Markowitz formulation

- Great theoretical success ... 1990 Nobel prize to Markowitz and Sharpe
- But ... "*in practice it is an error prone procedure that often results in error-maximized and investment-irrelevant portfolios*"

  R. O. Michaud, **Efficient Asset Management**, HBS Press, 1998

  Similar conclusions: Chopra & Ziemba (1993), Broadie (1993).

Comments on Markowitz formulation

- Great theoretical success ... 1990 Nobel prize to Markowitz and Sharpe
- But ... "*in practice it is an error prone procedure that often results in error-maximized and investment-irrelevant portfolios*"
  R. O. Michaud, **Efficient Asset Management**, HBS Press, 1998
  Similar conclusions: Chopra & Ziemba (1993), Broadie (1993).
- $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ are estimated from noisy data ... the optimal portfolios amplify errors

Comments on Markowitz formulation

- Great theoretical success ... 1990 Nobel prize to Markowitz and Sharpe
- But ... "*in practice it is an error prone procedure that often results in error-maximized and investment-irrelevant portfolios*"
  `R. O. Michaud,` **Efficient Asset Management**`, HBS Press, 1998`
  Similar conclusions: Chopra & Ziemba (1993), Broadie (1993).
- $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ are estimated from noisy data ... the optimal portfolios amplify errors
- Solutions:
  - bounds on the portfolio components: Chopra (1993), Frost & Savarino (1988)
  - James-Stein estimates for the mean: Chopra et al (1993)
  - Bayesian estimation: Chopra (1993), Frost et al (1986), Black-Litterman
  - Resampling $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: Michaud (1989)
  - Stochastic programming: Ziemba & Mulvey (1998)

Comments on Markowitz formulation

- Great theoretical success ... 1990 Nobel prize to Markowitz and Sharpe
- But ... "*in practice it is an error prone procedure that often results in error-maximized and investment-irrelevant portfolios*"
  R. O. Michaud, **Efficient Asset Management**, HBS Press, 1998
  Similar conclusions: Chopra & Ziemba (1993), Broadie (1993).
- $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ are estimated from noisy data ... the optimal portfolios amplify errors
- Solutions:
  - bounds on the portfolio components: Chopra (1993), Frost & Savarino (1988)
  - James-Stein estimates for the mean: Chopra et al (1993)
  - Bayesian estimation: Chopra (1993), Frost et al (1986), Black-Litterman
  - Resampling $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: Michaud (1989)
  - Stochastic programming: Ziemba & Mulvey (1998)
- Problems:
  - No guarantees on portfolio *performance*
  - Sampling based methods become inefficient as number of assets grow

Uncertain factor models

- Market return $\mathbf{r} = \boldsymbol{\mu} + \mathbf{V}^T \mathbf{f} + \boldsymbol{\epsilon}$ where
  - mean asset return: $\boldsymbol{\mu} \in \mathbf{R}^n$
  - factor returns: $\mathbf{f} \in \mathbf{R}^m$
  - factor loading: $\mathbf{V} \in \mathbf{R}^{m \times n}$
  - residual returns: $\boldsymbol{\epsilon} \in \mathbf{R}^n$

Uncertain factor models

- Market return $\mathbf{r} = \boldsymbol{\mu} + \mathbf{V}^T \mathbf{f} + \boldsymbol{\epsilon}$ where
  - mean asset return: $\boldsymbol{\mu} \in S_m$
  - factor returns: $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \mathbf{F})$, $\mathbf{F}$ known and stable (can be relaxed)
  - residual returns: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$, $\mathbf{D} \in S_d$
  - factor loading: $\mathbf{V} \in \mathbf{R}^{m \times n}$, $\mathbf{V} \in S_v$
- The *uncertainty structure* for the market parameters:
  - $S_m = \{\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\nu} : |\nu_i| \leq \gamma_i, i = 1, \ldots, n\}$
  - $S_v = \{\mathbf{V} = \mathbf{V}_0 + \mathbf{W} : \|\mathbf{W}_i\|_g \leq \rho_i, i = 1, \ldots, n\}$, $\mathbf{W}_i$ = i-th column of $\mathbf{V}$
  - $S_d = \{\mathbf{D} = \mathbf{diag}(\mathbf{d}) : \underline{d}_i \leq d_i \leq \overline{d}_i, i = 1, \ldots, n\}$

  why ? how to parametrize ? Answer: statistical results from linear regression

Uncertain factor models

- Market return $\mathbf{r} = \boldsymbol{\mu} + \mathbf{V}^T \mathbf{f} + \boldsymbol{\epsilon}$ where
  - mean asset return: $\boldsymbol{\mu} \in S_m$
  - factor returns: $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \mathbf{F})$, $\mathbf{F}$ known and stable (can be relaxed)
  - residual returns: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$, $\mathbf{D} \in S_d$
  - factor loading: $\mathbf{V} \in \mathbf{R}^{m \times n}$, $\mathbf{V} \in S_v$
- The *uncertainty structure* for the market parameters:
  - $S_m = \{\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\nu} : |\nu_i| \leq \gamma_i, i = 1, \ldots, n\}$
  - $S_v = \{\mathbf{V} = \mathbf{V}_0 + \mathbf{W} : \|\mathbf{W}_i\|_g \leq \rho_i, i = 1, \ldots, n\}$, $\mathbf{W}_i$ = i-th column of $\mathbf{V}$
  - $S_d = \{\mathbf{D} = \mathbf{diag}(\mathbf{d}) : \underline{d}_i \leq d_i \leq \overline{d}_i, i = 1, \ldots, n\}$

  why ? how to parametrize ? Answer: statistical results from linear regression
- Robust recipe
  - Given return data $\{\mathbf{r}^t : t = 1, \ldots, p\}$, parametrize the *uncertainty structure*, i.e. choose $(\boldsymbol{\mu}_0, \mathbf{V}_0)$, $\mathbf{G}$, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$, $\underline{\mathbf{d}}$, $\overline{\mathbf{d}}$
  - Given a particular choice of (S_d, S_m, S_v), choose a "risk-return" optimal $\boldsymbol{\phi}^*$

Robust portfolio selection

- For fixed $(\boldsymbol{\mu} \in S_m, \mathbf{V} \in S_v, \mathbf{D} \in S_d)$ the market return

$$\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D})$$

  and portfolio return

$$r_\phi \sim \mathcal{N}(\boldsymbol{\mu}^T \boldsymbol{\phi}, \boldsymbol{\phi}^T (\mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D}) \boldsymbol{\phi})$$

Robust portfolio selection

- For fixed $(\boldsymbol{\mu} \in S_m, \mathbf{V} \in S_v, \mathbf{D} \in S_d)$ the market return

$$\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D})$$

  and portfolio return

$$r_\phi \sim \mathcal{N}(\boldsymbol{\mu}^T \boldsymbol{\phi}, \boldsymbol{\phi}^T (\mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D}) \boldsymbol{\phi})$$

- Robust minimum variance portfolio selection: minimax formulation

$$\begin{array}{rl} \text{min} & \max_{\{\mathbf{V} \in S_v, \mathbf{D} \in S_d\}} \left\{ \boldsymbol{\phi}^T (\mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D}) \boldsymbol{\phi} \right\} \\ \text{subject to} & \min_{\{\boldsymbol{\mu} \in S_m\}} \left\{ \boldsymbol{\mu}^T \boldsymbol{\phi} \right\} \geq \alpha, \\ & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Robust portfolio selection

- For fixed $(\boldsymbol{\mu} \in S_m, \mathbf{V} \in S_v, \mathbf{D} \in S_d)$ the market return

$$\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V}^T\mathbf{F}\mathbf{V} + \mathbf{D})$$

  and portfolio return

$$r_\phi \sim \mathcal{N}(\boldsymbol{\mu}^T\boldsymbol{\phi}, \boldsymbol{\phi}^T(\mathbf{V}^T\mathbf{F}\mathbf{V} + \mathbf{D})\boldsymbol{\phi})$$

- Robust maximum Sharpe ratio portfolio selection

$$\begin{array}{rl} \text{max} & \text{min}_{\{\boldsymbol{\mu}\in S_m, \mathbf{V}\in S_v, \mathbf{D}\in S_d\}} \left\{ \frac{\boldsymbol{\mu}^T\boldsymbol{\phi} - r_f}{\sqrt{\boldsymbol{\phi}^T(\mathbf{V}^T\mathbf{F}\mathbf{V}+\mathbf{D})\boldsymbol{\phi}}} \right\} \\ \text{subject to} & \mathbf{1}^T\boldsymbol{\phi} = 1. \end{array}$$

Robust portfolio selection

- For fixed $(\boldsymbol{\mu} \in S_m, \mathbf{V} \in S_v, \mathbf{D} \in S_d)$ the market return

$$\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D})$$

  and portfolio return

$$r_\phi \sim \mathcal{N}(\boldsymbol{\mu}^T \boldsymbol{\phi}, \boldsymbol{\phi}^T (\mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D}) \boldsymbol{\phi})$$

- Robust Value-at-risk portfolio selection

$$\begin{array}{rl} \text{max} & \text{min}_{\{\boldsymbol{\mu} \in S_m\}} \left\{ \boldsymbol{\mu}^T \boldsymbol{\phi} \right\} \\ \text{subject to} & \text{max}_{\{\boldsymbol{\mu} \in S_m, \mathbf{V} \in S_v, \mathbf{D} \in S_d\}} \left\{ \mathbf{P}\{ r_\phi \leq \alpha \} \right\} \leq \beta, \\ & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Linear regression and uncertainty sets

- Data: Collect data over p periods
  - asset returns: $\{\mathbf{r}^t : t = 1, \ldots, p\}$,
  - factor returns: $\{\mathbf{f}^t : t = 1, \ldots, p\}$

Linear regression and uncertainty sets

- Data: Collect data over p periods
  - asset returns: $\{\mathbf{r}^t : t = 1, \ldots, p\}$,
  - factor returns: $\{\mathbf{f}^t : t = 1, \ldots, p\}$
- Collect terms corresponding to a particular asset i:

$$\mathbf{y}_i = \mathbf{A}\mathbf{x}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \ldots, m.$$

where

$$\mathbf{y}_i = \begin{bmatrix} r_i^1 \\ r_i^2 \\ \vdots \\ r_i^p \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & f_1^1 & f_2^1 & \ldots & f_n^1 \\ 1 & f_1^2 & f_2^2 & \ldots & f_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & f_1^p & f_2^p & \ldots & f_n^p \end{bmatrix} \quad \mathbf{x}_i = \begin{bmatrix} \mu_i \\ V_{1i} \\ V_{2i} \\ \vdots \\ V_{mi} \end{bmatrix}$$

Linear regression and uncertainty sets

- Data: Collect data over p periods
  - asset returns: $\{\mathbf{r}^t : t = 1, \ldots, p\}$,
  - factor returns: $\{\mathbf{f}^t : t = 1, \ldots, p\}$
- Collect terms corresponding to a particular asset i:

$$\mathbf{y}_i = \mathbf{A}\mathbf{x}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \ldots, m.$$

- Least squares estimate $\bar{\mathbf{x}}_i$ of true $\mathbf{x}_i$: $\overline{\mathbf{x}}_i = \begin{bmatrix} \overline{\mu}_i \\ \overline{\mathbf{V}}_i \end{bmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}_i$

Linear regression and uncertainty sets

- Data: Collect data over p periods
  - asset returns: $\{\mathbf{r}^t : t = 1, \ldots, p\}$,
  - factor returns: $\{\mathbf{f}^t : t = 1, \ldots, p\}$
- Collect terms corresponding to a particular asset i:

$$\mathbf{y}_i = \mathbf{A}\mathbf{x}_i + \boldsymbol{\epsilon}_i, \quad i = 1, \ldots, m.$$

- Least squares estimate $\bar{\mathbf{x}}_i$ of true $\mathbf{x}_i$: $\overline{\mathbf{x}}_i = \begin{bmatrix} \overline{\mu}_i \\ \overline{\mathbf{V}}_i \end{bmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}_i$
- Set “centers” $\boldsymbol{\mu}_0 = \bar{\boldsymbol{\mu}}$ and $\mathbf{V}_0 = \bar{\mathbf{V}}$

Norm $\mathbf{G}$*, and bounds* $\boldsymbol{\rho}, \boldsymbol{\gamma}$

- For $\mathbf{Q} \in \mathbf{R}^{J\times(m+1)}$,
  $\mathcal{Z} = (\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i))^T (J s_i^2 \mathbf{Q}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{Q}^T)^{-1} (\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i)) \sim \mathcal{F}_J$
  where
  - $\mathbf{x}_i$: *true* value of the parameters
  - $s_i^2 = \frac{\|\mathbf{y}_i - \mathbf{A}\bar{\mathbf{x}}_i\|^2}{p-m-1}$: sample error variance
  - $\mathcal{F}_J$: F-distribution with J dof in num and $(p-m-1)$ dof in denom

Norm $\mathbf{G}$*, and bounds* $\boldsymbol{\rho}, \gamma$

- For $\mathbf{Q} \in \mathbf{R}^{J \times (m+1)}$,
  $\mathcal{Z} = (\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i))^T (J s_i^2 \mathbf{Q}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{Q}^T)^{-1} (\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i)) \sim \mathcal{F}_J$
- Pick a confidence level $\omega \in (0,1)$. Let $c_J(\omega) = F_{\mathcal{F}_J}^{-1}(\omega)$ be the ω-critical value.

Norm $\mathbf{G}$*, and bounds* $\boldsymbol{\rho}, \boldsymbol{\gamma}$

- For $\mathbf{Q} \in \mathbf{R}^{J\times(m+1)}$,
  $\mathcal{Z} = (\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i))^T (J s_i^2 \mathbf{Q}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{Q}^T)^{-1}(\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i)) \sim \mathcal{F}_J$
- Pick a confidence level $\omega \in (0,1)$. Let $c_J(\omega) = F^{-1}_{\mathcal{F}_J}(\omega)$ be the ω-critical value.
- Choose $\mathbf{Q} = \mathbf{e}_1^T$
  - Then $\mathbf{Q}\bar{\mathbf{x}}_i = \bar{\mu}_i$ and $\mathbf{Q}\mathbf{x}_i = \mu_i$ and $\mathcal{Z}$ (above) implies

$$\mathbf{P}\left(|\mu_i - \bar{\mu}_i| \leq \sqrt{s_i^2 (\mathbf{A}^T\mathbf{A})^{-1}_{11} c_1(\omega)}\right) = \omega$$

  - Define $\gamma_i = \sqrt{s_i^2 (\mathbf{A}^T\mathbf{A})^{-1}_{11} c_1(\omega)}$.
  - With probability $p = \omega^n$ the mean vector $\boldsymbol{\mu}$ lies in the set

$$S_m = \left\{\boldsymbol{\mu} : \boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\nu}, |\nu_i| \leq \gamma_i\right\}$$

Norm $\mathbf{G}$*, and bounds* $\boldsymbol{\rho}, \gamma$

- For $\mathbf{Q} \in \mathbf{R}^{J\times(m+1)}$,
  $\mathcal{Z} = (\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i))^T (J s_i^2 \mathbf{Q}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{Q}^T)^{-1}(\mathbf{Q}(\mathbf{x}_i - \bar{\mathbf{x}}_i)) \sim \mathcal{F}_J$
- Pick a confidence level $\omega \in (0,1)$. Let $c_J(\omega) = F^{-1}_{\mathcal{F}_J}(\omega)$ be the ω-critical value.
- Choose $Q = \left[\mathbf{e}_2 \; \mathbf{e}_3 \; \ldots \mathbf{e}_{m+1}\right]^T \in \mathbf{R}^{m\times(m+1)}$
  - Then $\mathbf{Q}\bar{\mathbf{x}}_i = \bar{\mathbf{V}}_i$, $\mathbf{Q}\mathbf{x}_i = \mathbf{V}_i$ and $\mathcal{Z}$ (above) implies

$$\mathbf{P}\left((\overline{\mathbf{V}}_i - \mathbf{V}_i)^T (\mathbf{Q}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{Q})^{-1} (\overline{\mathbf{V}}_i - \mathbf{V}_i) \le m c_m(\omega) s_i^2 \right) = \omega$$

  - Set $\mathbf{G} = (\mathbf{Q}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{Q})^{-1}$, and $\rho_i = \sqrt{m c_m(\omega) s_i^2}$.
  - With probability $p = \omega^n$, $\mathbf{V}$ lies in the set

$$S_v = \left\{ \mathbf{V}_0 + \mathbf{W} : \left\|\mathbf{W}_i\right\|_g \le \rho_i \right\},$$

where $\mathbf{W}_i$ is the i-th column of $\mathbf{W}$ and $\left\|\mathbf{w}\right\|_g = \sqrt{\mathbf{w}^T\mathbf{G}\mathbf{w}}$

Norm $\mathbf{G}$*, and bounds* $\boldsymbol{\rho}, \boldsymbol{\gamma}$

- **Conclusion**: Sets S_m and S_v defined by data and desired confidence level ω.

Norm $\mathbf{G}$*, and bounds* $\boldsymbol{\rho}, \boldsymbol{\gamma}$

- **Conclusion**: Sets S_m and S_v defined by data and desired confidence level ω.
- What about S_d or equivalently $\underline{\mathbf{d}}$ and $\overline{\mathbf{d}}$?
  - defined by confidence regions around s_i^2
  - have to do some bootstrapping

Robust minimum variance problem

- Optimization problem

$$
\begin{array}{lrcl}
\text{min} & \nu + \delta & & \\
\text{s.t.} & \max_{\mathbf{V} \in S_v} \{\boldsymbol{\phi}^T \mathbf{V}^T \mathbf{F} \mathbf{V} \boldsymbol{\phi}\} & \leq & \nu, \\
& \boldsymbol{\phi}^T \overline{\mathbf{D}} \boldsymbol{\phi} & \leq & \delta, \\
& \min_{\boldsymbol{\mu} \in S_m} \{\boldsymbol{\mu}^T \boldsymbol{\phi}\} & \geq & \alpha, \\
& \mathbf{1}^T \boldsymbol{\phi} & = & 1.
\end{array}
$$

Robust minimum variance problem

- Optimization problem

$$\begin{array}{lrcl} \min & \nu + \delta & & \\ \text{s.t.} & \max_{\mathbf{V}\in S_v} \left\{\boldsymbol{\phi}^T \mathbf{V}^T \mathbf{F} \mathbf{V} \boldsymbol{\phi}\right\} & \leq & \nu, \\ & \boldsymbol{\phi}^T \overline{\mathbf{D}} \boldsymbol{\phi} & \leq & \delta, \\ & \min_{\boldsymbol{\mu}\in S_m} \left\{\boldsymbol{\mu}^T \boldsymbol{\phi}\right\} & \geq & \alpha, \\ & \mathbf{1}^T \boldsymbol{\phi} & = & 1. \end{array}$$

- Worst return: $\min_{\boldsymbol{\mu}\in S_m} \left\{\boldsymbol{\phi}^T \boldsymbol{\mu}\right\} = \boldsymbol{\mu}_0^T \boldsymbol{\phi} - \boldsymbol{\gamma}^T |\boldsymbol{\phi}|$

Robust minimum variance problem

- Optimization problem

$$\begin{array}{lrcl} \text{min} & \nu + \delta & & \\ \text{s.t.} & \max_{\mathbf{V} \in S_v} \left\{ \boldsymbol{\phi}^T \mathbf{V}^T \mathbf{F} \mathbf{V} \boldsymbol{\phi} \right\} & \leq & \nu, \\ & \boldsymbol{\phi}^T \overline{\mathbf{D}} \boldsymbol{\phi} & \leq & \delta, \\ & \min_{\boldsymbol{\mu} \in S_m} \left\{ \boldsymbol{\mu}^T \boldsymbol{\phi} \right\} & \geq & \alpha, \\ & \mathbf{1}^T \boldsymbol{\phi} & = & 1. \end{array}$$

- Worst return: $\min_{\boldsymbol{\mu} \in S_m} \left\{ \boldsymbol{\phi}^T \boldsymbol{\mu} \right\} = \boldsymbol{\mu}_0^T \boldsymbol{\phi} - \boldsymbol{\gamma}^T |\boldsymbol{\phi}|$
- Worst variance: $\max_{\mathbf{V} \in S_v} \left\{ \boldsymbol{\phi}^T \mathbf{V}^T \mathbf{F} \mathbf{V} \boldsymbol{\phi} \right\} = \max_{\{\mathbf{W} : \|\mathbf{W}_i\|_g \leq \rho_i\}} \|\mathbf{V}_0 \boldsymbol{\phi} + \mathbf{W} \boldsymbol{\phi}\|_f^2$
  - Optimal solution at boundary

$$\max_{\{\mathbf{W} : \|\mathbf{W}_i\|_g \leq \rho_i\}} \|\mathbf{V}_0 \boldsymbol{\phi} + \mathbf{W} \boldsymbol{\phi}\|_f^2 = \max_{\{\mathbf{w} : \|\mathbf{w}\|_g \leq 1\}} \|\mathbf{V}_0 \boldsymbol{\phi} + r\mathbf{w}\|_f^2, \quad r = \boldsymbol{\rho}^T |\boldsymbol{\phi}| .$$

Worst case variance

- $\mathcal{S}$-procedure: $\|\mathbf{V}_0\boldsymbol{\phi} + r\mathbf{w}\|_f^2 \leq \nu$ for all $\|w\|_g \leq 1$ iff $\exists\, \tau \geq 0$ with

$$\mathbf{M} = \begin{bmatrix} \nu - \tau - \boldsymbol{\phi}^T\mathbf{V}_0^T\mathbf{F}\mathbf{V}_0\boldsymbol{\phi} & r\mathbf{F}\mathbf{V}_0\boldsymbol{\phi} \\ r\boldsymbol{\phi}^T\mathbf{F}\mathbf{V}_0 & \tau\mathbf{G} - r^2\mathbf{F} \end{bmatrix} \succeq \mathbf{0}$$

Worst case variance

- $\mathcal{S}$-procedure: $\|\mathbf{V}_0\boldsymbol{\phi} + r\mathbf{w}\|_f^2 \leq \nu$ for all $\|w\|_g \leq 1$ iff $\exists\, \tau \geq 0$ with

$$\mathbf{M} = \begin{bmatrix} \nu - \tau - \boldsymbol{\phi}^T \mathbf{V}_0^T \mathbf{F} \mathbf{V}_0 \boldsymbol{\phi} & r\mathbf{F}\mathbf{V}_0\boldsymbol{\phi} \\ r\boldsymbol{\phi}^T \mathbf{F}\mathbf{V}_0 & \tau\mathbf{G} - r^2\mathbf{F} \end{bmatrix} \succeq \mathbf{0}$$

- Let $\mathbf{H} = \mathbf{G}^{-\frac{1}{2}}\mathbf{F}\mathbf{G}^{-\frac{1}{2}} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$. Then $\mathbf{M} \succeq \mathbf{0}$ iff

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Q}^T\mathbf{G}^{\frac{1}{2}} \end{bmatrix} \mathbf{M} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{G}^{\frac{1}{2}}\mathbf{Q} \end{bmatrix} = \begin{bmatrix} \nu - \tau - \mathbf{w}^T\mathbf{w} & -r\mathbf{w}^T\boldsymbol{\Lambda}^{\frac{1}{2}} \\ -r\boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{w} & \tau\mathbf{I} - r^2\boldsymbol{\Lambda} \end{bmatrix} \succeq \mathbf{0}$$

Worst case variance

- $\mathcal{S}$-procedure: $\|\mathbf{V}_0\boldsymbol{\phi} + r\mathbf{w}\|_f^2 \le \nu$ for all $\|w\|_g \le 1$ iff $\exists\, \tau \ge 0$ with

$$\mathbf{M} = \begin{bmatrix} \nu - \tau - \boldsymbol{\phi}^T\mathbf{V}_0^T\mathbf{F}\mathbf{V}_0\boldsymbol{\phi} & r\mathbf{F}\mathbf{V}_0\boldsymbol{\phi} \\ r\boldsymbol{\phi}^T\mathbf{F}\mathbf{V}_0 & \tau\mathbf{G} - r^2\mathbf{F} \end{bmatrix} \succeq \mathbf{0}$$

- Let $\mathbf{H} = \mathbf{G}^{-\frac{1}{2}}\mathbf{F}\mathbf{G}^{-\frac{1}{2}} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$. Then $\mathbf{M} \succeq \mathbf{0}$ iff

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Q}^T\mathbf{G}^{\frac{1}{2}} \end{bmatrix} \mathbf{M} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{G}^{\frac{1}{2}}\mathbf{Q} \end{bmatrix} = \begin{bmatrix} \nu - \tau - \mathbf{w}^T\mathbf{w} & -r\mathbf{w}^T\boldsymbol{\Lambda}^{\frac{1}{2}} \\ -r\boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{w} & \tau\mathbf{I} - r^2\boldsymbol{\Lambda} \end{bmatrix} \succeq \mathbf{0}$$

- Equivalently, $\tau \ge r^2\lambda_{\max}(\mathbf{H})$, and Schur complement $\tau\mathbf{I} - r^2\boldsymbol{\Lambda}$

$$\beta - \tau - \mathbf{w}^T\mathbf{w} - r^2\Big(\sum_{i:\tau \ne r^2\lambda_i} \frac{\lambda_i w_i^2}{\tau - r^2\lambda_i}\Big) \ge 0.$$

Robust minimum variance problem

- Some more linear algebra implies min variance problem equivalent to

$$
\begin{array}{lrcl}
\text{min} & \multicolumn{3}{l}{\tau + \mathbf{1}^T \mathbf{t} + \delta} \\
\text{s.t.} & \mathbf{w} & = & \mathbf{Q}^T \mathbf{V}_0 \boldsymbol{\phi} \\
& r & = & \sum_{i=1}^n \rho_i \left| \phi_i \right| \\
& \mathbf{1}^T \boldsymbol{\phi} & = & 1 \\
& \mathbf{1}^T \mathbf{t} & \leq & \nu - \tau \\
& w_i^2 & \leq & t_i (1 - \sigma \lambda_i), \quad i = 1, \ldots, m \\
& r^2 & \leq & \sigma \tau \\
& \sigma & \leq & \frac{1}{\lambda_{\max}(\mathbf{H})} \\
& \boldsymbol{\phi}^T \mathbf{D} \boldsymbol{\phi} & \leq & \delta
\end{array}
$$

Robust minimum variance problem

- Some more linear algebra implies min variance problem equivalent to

$$
\begin{array}{lrcl}
\text{min} & \multicolumn{3}{l}{\tau + \mathbf{1}^T\mathbf{t} + \delta} \\
\text{s.t.} & \mathbf{w} & = & \mathbf{Q}^T\mathbf{V}_0\boldsymbol{\phi} \\
& r & = & \sum_{i=1}^n \rho_i \, |\phi_i| \\
& \mathbf{1}^T\boldsymbol{\phi} & = & 1 \\
& \mathbf{1}^T\mathbf{t} & \leq & \nu - \tau \\
& w_i^2 & \leq & t_i(1-\sigma\lambda_i), \quad i = 1,\ldots,m \\
& r^2 & \leq & \sigma\tau \\
& \sigma & \leq & \frac{1}{\lambda_{\max}(\mathbf{H})} \\
& \boldsymbol{\phi}^T\mathbf{D}\boldsymbol{\phi} & \leq & \delta
\end{array}
$$

- This optimization problem is a Second-Order Cone Program (SOCP)
  - linear objective
  - linear + conic section constraints

Robust minimum variance problem

- Some more linear algebra implies min variance problem equivalent to

$$
\begin{array}{lrcl}
\text{min} & \multicolumn{3}{l}{\tau + \mathbf{1}^T \mathbf{t} + \delta} \\
\text{s.t.} & \mathbf{w} & = & \mathbf{Q}^T \mathbf{V}_0 \boldsymbol{\phi} \\
& r & = & \sum_{i=1}^n \rho_i \left|\phi_i\right| \\
& \mathbf{1}^T \boldsymbol{\phi} & = & 1 \\
& \mathbf{1}^T \mathbf{t} & \leq & \nu - \tau \\
& w_i^2 & \leq & t_i(1 - \sigma\lambda_i), \quad i = 1, \ldots, m \\
& r^2 & \leq & \sigma\tau \\
& \sigma & \leq & \frac{1}{\lambda_{\max}(\mathbf{H})} \\
& \boldsymbol{\phi}^T \mathbf{D} \boldsymbol{\phi} & \leq & \delta
\end{array}
$$

- This optimization problem is a Second-Order Cone Program (SOCP)
- Robust maximum Sharpe ratio and robust VaR problems are also SOCPs

Robust minimum variance problem

- Some more linear algebra implies min variance problem equivalent to

$$
\begin{array}{lrcl}
\text{min} & \tau + \mathbf{1}^T\mathbf{t} + \delta & & \\
\text{s.t.} & \mathbf{w} & = & \mathbf{Q}^T\mathbf{V}_0\boldsymbol{\phi} \\
 & r & = & \sum_{i=1}^n \rho_i\,|\phi_i| \\
 & \mathbf{1}^T\boldsymbol{\phi} & = & 1 \\
 & \mathbf{1}^T\mathbf{t} & \leq & \nu - \tau \\
 & w_i^2 & \leq & t_i(1-\sigma\lambda_i), \quad i = 1,\ldots,m \\
 & r^2 & \leq & \sigma\tau \\
 & \sigma & \leq & \frac{1}{\lambda_{\max}(\mathbf{H})} \\
 & \boldsymbol{\phi}^T\mathbf{D}\boldsymbol{\phi} & \leq & \delta
\end{array}
$$

- This optimization problem is a Second-Order Cone Program (SOCP)
- Robust maximum Sharpe ratio and robust VaR problems are also SOCPs
- Worst case complexity of SOCPs comparable for quadratic programs

Quick catchup

- Classical strategies are sensitive to parameter perturbation
- Robust strategies attempt to correct this via *uncertainty sets*
- Uncertainty sets defined by the data and desired confidence level
- Resulting optimization problem is an SOCP ... can be solved efficiently

Quick catchup

- Classical strategies are sensitive to parameter perturbation
- Robust strategies attempt to correct this via *uncertainty sets*
- Uncertainty sets defined by the data and desired confidence level
- Resulting optimization problem is an SOCP ... can be solved efficiently

Robust investment strategy

- Collect data: asset returns $\mathbf{r}$ and factor returns $\mathbf{f}$.
- Compute the least squares estimates $\boldsymbol{\mu}_0$, $\mathbf{V}_0$ and $\mathbf{F}$
- Choose a confidence level ω and define S_m, S_v, and S_d
- Solve the SOCP corresponding to the robust problem of interest

Quick catchup

- Classical strategies are sensitive to parameter perturbation
- Robust strategies attempt to correct this via *uncertainty sets*
- Uncertainty sets defined by the data and desired confidence level
- Resulting optimization problem is an SOCP ... can be solved efficiently

Robust investment strategy

- Collect data: asset returns $\mathbf{r}$ and factor returns $\mathbf{f}$.
- Compute the least squares estimates $\boldsymbol{\mu}_0$, $\mathbf{V}_0$ and $\mathbf{F}$
- Choose a confidence level ω and define S_m, S_v, and S_d
- Solve the SOCP corresponding to the robust problem of interest

Our modifications:

- Replaced usual mean-variance portfolio selection by a robust version.
- Risk-aversion dictates ω: high $\omega \equiv$ conservative portfolios

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible (Wang (2002))

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible (Wang (2002))
- Probabilistic guarantee on performance
  - $\phi^*(\omega)$: *solution* of the robust max Sharpe ratio problem at confidence ω
  - $s^*(\omega)$: *value* of the robust max Sharpe ratio problem at confidence ω
  - Result: *realized* Sharpe ratio of $\phi^*(\omega) \geq s^*(\omega)$ with probability ω

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible (Wang (2002))
- Probabilistic guarantee on performance
- Uncertain $\mathbf{F}$: $\mathbf{F} \mid$ data $\sim \mathbf{W}^{-1}_{(p-1)}((p-1)\mathbf{F}_p)$, $\mathbf{F}_p$ = sample covariance matrix.

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible (Wang (2002))
- Probabilistic guarantee on performance
- Uncertain $\mathbf{F}$: $\mathbf{F} \mid$ data $\sim \mathbf{W}^{-1}_{(p-1)}((p-1)\mathbf{F}_p)$, $\mathbf{F}_p$ = sample covariance matrix. Implies the following uncertainty structures

$$\mathbf{F} \in S_{f-1} = \left\{\mathbf{F}^{-1} = \mathbf{F}_0^{-1} + \boldsymbol{\Delta} : \left\|\mathbf{N}^{\frac{1}{2}}\boldsymbol{\Delta}\mathbf{N}^{\frac{1}{2}}\right\| \leq \epsilon\right\}$$

$$\mathbf{F} \in S_f = \left\{\mathbf{F} = \mathbf{F}_0 + \boldsymbol{\Delta} : \left\|\mathbf{F}_0^{-\frac{1}{2}}\boldsymbol{\Delta}\mathbf{F}_0^{-\frac{1}{2}}\right\| \leq \epsilon\right\}$$

The min-max problems still remain SOCPs.

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible (Wang (2002))
- Probabilistic guarantee on performance
- Uncertain $\mathbf{F}$: $\mathbf{F} \mid$ data $\sim \mathbf{W}^{-1}_{(p-1)}((p-1)\mathbf{F}_p)$, $\mathbf{F}_p$ = sample covariance matrix. Implies the following uncertainty structures

$$\mathbf{F} \in S_{f-1} = \left\{ \mathbf{F}^{-1} = \mathbf{F}_0^{-1} + \mathbf{\Delta} : \left\| \mathbf{N}^{\frac{1}{2}} \mathbf{\Delta} \mathbf{N}^{\frac{1}{2}} \right\| \leq \epsilon \right\}$$

$$\mathbf{F} \in S_f = \left\{ \mathbf{F} = \mathbf{F}_0 + \mathbf{\Delta} : \left\| \mathbf{F}_0^{-\frac{1}{2}} \mathbf{\Delta} \mathbf{F}_0^{-\frac{1}{2}} \right\| \leq \epsilon \right\}$$

The min-max problems still remain SOCPs.

- Dynamics

Properties and extensions

- Analog of Markowitz portfolio selection for uncertain markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible (Wang (2002))
- Probabilistic guarantee on performance
- Uncertain $\mathbf{F}$: $\mathbf{F}$ | data $\sim \mathbf{W}^{-1}_{(p-1)}((p-1)\mathbf{F}_p)$, $\mathbf{F}_p$ = sample covariance matrix. Implies the following uncertainty structures

$$\mathbf{F} \in S_{f-1} = \left\{\mathbf{F}^{-1} = \mathbf{F}_0^{-1} + \mathbf{\Delta} : \left\|\mathbf{N}^{\frac{1}{2}}\mathbf{\Delta}\mathbf{N}^{\frac{1}{2}}\right\| \leq \epsilon\right\}$$

$$\mathbf{F} \in S_f = \left\{\mathbf{F} = \mathbf{F}_0 + \mathbf{\Delta} : \left\|\mathbf{F}_0^{-\frac{1}{2}}\mathbf{\Delta}\mathbf{F}_0^{-\frac{1}{2}}\right\| \leq \epsilon\right\}$$

  The min-max problems still remain SOCPs.
- Dynamics
  - The sets S_m, S_v and S_d can be efficiently updated ... Kalman filtering
  - Extends to a multi-period model ... *robust* dynamic programming

Evaluation of robust portfolios

- Focused on the Robust Maximum Sharpe Ratio problem

Evaluation of robust portfolios

- Focused on the Robust Maximum Sharpe Ratio problem
  - did not want user-defined variables that had to be tuned
  - can compare with results in the literature

Evaluation of robust portfolios

- Focused on the Robust Maximum Sharpe Ratio problem
- Two kinds of experiments:
  - Simulation experiments
  - Sample-path behavior on real market data

Evaluation of robust portfolios

- Focused on the Robust Maximum Sharpe Ratio problem
- Two kinds of experiments:
  - Simulation experiments
  - Sample-path behavior on real market data
- Simulation experiments: $(n = 500, m = 40)$
  - Randomly generated $\mathbf{F}$, $\mathbf{V}$ and $\mathbf{D} = \mathbf{diag}(d_i)$
  - Randomly generated returns $\mathbf{r}$ and $\mathbf{f}$ using the linear model
  - Solved for $(\boldsymbol{\mu}_0, \mathbf{V}_0)$, $\mathbf{G}$, and $(\boldsymbol{\rho}, \boldsymbol{\gamma})$.

Evaluation of robust portfolios

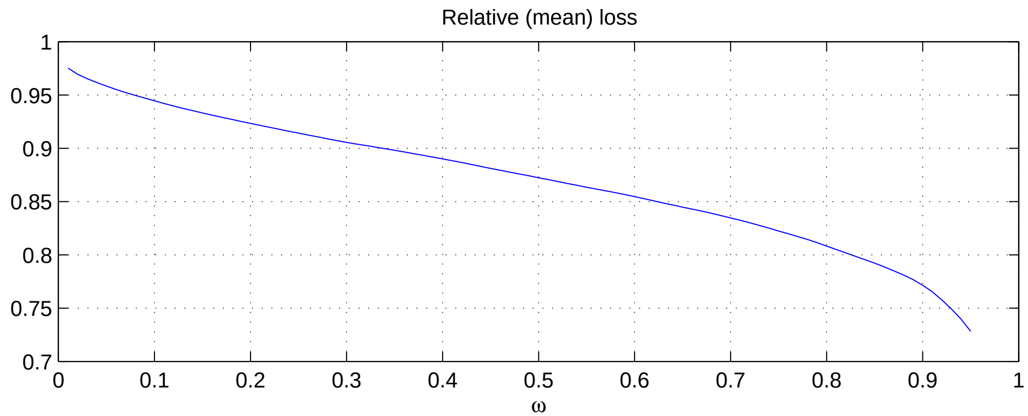
- Focused on the Robust Maximum Sharpe Ratio problem
- Two kinds of experiments:
  - Simulation experiments
  - Sample-path behavior on real market data
- Simulation experiments: $(n = 500, m = 40)$
  - Randomly generated $\mathbf{F}$, $\mathbf{V}$ and $\mathbf{D} = \mathbf{diag}(d_i)$
  - Randomly generated returns $\mathbf{r}$ and $\mathbf{f}$ using the linear model
  - Solved for $(\boldsymbol{\mu}_0, \mathbf{V}_0)$, $\mathbf{G}$, and $(\boldsymbol{\rho}, \boldsymbol{\gamma})$.
  - Results
    - Cost of robustness as a function of the confidence level ω
    - Gains from robustness as a function of the noise variance
    - Comparison of running time

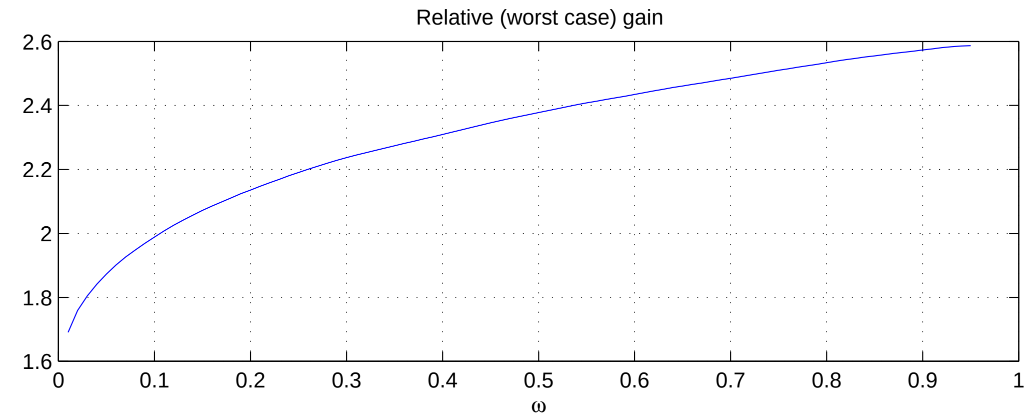
Evaluation of robust portfolios

- Focused on the Robust Maximum Sharpe Ratio problem
- Two kinds of experiments:
  - Simulation experiments
  - Sample-path behavior on real market data
- Simulation experiments: $(n = 500, m = 40)$
  - Randomly generated $\mathbf{F}$, $\mathbf{V}$ and $\mathbf{D} = \mathbf{diag}(d_i)$
  - Randomly generated returns $\mathbf{r}$ and $\mathbf{f}$ using the linear model
  - Solved for $(\boldsymbol{\mu}_0, \mathbf{V}_0)$, $\mathbf{G}$, and $(\boldsymbol{\rho}, \boldsymbol{\gamma})$.
  - Results
    - Cost of robustness as a function of the confidence level ω
    - Gains from robustness as a function of the noise variance
    - Comparison of running time
- Sample path experiments: cumulative returns

Performance as a function of ω

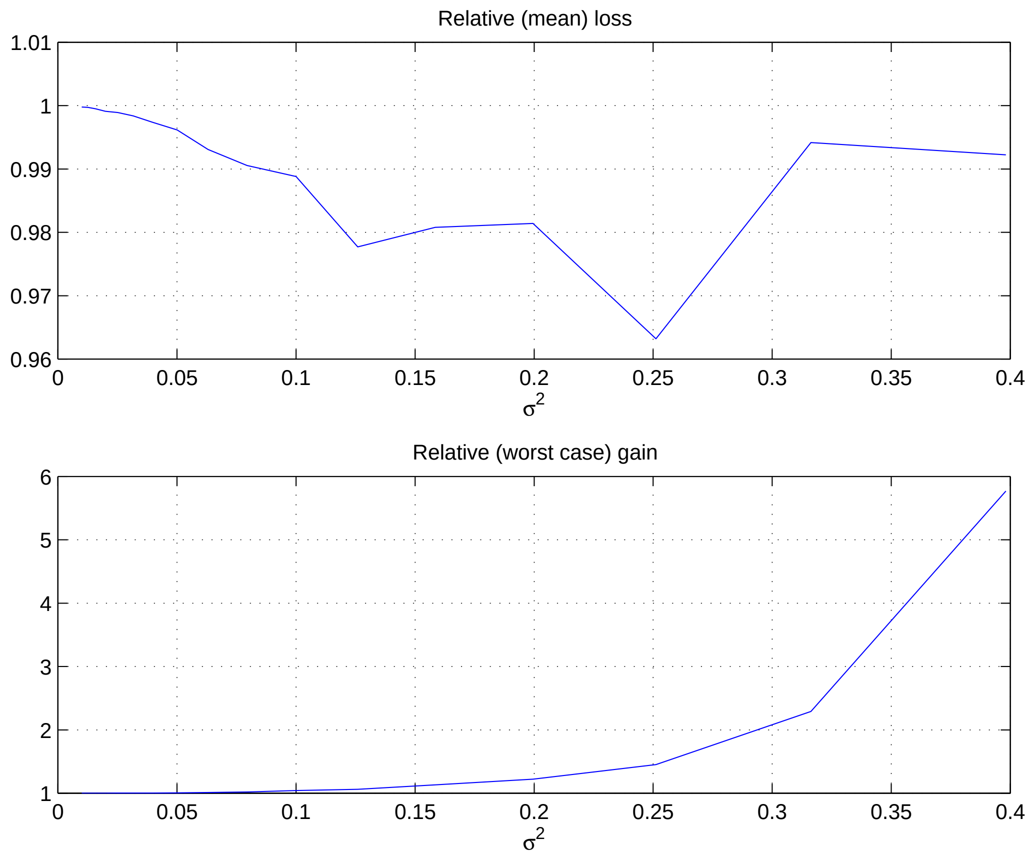
- $\mathbf{D} = 0.1\,\mathbf{diag}(\mathbf{V}_0^T \mathbf{F} \mathbf{V}_0)$, i.e. the factor model explains 90% of the variance.
- $\text{loss} = \dfrac{\text{Sharpe ratio of robust portfolio}}{\text{Sharpe ratio of classical portfolio}}$ & $\text{gain} = \dfrac{\text{Worst case Sharpe ratio of robust portfolio}}{\text{Worst case Sharpe ratio of classical portfolio}}$





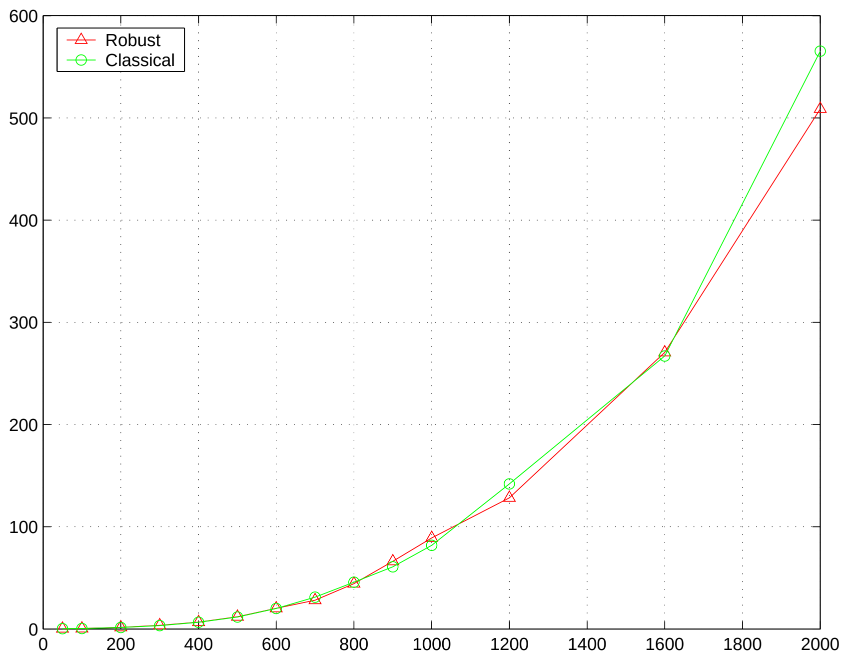
Performance as a function of noise variance

- $\omega = 0.95$ and $\overline{\mathbf{D}} = \sigma^2 \,\mathbf{diag}(\mathbf{V}_0^T \mathbf{F} \mathbf{V}_0)$



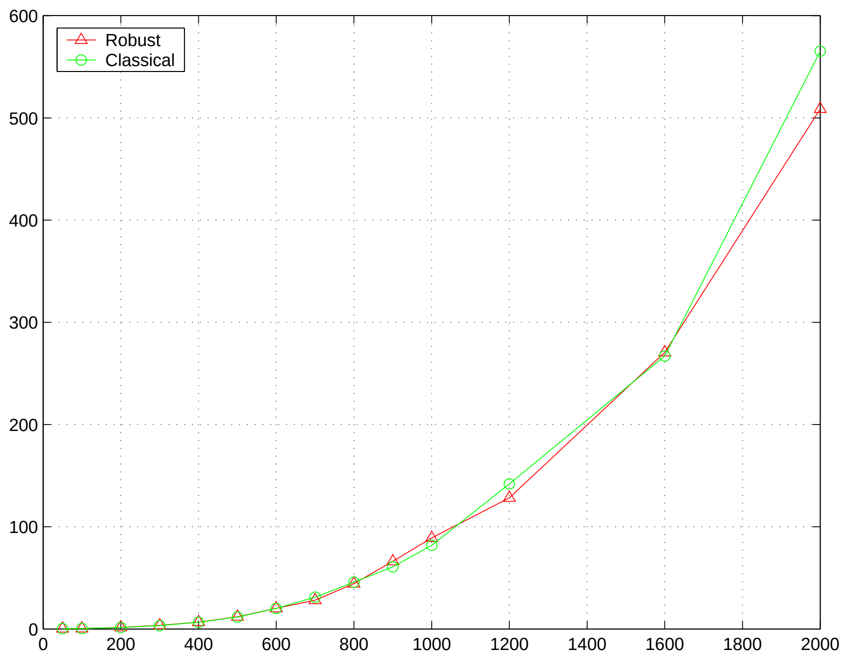
Comparison of running times

- $m = \lceil 0.1n \rceil$, $\omega = 0.95$ and $\overline{\mathbf{D}} = \sigma^2\,\mathbf{diag}(\mathbf{V}_0^T \mathbf{F} \mathbf{V}_0)$
- SeDuMi V1.03 within Matlab6.1 R12 on a Dell Precision workstation running RedHat 7.1 ... Running times averaged over 100 random instances



Comparison of running times

- $m = \lceil 0.1n \rceil$, $\omega = 0.95$ and $\overline{\mathbf{D}} = \sigma^2\,\mathbf{diag}(\mathbf{V}_0^T \mathbf{F} \mathbf{V}_0)$
- SeDuMi V1.03 within Matlab6.1 R12 on a Dell Precision workstation running RedHat 7.1 ... Running times averaged over 100 random instances



- Running times are almost identical ... approximately quadratic

Performance on real market data

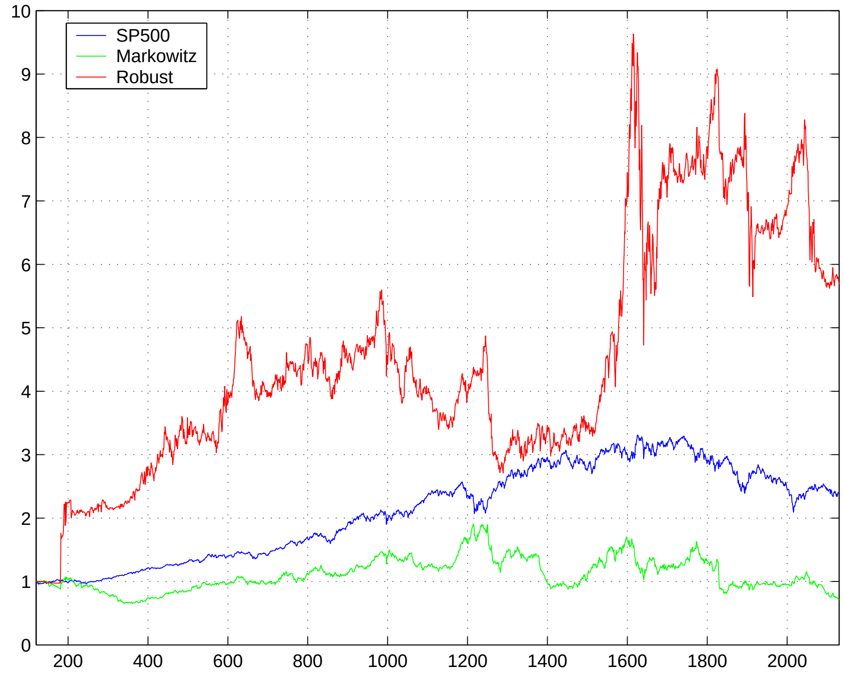
- Market period: December 31st, 1993 - March 26, 2002
- Assets: SP500 index
- Factors: DJA, SPX, NDX, RUT, TYX + top few eigenvectors of $\mathbf{\Sigma}_R$

Performance on real market data

- Market period: December 31st, 1993 - March 26, 2002
- Assets: SP500 index
- Factors: DJA, SPX, NDX, RUT, TYX + top few eigenvectors of $\boldsymbol{\Sigma}_R$
- Experimental procedure:
  - Data divided into investment periods of length p days
  - For each period, estimated the asset covariance $\boldsymbol{\Sigma}_R$ and kept “top” eigenvectors
  - Estimated $\mathbf{V}_0$, $\boldsymbol{\mu}_0$, $\mathbf{G}$, $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ over a history $h = 4p$
  - Set $\overline{d}_i = s_i^2$ and $r_f =$ average T-bill rate
  - Robust (resp. classical) portfolio $\boldsymbol{\phi}_r^t$ (resp. $\boldsymbol{\phi}_m^t$) selected by robust (resp. classical) Sharpe ratio problem
  - Portfolio $\boldsymbol{\phi}_r^t$ and $\boldsymbol{\phi}_m^t$ held constant for period $t)$ and then rebalanced

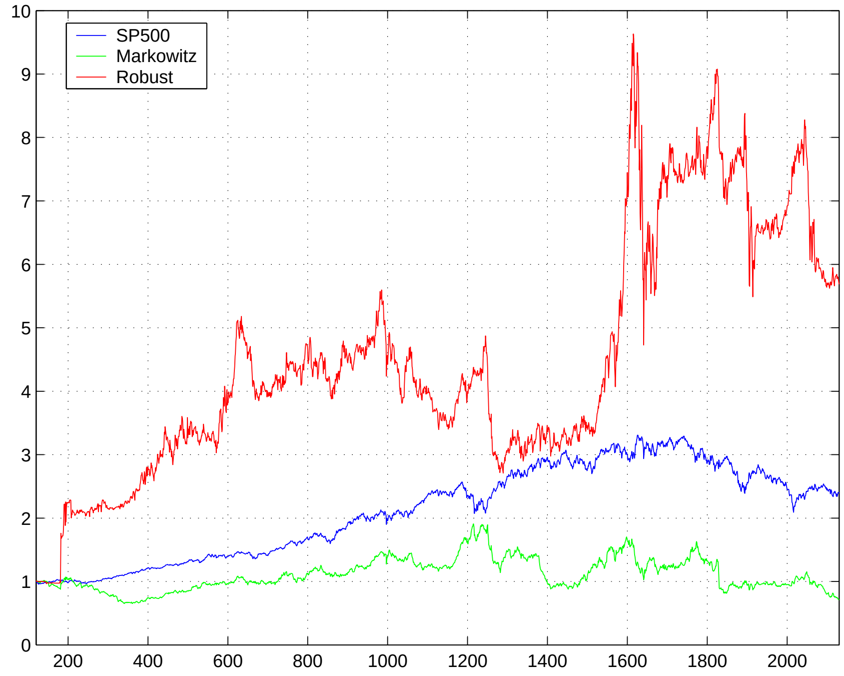
Comparison of overall performance

- Cumulative daily returns for Robust and Markowitz strategies



Comparison of overall performance

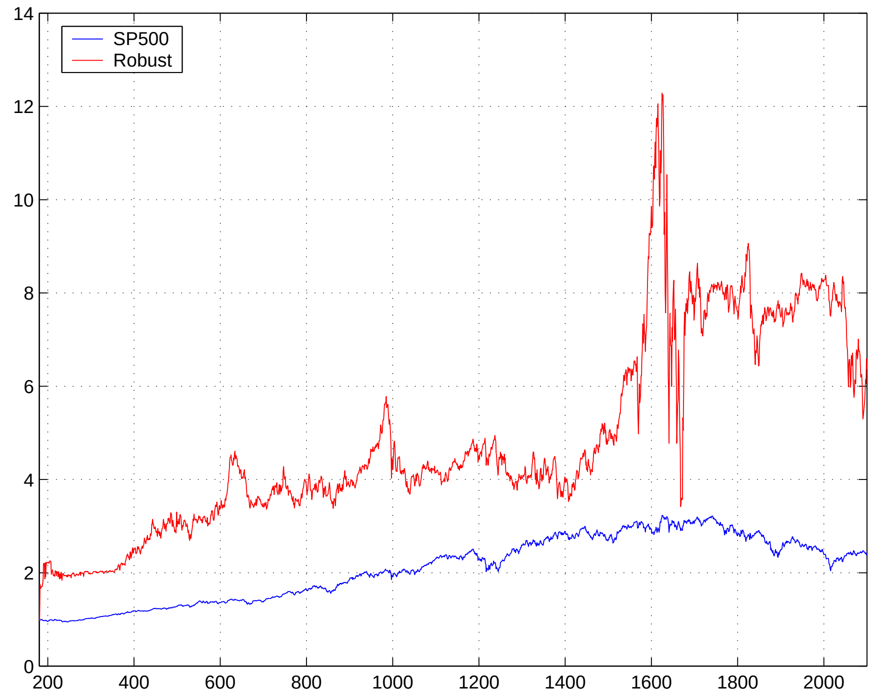
- Cumulative daily returns for Robust and Markowitz strategies



- Results averaged over 5 different start times
- Need a different p and h for bull/bear periods

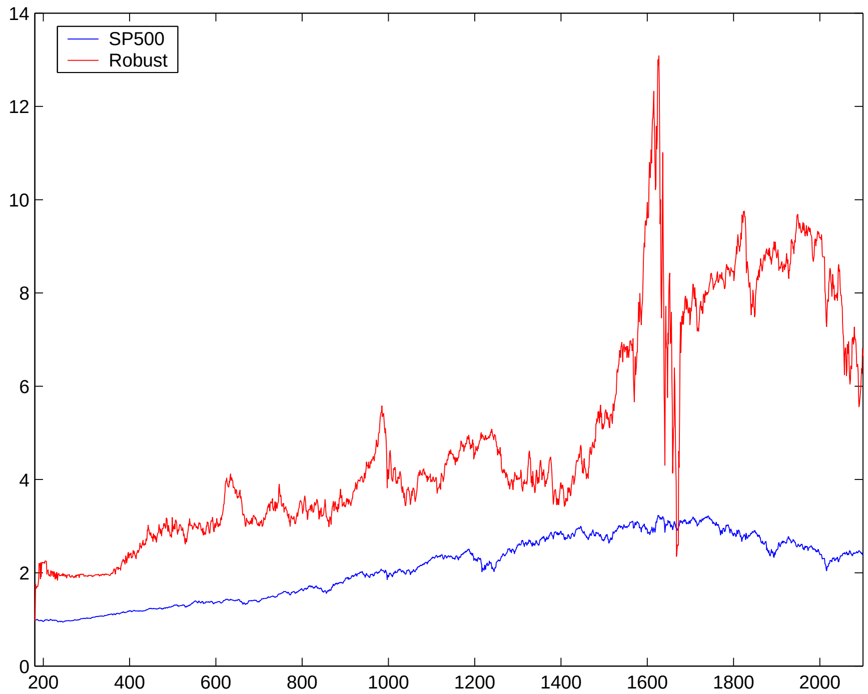
Comparison of overall performance

- Two policies: Policy 1: $(p = 30, h = 2)$, Policy 2: $(p = 30, h = 4)$
  - $R_{k-1}^{(j)}$: cumulative return of policy j over period $k - 1$
  - Invest $\theta^{(j)} = \dfrac{e^{\alpha(R_{k-1}^{(j)} - 1)}}{e^{\alpha(R_{k-1}^{(1)} - 1)} + e^{\alpha(R_{k-1}^{(2)} - 1)}}$ in policy j
- Cumulative daily returns for $\alpha = 5$



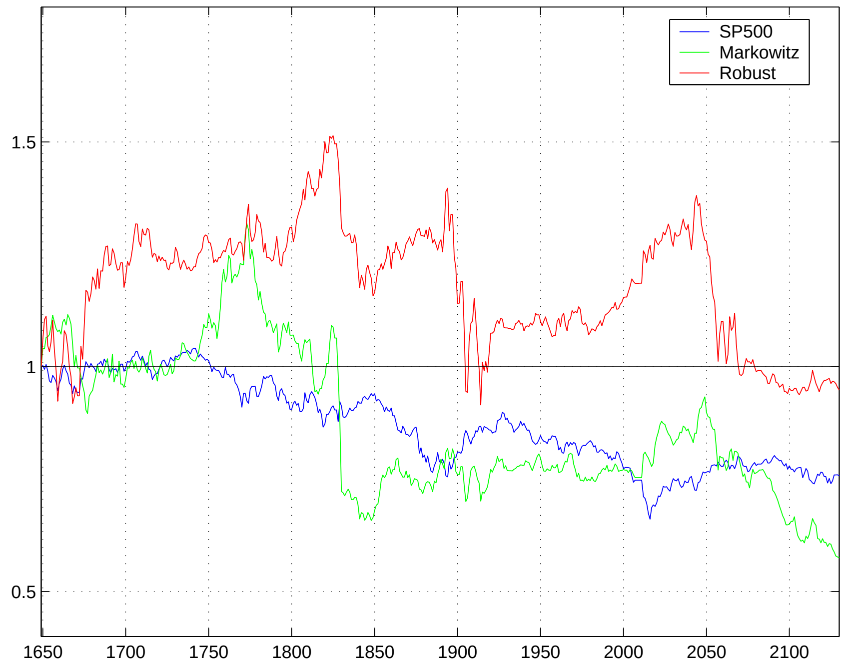
Comparison of overall performance

- Two policies: Policy 1: $(p = 30, h = 2)$, Policy 2: $(p = 30, h = 4)$
  - $R^{(j)}_{k-1}$: cumulative return of policy j over period $k - 1$
  - Invest $\theta^{(j)} = \dfrac{e^{\alpha(R^{(j)}_{k-1}-1)}}{e^{\alpha(R^{(1)}_{k-1}-1)} + e^{\alpha(R^{(2)}_{k-1}-1)}}$ in policy j
- Cumulative daily returns for $\alpha = 100$



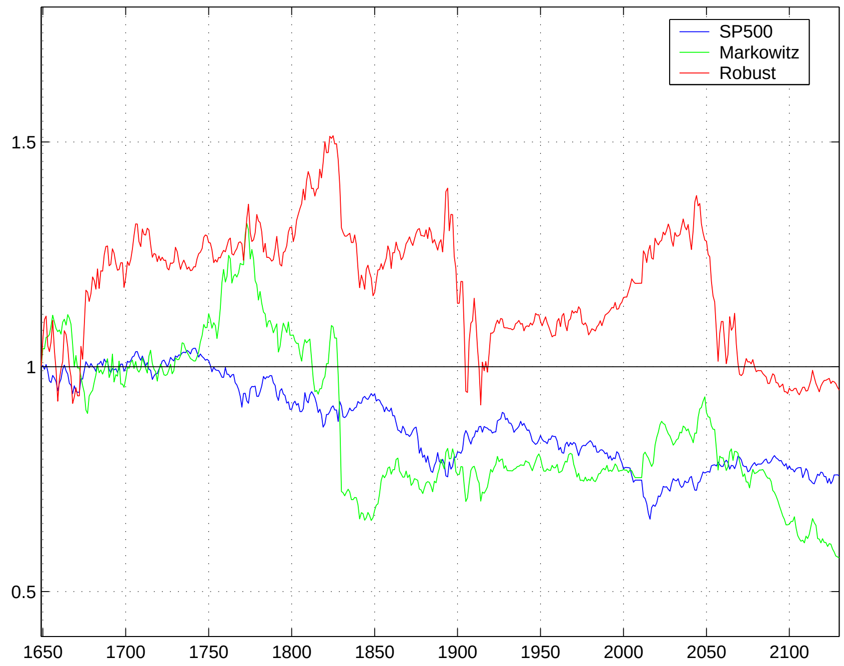
Comparison over down period

- Cumulative daily returns after the SP500 peak: $(p = 30, h = 4)$



Comparison over down period

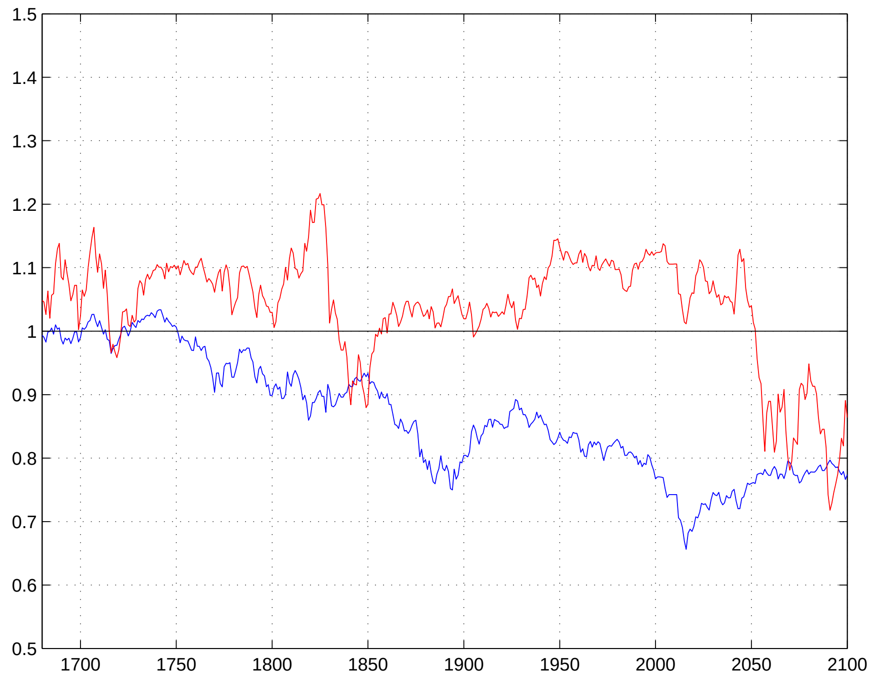
- Cumulative daily returns after the SP500 peak: $(p = 30, h = 4)$



- Markowitz strategy follows the market
- The *myopic* nature is apparent .. returns lurch up/down

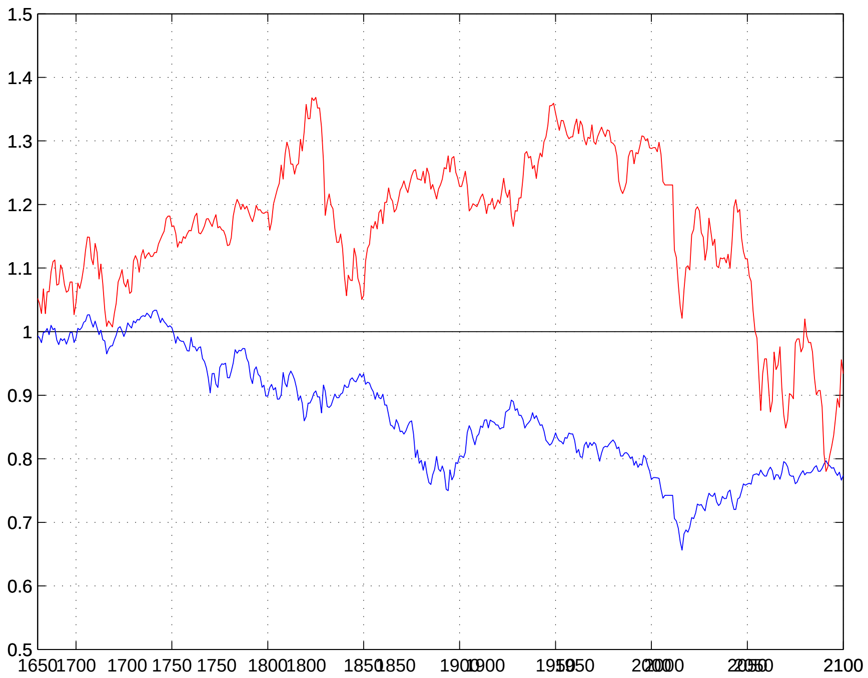
Comparison over down period

- Cumulative daily returns after the SP500 peak: Mixed strategy $\alpha = 5$



Comparison over down period

- Cumulative daily returns after the SP500 peak: Mixed strategy $\alpha = 100$



Summary

- Replaced point estimates $(\boldsymbol{\mu}_0, \mathbf{V}_0)$ by set estimates (S_m, S_v)
  This translates to confidence in the performance of the portfolios.

Summary

- Replaced point estimates $(\boldsymbol{\mu}_0, \mathbf{V}_0)$ by set estimates (S_m, S_v)
  This translates to confidence in the performance of the portfolios.
- The robust analog of classical portfolio selection problems are SOCPs
  - efficient, off-the-shelf solution algorithms available
  - complexity almost the same as quadratic programming
  - complexity does not increase when SOC/linear side constraints added

Summary

- Replaced point estimates $(\boldsymbol{\mu}_0, \mathbf{V}_0)$ by set estimates (S_m, S_v)
  This translates to confidence in the performance of the portfolios.
- The robust analog of classical portfolio selection problems are SOCPs
  - efficient, off-the-shelf solution algorithms available
  - complexity almost the same as quadratic programming
  - complexity does not increase when SOC/linear side constraints added
- Simulation experiments suggest
  - robustness does not entail a heavy toll on the mean behavior
  - robustness especially important when the observations are noisy

Summary

- Replaced point estimates $(\boldsymbol{\mu}_0, \mathbf{V}_0)$ by set estimates (S_m, S_v)
  This translates to confidence in the performance of the portfolios.
- The robust analog of classical portfolio selection problems are SOCPs
  - efficient, off-the-shelf solution algorithms available
  - complexity almost the same as quadratic programming
  - complexity does not increase when SOC/linear side constraints added
- Simulation experiments suggest
  - robustness does not entail a heavy toll on the mean behavior
  - robustness especially important when the observations are noisy
- Experiments on real market data suggest that returns are sensitive to
  - Rebalance frequency
  - Rebalance points
  - Quality of factors and number eigenvectors

Summary

- Replaced point estimates $(\boldsymbol{\mu}_0, \mathbf{V}_0)$ by set estimates (S_m, S_v)
  This translates to confidence in the performance of the portfolios.
- The robust analog of classical portfolio selection problems are SOCPs
  - efficient, off-the-shelf solution algorithms available
  - complexity almost the same as quadratic programming
  - complexity does not increase when SOC/linear side constraints added
- Simulation experiments suggest
  - robustness does not entail a heavy toll on the mean behavior
  - robustness especially important when the observations are noisy
- Experiments on real market data suggest that returns are sensitive to
  - Rebalance frequency
  - Rebalance points
  - Quality of factors and number eigenvectors
- Transaction costs: Cost of robust strategy is slightly larger than Classical strategy