Distinguishing separable and entangled states in quantum mechanics

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Quantum entanglement

- Entanglement is one of the most distinctive features of quantum physics.
- A strong “linkage” between two systems, that cannot be explained through probability.
- If two systems have interacted in the past, it may not be possible to assign independent states.
- EPR “paradox”, Bell inequalities.

Verified experimentally.
Hidden variable theories must be non-local.

Entanglement as a resource

- Quantum computation: some computational problems have lower complexity if entangled states are available.
- Quantum key distribution in cryptography.
- Teleportation: entanglement + classical communication.

All require entangled states. How?

EX: Quantum teleportation

(Bennett et al., 1993)

- Prepare an entangled pair of particles.
- Give one to Alice, other one to Bob.
- Alice performs a joint measurement on the data and her entangled particle.
- Alice sends the “random” result to Bob using a classical channel.
- Bob performs a measurement according to Alice’s instructions.
- Bob obtains Alice’s data. Alice’s data is destroyed.

Entanglement as a resource

- Teleportation: entanglement + classical communication.
- Quantum key distribution in cryptography.
- Quantum computing: some computational problems may have lower complexity if entangled states are available.

Q: How to determine whether or not a given quantum state is entangled ?

Outline

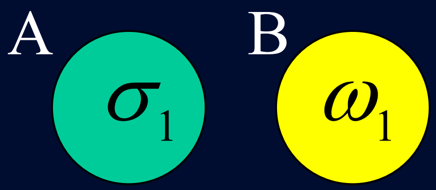
- Entangled vs. separable states
- Characterizations
- Positive maps and bihermitian forms
- Entanglement witnesses
- State extensions
- SDP approach
- Conclusions

Mathematical description

QM state described by PSD Hermitian matrices ρ (density matrix, mixed states)

- States of multipartite systems are described by operators on the tensor product of vector spaces

$$\rho = \sigma_1 \otimes \omega_1$$



- **Product states:** each subsystem is in a definite state

Separable states:

convex combination of product states.

$$\rho = \sum p_i \sigma_i \otimes \omega_i, \;\; p \geq 0_i, \sum p_i = 1$$

Interpretation: statistical ensemble of locally prepared states.

Entangled states: those that cannot be written as a convex combination of product states.

Deciding entanglement

Decision problem: find a decomposition of the state ρ as a convex combination of product states or prove that no such decomposition exists.

Semialgebraic, thus decidable.
But NP-hard (Gurvits 2002).

Not clear how to proceed. In particular, how to *certify* entanglement?

Starting point: necessary conditions for separability.

The PPT criterion

(Peres-Horodecki 1996)

Consider the map ("partial transpose") defined by:

$$PT(\sigma \otimes \omega) = \sigma \otimes \omega^T$$

Extend by linearity. Two-block example:

$$PT\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}\right) = \begin{bmatrix} M_{11} & M_{12}^T \\ M_{12}^T & M_{22} \end{bmatrix}$$

If ρ is separable, then PT(ρ) is PSD.

A necessary condition for separability.

Entangled states exist!

If PT(ρ) is *not* PSD , then ρ has to be entangled!

A sufficient condition for entanglement.

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad PT(\rho) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

NOT PSD

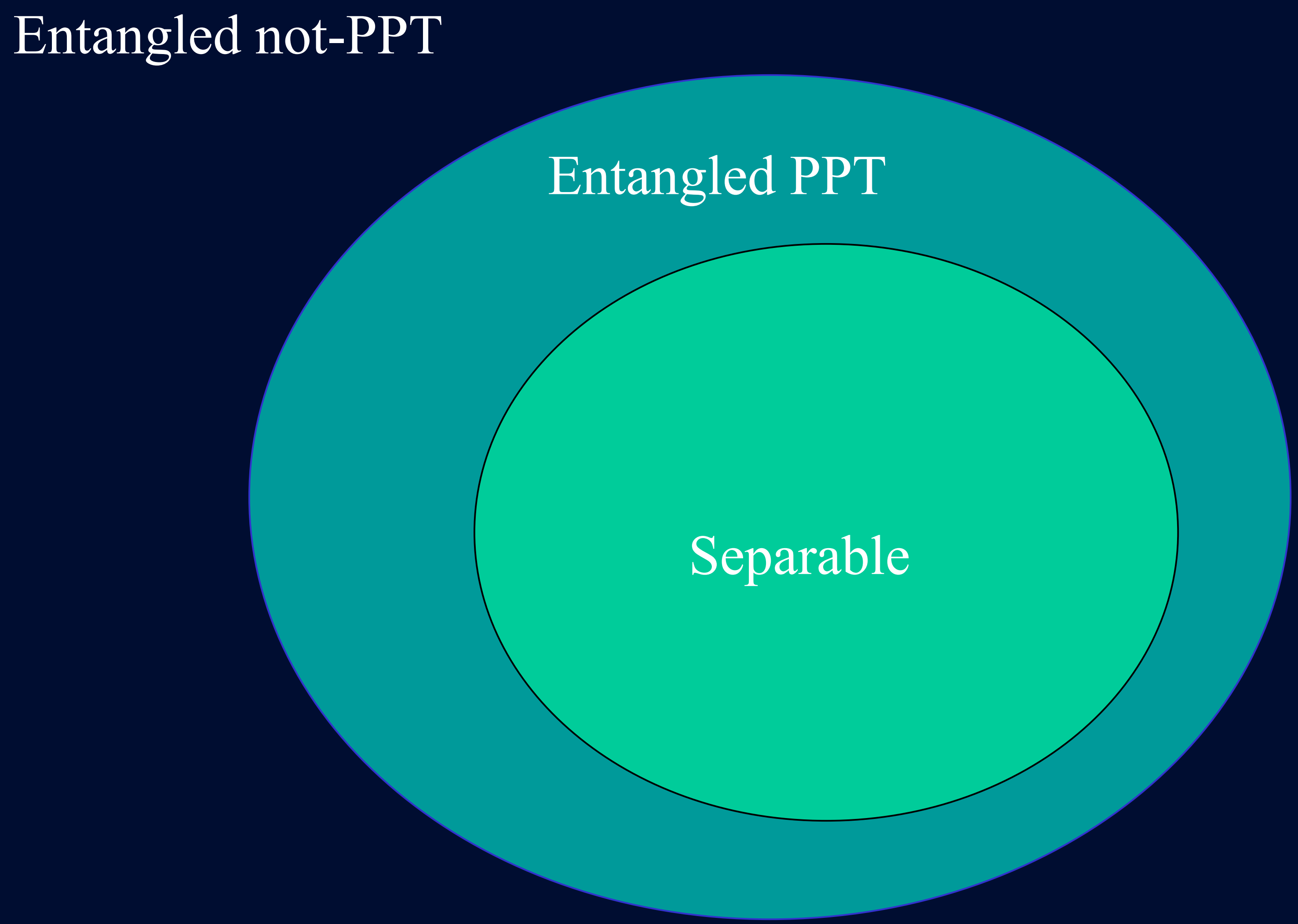
Therefore, ρ cannot be separable. It is entangled.

PPT is easy to verify. Correct solution for 2x2, 2x3.

But, not the whole story.

In general, *not* necessary and sufficient.

There exist entangled states, with positive PT.


Entangled not-PPT
Entangled PPT
Separable

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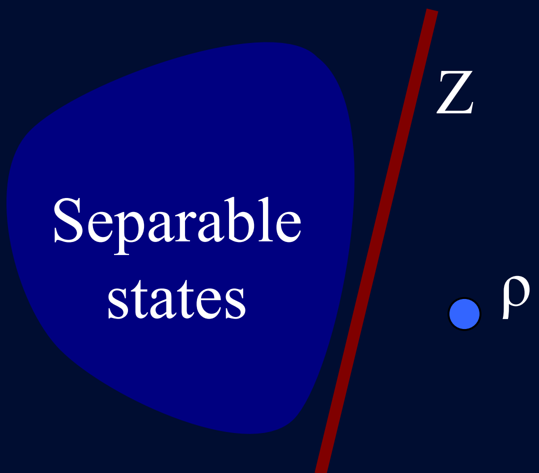
How can we obtain sharper characterizations of entangled states?

Better “operational criteria” to classify states?

Separability and convexity



(Hahn-Banach Theorem)

Z is an "entanglement witness," a generalization of Bell's inequalities

Theorem (Horodecki 1996): a state ρ is entangled if and only if there exists a Hermitian Z such that:

$$\mathrm{Tr}\left[\rho Z\right] < 0$$

$$\forall x, y\ \mathrm{Tr}\left[\left(xx^{*} \otimes yy^{*}\right) Z\right] = \sum Z_{ij;kl} x_i x^{*}_j y_k y^{*}_l \geq 0$$ **Hard!**

An aside: positive maps

- A matrix map is *positive* if

$$\rho \geq 0 \Rightarrow L(\rho) \geq 0$$

i.e., takes the PSD cone into itself.

- There is a bijection between matrix maps and bihermitian forms:

$$p(x, y) = y^* L(xx^*) y$$

- Therefore, checking positivity of a matrix map is equivalent to nonnegativity of bihermitian forms. Also equivalent to characterization of EWs.

Our approach

(Phys. Rev. Lett., May 2002

Complementary interpretations.

- Finding state extensions.
- Search for specific classes of EWs.

The underlying theme:

To replace polynomial nonnegativity by something more tractable, sum of squares decompositions.

State extensions

- More consequences of separability: state extensions.

If $\rho \in H_A \otimes H_B$ is separable, then

$$\rho = \sum p_i \, x_i x_i^* \otimes y_i y_i^*$$

We can always extend it to $\tilde{\rho} \in H_A \otimes H_B \otimes H_A$

$$\tilde{\rho} = \sum p_i \, x_i x_i^* \otimes y_i y_i^* \otimes x_i x_i^*$$

- The extended tripartite state is separable for all partitions (A|BA),(AA|B).

$$\tilde{\rho} \in H_A \otimes H_B \otimes H_A$$

- In particular, should be PPT wrt to all partitions.

If ρ is separable then there is an extension:

- Reduces to ρ under projection on first two components.
- Is symmetric in the 1st and 3rd components.
- Is PPT for all partitions.

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Searching for such extension is an SDP. Why?

Take a generic matrix $\tilde{\rho} \in H_A \otimes H_B \otimes H_A$

- The first two conditions are affine equations.
- The third one gives two PSD constraints.

If such extension does not exist, ρ is entangled.

Infeasibility certificates for this SDP provide entanglement witnesses!

Going to the other side...

- Do these witnesses have "easy" interpretations?
- How does all this relate (if at all) with PPT?

Recall: To find witnesses, look for Z

$$\mathrm{Tr}\left[\rho Z\right] < 0$$

$$\forall x, y\ \mathrm{Tr}\left[\left(xx^{*} \otimes yy^{*}\right) Z\right] = \sum Z_{ij;kl} x_i x_j^{*} y_k y_l^{*} \geq 0$$

First Relaxation

Restrict attention to a special type of Z:

The bihermitian form Z is a sum of squared magnitudes.

$$\mathrm{Tr}\left[\rho Z\right] = \sum Z_{ij;kl} x_i x_j^* y_k y_l^* = \sum \left| G_{m;ik} x_i y_k \right|^2 + \sum \left| H_{m;ik} x_i y_k^* \right|^2$$

$$= \left(x \otimes y\right)^* G \left(x \otimes y\right) + \left(x^* \otimes y^{\mathrm{T}}\right) H \left(x \otimes \left(y^*\right)^{\mathrm{T}}\right)$$

$$= \mathrm{Tr}\left[\rho G + \rho^{\mathrm{T}_2} H\right]$$

$$\begin{aligned} &\text{minimize} \quad \mathrm{Tr}\left[\rho G + \rho^{\mathrm{T}_2} H\right] \\ &\text{subject to} \quad G \ge 0, \quad H \ge 0 \end{aligned}$$

If minimum is less than zero, ρ is entangled

First Relaxation

- Relax nonnegativity to SOS.
- Exactly equivalent to the PPT criterion!
- A reason: when checking SOS for bihermitian forms, the SDP is trivial. Only need to check if two matrices are PSD.
- Corresponds to an easily parameterized class of positive maps: the *decomposable* ones

$$L(X) = \sum_i A_i X A_i^* + \sum_i B_i X^T B_i^*$$

Nice, but we know already that can do better...

Higher relaxations

Broaden the class of allowed Z to those for which

$$\sum |x_i|^2 \sum Z_{ij;kl} x_i x^*_j y_k y^*_l$$

is a sum of squared magnitudes.

Also semidefinite programs.

Strictly stronger than PPT.

Can search directly for the witness Z.

Second Relaxation

$$\sum |x_i|^2 \sum Z_{ij;kl} x_i x_j^* y_k y_l^* = \sum \left| G_{m;ijk} x_i y_k x_j \right|^2 + \sum \left| H_{m;ijk} x_i^* y_k x_j \right|^2$$

$$+ \sum \left| K_{m;ijk} x_i y_k^* x_j \right|^2$$

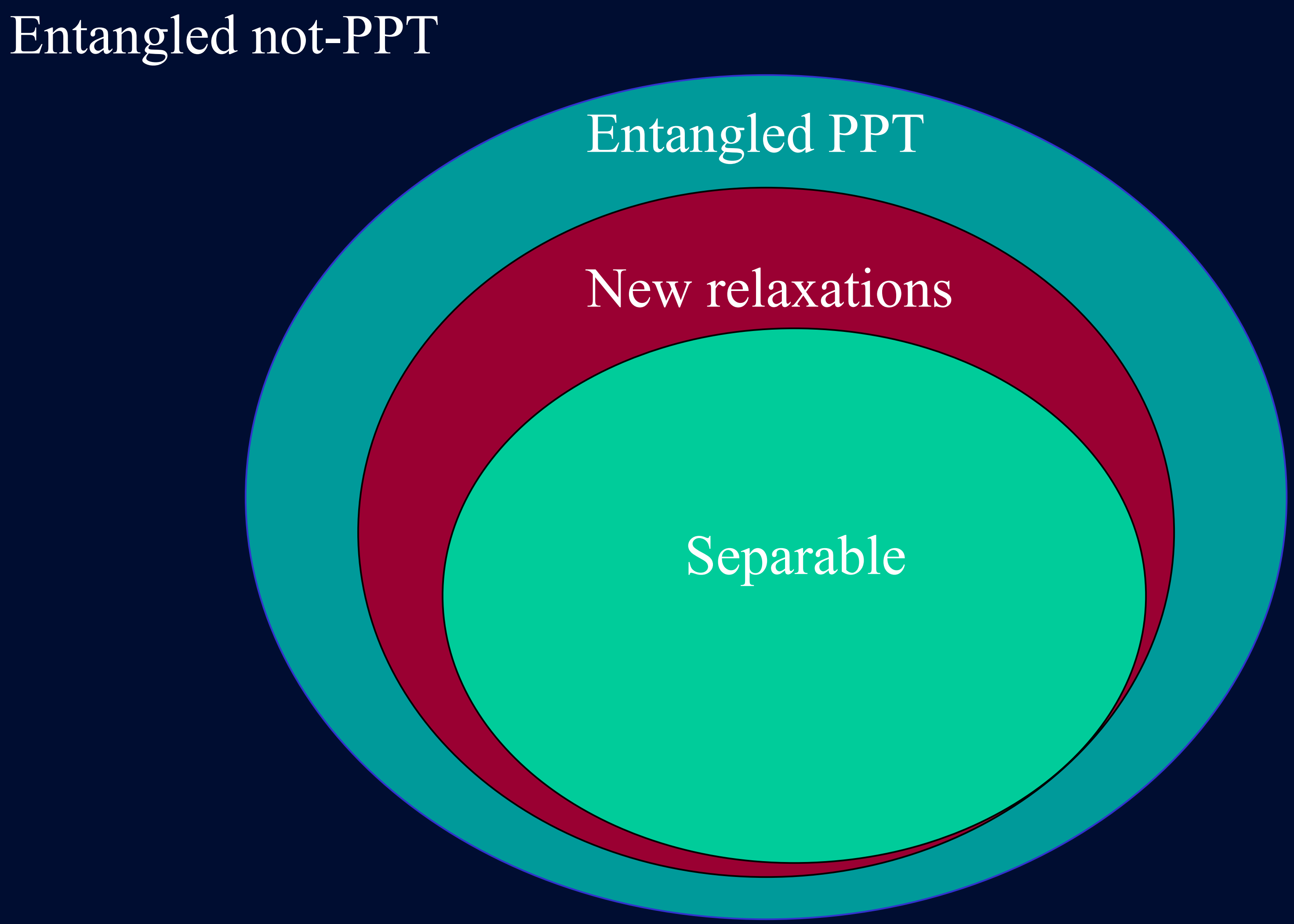
minimize $\mathrm{Tr}[\rho Z]$

subject to $G \geq 0 \quad H \geq 0 \quad K \geq 0$

$$(x \otimes y \otimes x)^* (Z \otimes I)(x \otimes y \otimes x) = (x \otimes y \otimes x)^* \left(G + H^{T_1} + K^{T_2}\right)(x \otimes y \otimes x)$$

If the minimum is less than zero then ρ **is entangled**.

The dual of the state extension SDP.


Entangled not-PPT
Entangled PPT
New relaxations
Separable

Hierarchy of conditions

- Using different factors, obtain a nested family of SDPs.
- A section of the SOS approximations to the cone of nonnegative polynomials.
- Essentially similar procedures for many other NP-hard cones: copositive matrices, etc.
- Special case of more general P-satz approach.
- More general constructions are complete.
- Appealing properties: invariance under LOCC?

Numerical results

- SDPs solved using SeDuMi (Jos Sturm).
- Tested many known families of PPT entangled states.
- These correspond to bihermitian forms that are not SOS.
- For all examples tried, the second-level test always succeeded.
- But, counterexamples *must* exist.
- In some cases, can obtain closed-form solutions for witnesses.
- Exploiting symmetry helps a lot.

Summary

- Can decide/certify entanglement via SDP/SOS.
- Hierarchy of tests.
- Dual interpretations
  - State extensions
  - Entanglement witnesses of specific kind
- Simplest case, reduces to PPT criterion.
- Second level, detects all states tried!
- Explicit witnesses, easily verifiable.