# Extensions of S-Procedure and Their Applications*

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# Minimizing 1-dim Quadratic Polynomials

- Let $q(x) = c + 2bx + ax^2$, $x \in \Re$. Then

$$q(x) = (1, x) \begin{bmatrix} c & b \\ b & a \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0, \ \forall\ x \in \Re \quad \Longleftrightarrow \quad \begin{bmatrix} c & b \\ b & a \end{bmatrix} \succeq \mathbf{0}.$$

- Moreover, we have

$$q(x) \geq 0, \ \forall\ x^2 \leq 1 \quad \Longleftrightarrow \quad \exists s \geq 0, \ \text{s.t. } q(x) \geq s(1 - x^2), \ \forall x \in \Re.$$

- In matrix form, we have

$$q(x) \geq 0, \ \forall\ x^2 \leq 1 \quad \Longleftrightarrow \quad \begin{bmatrix} c - s & b \\ b & a + s \end{bmatrix} \succeq \mathbf{0}, \ \text{for some } s \geq 0.$$

# Minimizing 1-dim Quadratic Polynomials

$$\begin{array}{ll} \text{minimize} & c + 2bx + ax^2 \\ \text{subject to} & x^2 \le 1. \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & q(x) - r = c - r + 2bx + ax^2 \ge 0, \ \forall x^2 \le 1. \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & \begin{bmatrix} c - (r+s) & b \\ b & a+s \end{bmatrix} \succeq \mathbf{0}, \ s \ge 0. \end{array}$$

$$\Uparrow$$

An SDP, solvable by interior point methods

# LMI Description

Let $D = [-1, 1]$, we obtain the LMI characterization of the nonnegative quadratic function cone

$$\begin{aligned} \mathcal{K}_{1,1}(D) &= \left\{ \begin{bmatrix} c & b \\ b & a \end{bmatrix} \middle| \, c + 2bx + ax^2 \geq 0, \ \forall \, x \in D \right\} \\ &= \left\{ \begin{bmatrix} c & b \\ b & a \end{bmatrix} \middle| \begin{bmatrix} c & b \\ b & a \end{bmatrix} \succeq s \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ s \geq 0 \right\} \end{aligned}$$

# Trust Region Subproblem

Consider the following well known trust region subproblem:

$$\begin{array}{ll} \text{minimize} & c + 2\mathbf{b}^T\mathbf{x} + \mathbf{x}^T\mathbf{A}\mathbf{x} \\ \text{subject to} & \|\mathbf{x}\| \le 1. \end{array} \tag{1}$$

- Occurs frequently in trust region type methods for nonlinear programming
- Polynomial time solvable ($\mathbf{A}$ not necessarily PSD.)
- Can be reformulated as an SDP (semi-definite program)

# S-Procedure (Yakubovich, 1977)

- Let $q(\mathbf{x}) = c + 2\mathbf{b}^T\mathbf{x} + \mathbf{x}^T\mathbf{A}\mathbf{x}$, $\mathbf{x} \in \Re^n$ then

$$q(\mathbf{x}) \geq 0,\ \forall\ \|\mathbf{x}\| \leq 1 \quad \Longleftrightarrow \quad \begin{bmatrix} c - s & \mathbf{b}^T \\ \mathbf{b} & \mathbf{A} + s\mathbf{I}_n \end{bmatrix} \succeq \mathbf{0}, \text{ for some } s \geq 0.$$

- Effective tool in robust optimization

- As a result, the trust region subproblem (1) can be reformulated as the following SDP:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & \begin{bmatrix} c - (r + s) & \mathbf{b}^T \\ \mathbf{b} & \mathbf{A} + s\mathbf{I}_n \end{bmatrix} \succeq \mathbf{0},\ \ s \geq 0. \end{array} \tag{2}$$

# An Application of S-Procedure

Consider a robust linear inequality constraint:

$$(\mathbf{a} + \Delta\mathbf{a})^T\mathbf{x} \le b + \Delta b, \quad \forall \, \|(\Delta\mathbf{a}, \Delta b)\| \le \epsilon.$$

Using S-procedure, we obtain the equivalent LMI characterization:

$$\begin{bmatrix} \mathbf{a}^T\mathbf{x} - b - s & \frac{1}{2}[\mathbf{x} \; 1]^T \\ \frac{1}{2}[\mathbf{x} \; 1] & \frac{s}{\epsilon^2}\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, \;\; s \ge 0.$$

$$\Updownarrow$$

$$\mathbf{a}^T\mathbf{x} - b - \epsilon\sqrt{\|\mathbf{x}\|^2 + 1} \ge 0.$$

$$\Uparrow$$

A Convex Second Order Cone Constraint

# Extensions of S-Procedure I

**Theorem 1. [Luo-Sturm-Zhang]**

*Let* $q(\mathbf{X}) = \mathbf{C} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{B} + \mathbf{X}^T\mathbf{A}\mathbf{X}$, *where* $\mathbf{X} \in \Re^{n\times m}$, $\mathbf{C} \in \mathcal{S}^m$, $\mathbf{B} \in \Re^{n\times m}$ *and* $\mathbf{A} \in \mathcal{S}^n$, *then*

$$\begin{array}{c} q(\mathbf{X}) \succeq \mathbf{0}, \ \textit{for all} \ \ \|\mathbf{X}\|_F = \sqrt{Tr(\mathbf{X}^T\mathbf{X})} \le 1 \\ \Updownarrow \\ \begin{bmatrix} \mathbf{C} - s\mathbf{I}_m & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A} + s\mathbf{I}_n \end{bmatrix} \succeq \mathbf{0}, \ \textit{for some} \ s \ge 0. \end{array} \tag{3}$$

- When $\mathbf{A} = \mathbf{0}$, this is robust LMI considered by El-Ghaoui (SIMAX).
- (3) gives a LMI characterization of the convex cone ...

$$\begin{aligned} \mathcal{K}_{m,n}(D) &= \left\{ \begin{bmatrix} \mathbf{C} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \middle| \mathbf{C} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{B} + \mathbf{X}^T\mathbf{A}\mathbf{X} \succeq \mathbf{0}, \ \forall \ \|\mathbf{X}\|_F \le 1 \right\} \\ &= \left\{ \begin{bmatrix} \mathbf{C} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \middle| \begin{bmatrix} \mathbf{C} - s\mathbf{I}_m & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A} + s\mathbf{I}_n \end{bmatrix} \succeq \mathbf{0}, \ s \ge 0 \right\} \end{aligned}$$

# Trust Region Subproblem – Matrix Form

Consider the generalized trust region subproblem in $\Re^{n\times m}$

$$\begin{array}{ll} \text{minimize} & \lambda_{\min}(\mathbf{C}+\mathbf{B}^T\mathbf{X}+\mathbf{X}^T\mathbf{B}+\mathbf{X}^T\mathbf{A}\mathbf{X}) \\ \text{subject to} & \|\mathbf{X}\|_F \le 1, \end{array} \tag{4}$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a matrix. Then, by Theorem 1, we can transform (4) as the following semi-definite program:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & \begin{bmatrix} \mathbf{C}-(r+s)\mathbf{I}_m & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A}+s\mathbf{I}_n \end{bmatrix} \succeq \mathbf{0},\ \ s \ge 0. \end{array} \tag{5}$$

# Extensions of S-Procedure II

**Theorem 2. [Luo-Sturm-Zhang]** *The following statements are equivalent:*

$$\mathbf{C} + \mathbf{X}^T\mathbf{B} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{A}\mathbf{X} \succeq \mathbf{0}, \quad \forall\, \mathbf{X} \text{ s.t. } \mathbf{I} - \mathbf{X}^T\mathbf{D}\mathbf{X} \succeq \mathbf{0}$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{C} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A} \end{bmatrix} - s \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} \end{bmatrix} \succeq \mathbf{0}, \text{ for some } s \geq 0.$$

*and if* $\mathbf{D} \succeq \mathbf{0}$, *then the above are further equivalent to*

$$\mathbf{C} + \mathbf{X}^T\mathbf{B} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{A}\mathbf{X} \succeq \mathbf{0}, \quad \forall\, \mathbf{X} \text{ s.t. } Tr(\mathbf{X}^T\mathbf{D}\mathbf{X}) \leq 1$$

# An Observation

- Notice that $\|\mathbf{X}\|_F \le 1$ implies $\mathbf{I} - \mathbf{X}^T\mathbf{X} \succeq \mathbf{0}$.
- In general, for $\mathbf{D} \succeq \mathbf{0}$,

$$\{\mathbf{X} \mid \mathbf{I} - \mathbf{X}^T\mathbf{D}\mathbf{X} \succeq \mathbf{0}\} = \mathcal{D}_1 \supset \mathcal{D}_2 = \{\mathbf{X} \mid \mathrm{Tr}\,(\mathbf{X}^T\mathbf{D}\mathbf{X}) \le 1\}.$$

- However, when $\mathbf{D} \succeq \mathbf{0}$, there holds

$$\boxed{\mathbf{C} + \mathbf{X}^T\mathbf{B} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{A}\mathbf{X} \succeq \mathbf{0}, \quad \forall\, \mathbf{X} \in \mathcal{D}_1}$$

$$\Updownarrow$$

$$\boxed{\begin{bmatrix} \mathbf{C} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A} \end{bmatrix} - s \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} \end{bmatrix} \succeq \mathbf{0}, \text{ for some } s \ge 0.}$$

$$\Updownarrow$$

$$\boxed{\mathbf{C} + \mathbf{X}^T\mathbf{B} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{A}\mathbf{X} \succeq \mathbf{0}, \quad \forall\, \mathbf{X} \in \mathcal{D}_2.}$$

# Extensions of S-Procedure III

**Theorem 3. [Luo-Sturm-Zhang]**

$$\left\{\begin{array}{l} \mathbf{H} \succeq \mathbf{0} \\ \mathbf{C} + \mathbf{X}^T\mathbf{B} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{A}\mathbf{X} \succeq \mathbf{0}, \\ \mathbf{H} - (\mathbf{F} + \mathbf{G}\mathbf{X})(\mathbf{C} + \mathbf{X}^T\mathbf{B} + \mathbf{B}^T\mathbf{X} + \mathbf{X}^T\mathbf{A}\mathbf{X})^+(\mathbf{F} + \mathbf{G}\mathbf{X})^T \succeq \mathbf{0}, \end{array}\right.$$

*for all* $\mathbf{I} - \mathbf{X}^T\mathbf{D}\mathbf{X} \succeq \mathbf{0}$.

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{H} & \mathbf{F} & \mathbf{G} \\ \mathbf{F}^T & \mathbf{C} & \mathbf{B}^T \\ \mathbf{G}^T & \mathbf{B} & \mathbf{A} \end{bmatrix} - s \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{D} \end{bmatrix} \succeq \mathbf{0}, \quad s \geq 0.$$

*where* $\mathbf{M}^+$ *stands for the pseudo inverse of* $\mathbf{M} \succeq \mathbf{0}$.

# Extensions of S-Procedure IV

**Theorem 4. [Luo-Sturm-Zhang]**

*Characterizing a robust QMI over Stiefel manifold:*

$$\begin{bmatrix} \mathbf{H} & \mathbf{F}+\mathbf{G}\mathbf{X} \\ (\mathbf{F}+\mathbf{G}\mathbf{X})^T & \mathbf{C}+\mathbf{X}^T\mathbf{B}+\mathbf{B}^T\mathbf{X}+\mathbf{X}^T\mathbf{A}\mathbf{X} \end{bmatrix} \succeq \mathbf{0}, \text{ for all } \mathbf{X}^T\mathbf{X}=\mathbf{I}.$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{H} & \mathbf{F} & \mathbf{G} \\ \mathbf{F}^T & \mathbf{C} & \mathbf{B}^T \\ \mathbf{G}^T & \mathbf{B} & \mathbf{A} \end{bmatrix} - s \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, \quad s \in \Re.$$

Note that $s$ is now **free**.

# The Case of General Robust QMIs

Characterize the set of $m \times m$ symmetric matrices $\mathbf{C}$, $\mathbf{B}_j$, $\mathbf{A}_{ij}$, $i, j = 1, \ldots, n$, such that

$$\mathbf{C} + 2\sum_{j=1}^{n} x_j \mathbf{B}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \mathbf{A}_{ij} \succeq \mathbf{0}$$

holds for all $\mathbf{x} \in D$, where $D \subseteq \Re^n$ is a given domain.

Let us define

$$\mathcal{K}_{m,n}(D) = \left\{ \begin{bmatrix} \mathbf{C} & \mathbf{B}_1 & \cdots & \mathbf{B}_n \\ \mathbf{B}_1 & \mathbf{A}_{11} & \cdots & \mathbf{A}_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_n & \mathbf{A}_{n1} & \cdots & \mathbf{A}_{nn} \end{bmatrix} \middle| \begin{array}{l} \mathbf{C} + 2\sum_{j=1}^{n} x_j \mathbf{B}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \mathbf{A}_{ij} \succeq \mathbf{0} \\ \text{for all } \mathbf{x} \in D \end{array} \right\}$$

# Characterization of Dual Cone

**Theorem 5. [Luo-Sturm-Zhang]**

- *We have*

$$\mathcal{K}^*_{m,n}(D) = \text{cone}\ \{(\mathbf{x} \otimes \mathbf{x}) \otimes (\mathbf{y} \otimes \mathbf{y}) \mid \textit{for all}\ \mathbf{x} \in \mathcal{H}(D) \subseteq \Re^{1+n},\ \textit{and}\ \mathbf{y} \in \Re^m\},$$

*where* $\otimes$ *stands for the Kronecker product of two matrices, and*

$$\mathcal{H}(D) = \textit{cl}\ \left\{ \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \;\middle|\; \mathbf{x}/t \in D \right\}$$

*is the homogenization of* $D$.

# Homogenization of $D$

- If $D = \Re^n$, then $\mathcal{H}(D) = \Re^{n+1}$, so

$$\mathcal{K}^*_{m,n}(D) = \text{cone}\,\{\mathbf{X} \otimes \mathbf{Y} \mid \text{for all } \mathbf{X} \in \mathcal{S}^{n+1}_+,\ \mathbf{Y} \in \mathcal{S}^m_+\}.$$

- If $D = \{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\}$, then $\mathcal{H}(D) = \{(t, \mathbf{x}^T)^T \mid \|x\| \leq t\}$, so

$$\begin{aligned}\mathcal{K}^*_{m,n}(D) &= \text{cone}\,\{(\mathbf{x} \otimes \mathbf{x}) \otimes \mathbf{Y} \mid \text{for all } \mathbf{x} \in \text{SOC}(n+1),\ \mathbf{Y} \in \mathcal{S}^m_+\} \\ &= \text{cone}\,\{\mathbf{X} \otimes \mathbf{Y} \mid \text{for all } \mathbf{X} \in \mathcal{S}^{n+1}_+,\ \text{Tr}\,(\mathbf{JX}) \geq 0,\ \mathbf{Y} \in \mathcal{S}^m_+\}.\end{aligned}$$

  where $\mathbf{J} = \text{Diag}(1, -1, -1, \ldots, -1)$.

- General characterization of $\mathcal{H}(D)$ exists for $D$ defined by a single quadratic (in)equality.

# Characterizing the Dual Cone $\mathcal{K}^*_{m,n}(\Re^n)$

- Notice that $\mathcal{K}^*_{m,n}$ lies in the linear subspace

$$\begin{aligned}\mathcal{L}_{m,n} &:= \text{Lin}\,\{\mathbf{X}\otimes\mathbf{Y} \mid \mathbf{X}\in\mathcal{S}^{1+n},\ \mathbf{Y}\in\mathcal{S}^m\} \\ &= \left\{ \begin{bmatrix} \mathbf{G}_{00} & \mathbf{G}_{01} & \cdots & \mathbf{G}_{0n} \\ \mathbf{G}_{10} & \mathbf{G}_{11} & \cdots & \mathbf{G}_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{G}_{n0} & \mathbf{G}_{n1} & \cdots & \mathbf{G}_{nn} \end{bmatrix} \in \mathcal{S}^{(1+n)\times m} \;\middle|\; \begin{array}{l} \mathbf{G}_{ij}^T = \mathbf{G}_{ij}, \\ 0\le i,j\le n \end{array} \right\}\end{aligned}$$

- We also know $\mathcal{K}^*_{m,n} \subset \mathcal{S}_+^{(n+1)m}$, implying $\mathcal{K}^*_{m,n} \subseteq \mathcal{S}_+^{(n+1)m} \cap \mathcal{L}_{m,n}$.

- **Theorem 6 [Luo-Sturm-Zhang]** There holds $\mathcal{K}^*_{m,1}(\Re^1) = \mathcal{S}_+^{2m} \cap \mathcal{L}_{m,1}$. In other words,

$$\begin{bmatrix} \mathbf{G}_{00} & \mathbf{G}_{01} \\ \mathbf{G}_{01} & \mathbf{G}_{11} \end{bmatrix} \in \text{cone}\,\{\mathbf{X}\otimes\mathbf{Y} \mid \mathbf{X}\in\mathcal{S}_+^2,\ \mathbf{Y}\in\mathcal{S}_+^m\}$$

$$\iff \mathbf{G}_{00},\ \mathbf{G}_{11},\ \mathbf{G}_{01} \text{ symmetric},\ \begin{bmatrix} \mathbf{G}_{00} & \mathbf{G}_{01} \\ \mathbf{G}_{01} & \mathbf{G}_{11} \end{bmatrix} \succeq \mathbf{0}.$$

# Negative Results

- Negative result

For general $n$ and $m$, checking whether or not a given QMI system is nonnegative over $\Re^n$ is NP-hard.

- In general, let

$$\mathcal{C}_1 = \left\{ \begin{bmatrix} A_1 \bullet U \\ \vdots \\ A_k \bullet U \end{bmatrix} \middle| \; U \in \mathcal{S}_+^n \right\}, \quad \text{and} \quad \mathcal{C}_2 = \left\{ \begin{bmatrix} B_1 \bullet V \\ \vdots \\ B_k \bullet V \end{bmatrix} \middle| \; V \in \mathcal{S}_+^m \right\}.$$

**Nemirovskii** showed that

Checking if a given $\mathcal{C}_1 \subseteq \mathcal{C}_2$ is NP-hard.

# Robust QMI System in $\Re^n$

- For example, if $n = 1$, we have

$$\boxed{\mathbf{C} + 2x\mathbf{B} + x^2\mathbf{A} \succeq \mathbf{0}, \quad \forall\, x \in \Re} \quad \Leftrightarrow \quad \boxed{\begin{bmatrix} \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \succeq \mathbf{0}.}$$

- Similar result holds for $m = 2$ (but $n$ is arbitrary), i.e.,:

$$\begin{bmatrix} c(\mathbf{x}) & b(\mathbf{x}) \\ b(\mathbf{x}) & a(\mathbf{x}) \end{bmatrix} \succeq \mathbf{0}, \ \forall \mathbf{x} \in \Re^n \quad \Leftrightarrow \quad \text{A LMI system}$$

where $a(x),\ b(x),\ c(x)$ are some scalar (non-homogeneous) quadratic functions.

- Moreover, the following co-centered robust QMI can also be characterized by an LMI:

$$\begin{bmatrix} \mathbf{x}^T\mathbf{C}\mathbf{x} + c & \mathbf{x}^T\mathbf{B}\mathbf{x} + b \\ \mathbf{x}^T\mathbf{B}\mathbf{x} + b & \mathbf{x}^T\mathbf{A}\mathbf{x} + a \end{bmatrix} \succeq \mathbf{0}, \quad \forall\ \|x\| \le 1 \ \Leftrightarrow \ \begin{bmatrix} \mathbf{C} + c\mathbf{I} & \mathbf{B} + b\mathbf{I} \\ \mathbf{B} + b\mathbf{I} & \mathbf{A} + a\mathbf{I} \end{bmatrix} \succeq \mathbf{0}.$$

# Applications in Robust Convex Optimization

- Robust convex optimization models in mathematical programming have received much attention recently (Ben-Tal and Nemirovskii).

**Convex program**

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & g_i(\mathbf{x}) \leq 0, \ \forall i \end{array}$$

$\Longrightarrow$

**Robust version**

$$\begin{array}{ll} \text{minimize} & \max_{\|\Delta\| \leq \epsilon} f(\mathbf{x}, \Delta) \\ \text{subject to} & g_i(\mathbf{x}, \Delta) \leq 0, \\ & \text{for all } \|\Delta\| \leq \epsilon_i, \end{array}$$

- The robust formulation is still **convex**, but has **infinitely** many constraints.
  $\Longrightarrow$ reformulation is necessary.
- Arguably all engineering design problems should be treated in a robust way.

# Robust Linear Programming

- Robust optimization models are used for applications where data are inaccurate or unreliable. For example,

$$\begin{array}{ll} \text{maximize} & \min_{\|\Delta \mathbf{c}\| \le \epsilon_0} (\mathbf{c} + \Delta \mathbf{c})^T \mathbf{x} \\ \text{subject to} & (\mathbf{a}_i + \Delta \mathbf{a}_i)^T \mathbf{x} \ge b_i + \Delta b_i, \\ & \text{for all } \|(\Delta \mathbf{a}_i, \Delta b_i)\| \le \epsilon_i, \end{array} \quad \Longleftarrow \quad \textbf{Robust LP}$$

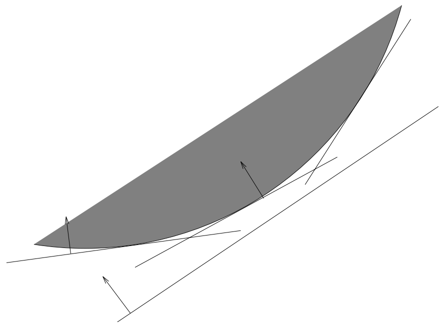
- Notice that the design variable $\mathbf{x}$ is required to satisfy the linear constraints for all small perturbations.
- Design variable $\mathbf{x}$ must then lie in a more restricted (but still convex) area.

# Robust Linear Constraint

- **Linear constraint:** $S = \{\mathbf{x} \mid \mathbf{a}^T\mathbf{x} \geq b\}$ represents a half space.
- **Robust linear constraint:**

$$\bar{S} = \{\mathbf{x} \mid (\mathbf{a} + \Delta\mathbf{a})^T\mathbf{x} \geq b + \Delta b, \quad \forall \, \|(\Delta\mathbf{a}, \Delta b)\| \leq \epsilon\}$$

is seen as the intersection of infinitely many half spaces



- In fact, robust feasible region can be characterized as

$$\bar{S} = \{\mathbf{x} \mid \mathbf{a}^T\mathbf{x} - b - \epsilon\sqrt{\|\mathbf{x}\|^2 + 1} \geq 0\}$$

$\Longrightarrow$ Robust linear programming is reduced to **SOCP** (Ben-Tal and Nemirovskii).

# A More General Robust LP Model

- Consider a robust linear program (more general than Ben-Tal/Nemirovskii)

$$\begin{array}{ll} \text{maximize} & \min_{\|\Delta\mathbf{x}\|\le\delta,\|\Delta\mathbf{c}\|\le\epsilon_0}(\mathbf{c}+\Delta\mathbf{c})^T(\mathbf{x}+\Delta\mathbf{x}) \\ \text{subject to} & (\mathbf{a}_i+\Delta\mathbf{a}_i)^T(\mathbf{x}+\Delta\mathbf{x}) \ge (b_i+\Delta b_i), \\ & \text{for all } \|(\Delta\mathbf{a}_i,\Delta b_i)\| \le \epsilon_i,\ \|\Delta\mathbf{x}\| \le \delta, \end{array}$$

- Two perturbations are considered.
  - The problem data $(\{\mathbf{a}_i\},\{b_i\},\mathbf{c})$ might be affected by unpredictable perturbation (e.g., measurement error)
  - Implementation errors due to finite arithmetic; $\mathbf{x}^{actual} := \mathbf{x}^{opt} + \Delta\mathbf{x}$, $\mathbf{x}^{opt}$ optimal solution.

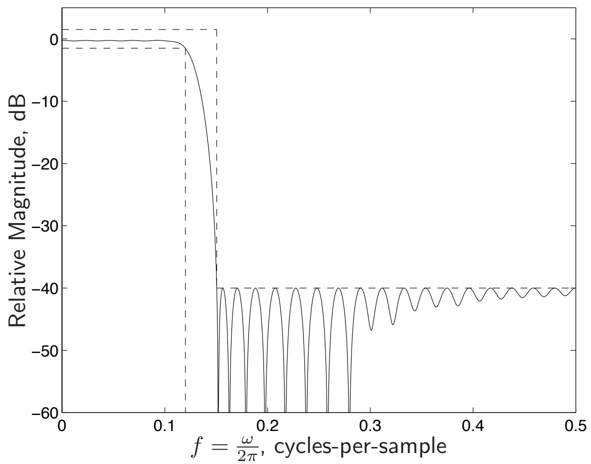
# Example: Design of a Linear Phase FIR Filter

- For a linear phase FIR filter $\mathbf{h} = (h_1 \ldots, h_n) \in \Re^n$, the frequency response is

$$H(e^{j\omega}) = e^{-jn\omega}(h_1 + h_2 \cos\omega + \cdots + h_n \cos(n\omega)) = e^{-jn\omega}(\mathbf{cos}\ \omega)^T \mathbf{h}.$$

- **Spectral envelope constraint:**

$$L(e^{-j\omega}) \leq (\mathbf{cos}\ \omega)^T \mathbf{h} \leq U(e^{-j\omega}), \quad \forall \omega \in [0, \pi] \qquad (6)$$



# Example: Design of a Linear Phase FIR Filter

- Finding a discrete $\mathbf{h}$ (say, 4-bit integer) satisfying (6) is NP-hard.
- Ignoring discrete structure of $\mathbf{h}$, we can find a $\mathbf{h}$ satisfying (6) in poly. time. However, rounding such solution to the nearest discrete $\mathbf{h}$ may degrade performance significantly.
- Strategy:

(a) discretize the frequency $[0, \pi]$;

(b) find a solution robust to discretization and rounding errors.

- This leads to the following notion of robustly feasible solution:

$$L(e^{-j\omega_i}) \le (\mathbf{cos}\ \omega_i + \Delta_i)^T(\mathbf{h} + \Delta\mathbf{h}) \le U(e^{-j\omega_i}),$$
$$\text{for all } \|\Delta_i\| \le \epsilon,\ \|\Delta\mathbf{h}\| \le \delta.$$

- $\Delta_i$ accounts for discretization error, while $\Delta\mathbf{h}$ takes care of rounding errors.

# Robustly Feasible Solution

Definition $\Rightarrow$

$$(\mathbf{a}_i + \Delta \mathbf{a}_i)^T (\mathbf{x} + \Delta \mathbf{x}) \geq (\mathbf{b}_i + \Delta \mathbf{b}_i),$$

for all $\|(\Delta \mathbf{a}_i, \Delta \mathbf{b}_i)\| \leq \epsilon_i, \ \|\Delta \mathbf{x}\| \leq \delta, \ \forall i$

$$\Updownarrow$$

$$\mathbf{a}_i^T (\mathbf{x} + \Delta \mathbf{x}) - \mathbf{b}_i - \epsilon_i \sqrt{\|\mathbf{x} + \Delta \mathbf{x}\|^2 + 1} \geq 0, \quad \forall \|\Delta \mathbf{x}\| \leq \delta, \ \forall i$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{I} & \sqrt{\epsilon_i} \begin{bmatrix} \mathbf{x} + \Delta \mathbf{x} \\ 1 \end{bmatrix} \\ \sqrt{\epsilon_i} \left[ (\mathbf{x} + \Delta \mathbf{x})^T \ 1 \right] & \mathbf{a}_i^T (\mathbf{x} + \Delta \mathbf{x}) - \mathbf{b}_i \end{bmatrix} \succeq \mathbf{0}, \ \forall \ \|\Delta \mathbf{x}\| \leq \delta, \ \forall i$$

# Robustly Feasible Solution: LMI formulation

Using our early results, we obtain the following equivalent LMI characterization of robustly feasible solution

$$\begin{bmatrix} \mathbf{I} & \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} & \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \\ \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T & \mathbf{a}_i^T\mathbf{x} - \mathbf{b}_i - \mu_i\delta & \frac{1}{2}\mathbf{a}_i^T \\ \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}^T & \frac{1}{2}\mathbf{a}_i & \mu_i\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, \quad \mu_i \geq 0, \quad \forall i.$$

# Robust Linear Programming: Continued

As for the objective, we introduce a new variable $t$ and a new constraint

$$\boxed{t - (\mathbf{c} + \Delta\mathbf{c})^T(\mathbf{x} + \Delta\mathbf{x}) \geq 0, \quad \forall\ \|\Delta\mathbf{c}\| \leq \epsilon_0,\ \ \|\Delta\mathbf{x}\| \leq \delta}$$

$$\Updownarrow$$

$$\boxed{t - \mathbf{c}^T(\mathbf{x} + \Delta\mathbf{x}) \geq \epsilon_0\|\mathbf{x} + \Delta\mathbf{x}\|, \quad \forall\ \|\Delta\mathbf{x}\| \leq \delta}$$

$$\Updownarrow$$

$$\boxed{\begin{bmatrix} \mathbf{I} & \sqrt{\epsilon_0}\mathbf{x} + \Delta\mathbf{x} \\ \sqrt{\epsilon_0}(\mathbf{x} + \Delta\mathbf{x})^T & t - \mathbf{c}^T(\mathbf{x} + \Delta\mathbf{x}) \end{bmatrix} \succeq \mathbf{0} \quad \forall\ \|\Delta\mathbf{x}\| \leq \delta}$$

$$\Updownarrow$$

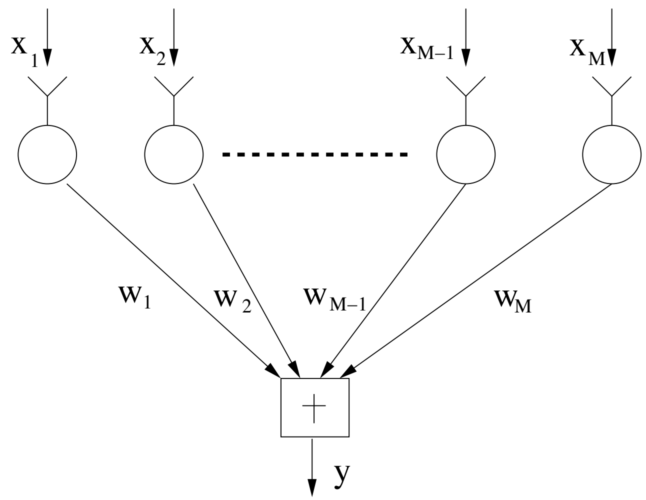
$$\boxed{\begin{bmatrix} \mathbf{I} & \sqrt{\epsilon_0}\mathbf{x}, & \sqrt{\epsilon_0}\mathbf{I} \\ \sqrt{\epsilon_0}\mathbf{x}^T & t - \mathbf{c}^T\mathbf{x} - \mu_0\delta & -\frac{1}{2}\mathbf{c}^T \\ \sqrt{\epsilon_0}\mathbf{I} & -\frac{1}{2}\mathbf{c} & \mu_0\mathbf{I} \end{bmatrix} \succeq \mathbf{0},\ \ \mu_0 \geq 0.}$$

# Robust LP: LMI formulation

$$
\begin{array}{ll}
\min & t \\
\text{s.t.} & \begin{bmatrix} \mathbf{I} & \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} & \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \\ \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{x}^T & 1 \end{bmatrix} & \mathbf{a}_i^T\mathbf{x} - \mathbf{b}_i - \mu_i\delta & \frac{1}{2}\mathbf{a}_i^T \\ \sqrt{\epsilon_i}\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} & \frac{1}{2}\mathbf{a}_i & \mu_i\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, \;\; \mu_i \geq 0, \;\; \forall i, \\
& \begin{bmatrix} \mathbf{I} & \sqrt{\epsilon_0}\mathbf{x}, & \sqrt{\epsilon_0}\mathbf{I} \\ \sqrt{\epsilon_0}\mathbf{x}^T & t - \mathbf{c}^T\mathbf{x} - \mu_0\delta & -\frac{1}{2}\mathbf{c}^T \\ \sqrt{\epsilon_0}\mathbf{I} & -\frac{1}{2}\mathbf{c} & \mu_0\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, \;\; \mu_0 \geq 0.
\end{array}
$$

Remark: This extends the work of Ben-Tal and Nemirovskii.

# Robust Beamforming Application




- Widely used in wireless communications, microphone array speech processing, radar, sonar, medical imaging, radio astronomy.
- The output of a narrowband beamformer is given by

$$y(k) = \mathbf{w}^H \mathbf{x}(k) \tag{7}$$

where $k$ is the sample index.

# Robust Beamforming

- The observation vector is given by

$$\mathbf{x}(k) = \mathbf{s}(k) + \mathbf{i}(k) + \mathbf{n}(k) = s(k)\mathbf{a} + \mathbf{i}(k) + \mathbf{n}(k) \tag{8}$$

  where $\mathbf{s}(k)$, $\mathbf{i}(k)$, and $\mathbf{n}(k)$ are the desired signal, interference, and noise components, respectively. Here, $s(k)$ is the signal waveform, and $\mathbf{a}$ is the signal steering vector.

- The robustness of a beamformer to a mismatch between the nominal (presumed) and real signal steering vectors becomes the main issue.
- Such mismatches can occur in practical situations as a consequence of look direction and signal pointing errors, imperfect array calibration and distorted antenna shape, array manifold mismodeling due to source wavefront distortions caused by environmental inhomogeneities, near-far problem, source spreading and local scattering.

# Robust Beamforming

- The optimal weight vector can be obtained through the maximization of the Signal-to-Interference-plus-Noise Ratio (SINR)

$$\text{SINR} = \frac{\sigma_s^2 \, |\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \mathbf{R}_{\mathrm{i+n}} \mathbf{w}} \tag{9}$$

  where $\mathbf{R}_{\mathrm{i+n}} = \mathrm{E}\left\{ (\mathbf{i}(t) + \mathbf{n}(t)) \, (\mathbf{i}(t) + \mathbf{n}(t))^H \right\}$.

- The maximization of (9) is equivalent to

$$\text{minimize}_{\mathbf{w}} \; \mathbf{w}^H \mathbf{R}_{\mathrm{i+n}} \mathbf{w} \qquad \text{subject to} \qquad \mathbf{w}^H \mathbf{a} = 1 \tag{10}$$

- The optimal weight vector is

$$\mathbf{w}_{\mathrm{opt}} = \alpha \, \mathbf{R}_{\mathrm{i+n}}^{-1} \, \mathbf{a} \tag{11}$$

  where $\alpha = \left(\mathbf{a}^H \mathbf{R}_{\mathrm{i+n}}^{-1} \mathbf{a}\right)^{-1}$ is the normalization constant (to be omitted for brevity).

# Robust Beamforming

- In practical applications, the exact signal-plus-noise covariance matrix $\mathbf{R}_{\mathrm{i+n}}$ is unavailable. Therefore, the sample covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}(n)\mathbf{x}(n)^H \tag{12}$$

  should be used. Here, $N$ is the training sample size.
- In this case, the problem (10) should be rewritten as

$$\text{minimize}_{\mathbf{w}}\ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \qquad \text{subject to} \qquad \mathbf{w}^H \mathbf{a} = 1. \tag{13}$$

# Robust Beamforming

- In practical applications, the steering vector distortions $\mathbf{e}$ can be bounded:

$$\|\mathbf{e}\| \leq \epsilon.$$

- Then, the actual signal steering vector belongs to the set

$$\mathcal{A}(\epsilon) = \{\mathbf{c} \mid \mathbf{c} = \mathbf{a} + \mathbf{e}\,,\ \|\mathbf{e}\| \leq \epsilon\}$$

- We impose a constraint that for all vectors in $\mathcal{A}(\epsilon)$, the array response should not be smaller than one, i.e.

$$|\mathbf{w}^H \mathbf{c}| \geq 1 \qquad \text{for all} \qquad \mathbf{c} \in \mathcal{A}(\epsilon)$$

# Formulation

- The robust formulation of adaptive beamformer is

$$\begin{array}{lll} \text{minimize}_{\mathbf{w}} & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} & \\ \text{subject to} & |\mathbf{w}^H(\mathbf{a}+\mathbf{e})| \geq 1, & \text{for all} \quad \|\mathbf{e}\| \leq \epsilon. \end{array}$$

- For each choice of $\mathbf{e}$, the condition $|\mathbf{w}^H(\mathbf{a}+\mathbf{e})| \geq 1$ represents a nonlinear and nonconvex constraint on $\mathbf{w}$.
- Since there are an infinite number of vectors $\mathbf{e}$ with $\|\mathbf{e}\| \leq \epsilon$, the robust beamforming problem is a semi-infinite nonconvex quadratic program.
- It is well known that the general nonconvex quadratically constrained quadratic programming problem is NP-hard and thus intractable.

# Convex Reformulation

- However, due to the special structure of the objective function and the constraints, the robust beamforming problem can be reformulated, surprisingly, as a convex second order cone program:

$$\begin{array}{ll} \text{minimize}_{\mathbf{w}} & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ \text{subject to} & \mathbf{w}^H \mathbf{a} \geq \epsilon \|\mathbf{w}\| + 1\,,\ \mathrm{Im}\left\{\mathbf{w}^H \mathbf{a}\right\} = 0. \end{array}$$

[**Vorobyov-Gershman-Luo, 2001**]

- The reformulation is based on S-Procedure type results and the homogeneous nature of the objective function.
- The above SOCP can be efficiently and easily solved via interior point method.
- Similar work has been done recently independently by Boyd's group.

# Simulation Examples

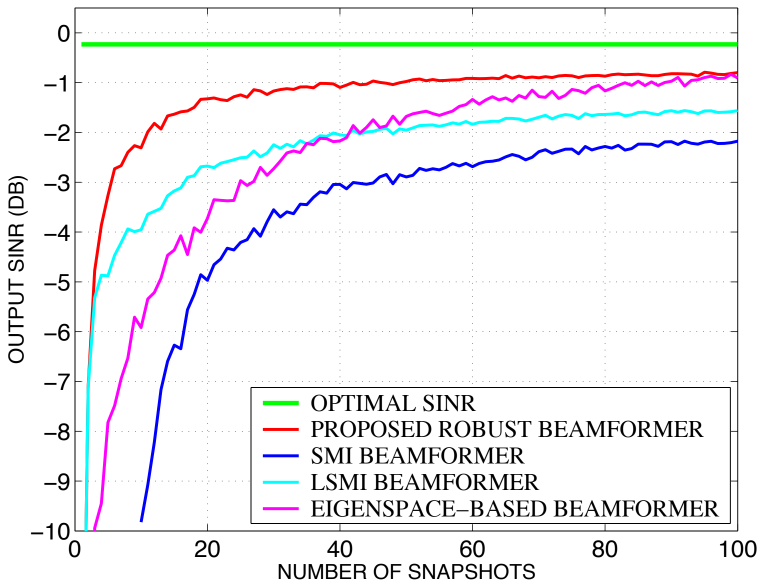
**Example 1: Steering vector mismatch due to local scattering**

- In this example, the presumed signal steering vector is a plane wave impinging on the array from $3^\circ$.
- The real steering vector is formed by five signal paths and is given by

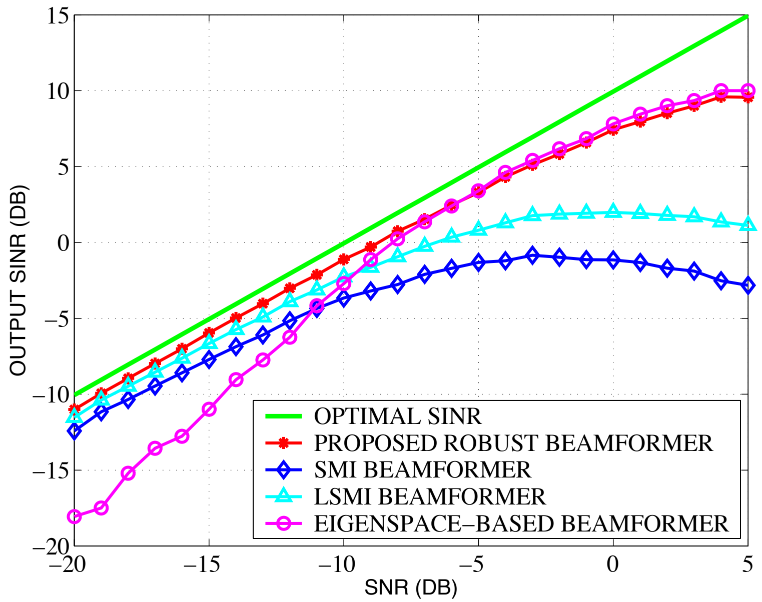
$$\tilde{\boldsymbol{a}} = \boldsymbol{a} + \sum_{i=1}^{4} e^{j\psi_i} \boldsymbol{b}(\theta_i) \tag{14}$$

  where $\boldsymbol{a}$ corresponds to the direct path, whereas $\boldsymbol{b}(\theta_i)$ $(i = 1, 2, 3, 4)$ correspond to the coherently scattered paths, with $\theta_i$, $i = 1, 2, 3, 4$ independently drawn.
- The phases $\psi_i$, $i = 1, 2, 3, 4$ are independently and uniformly drawn from the interval $[0, 2\pi]$.

# Performance Comparison




(a) output SINR v.s. sample size $N$; 1st example



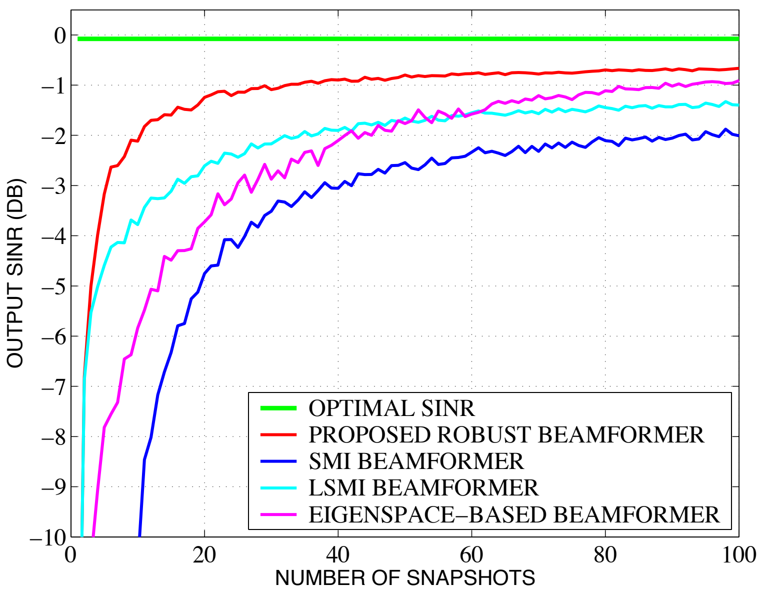

(b) output SINR versus SNR; 1st example

# Simulation Examples

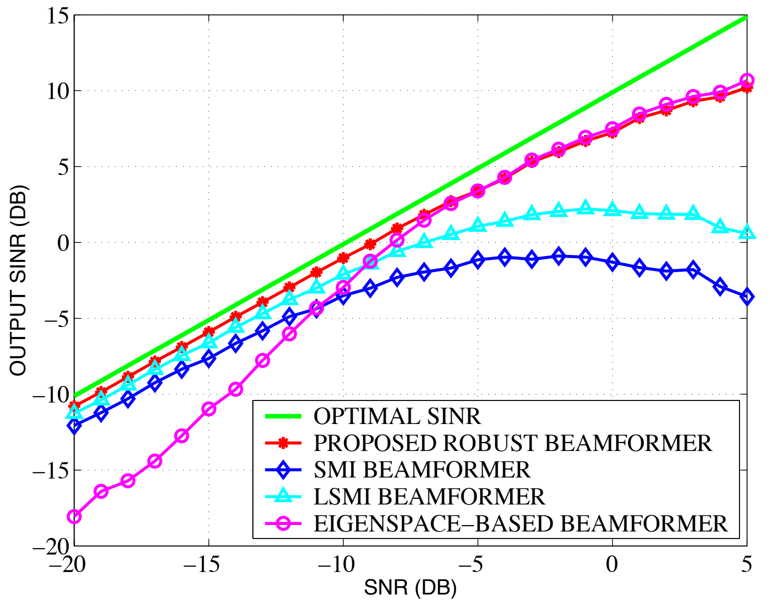
**Example 2: Near-far steering vector mismatch**

- In this example, we model the so-called near-far steering vector mismatch of the desired signal, whereby the presume steering vector of the signal is a plane wave impinging on the array from the normal direction $0^\circ$, whereas the real steering vector corresponds to the source located in the near field of the antenna at the distance $D^2/\lambda = (M-1)^2\lambda/4$ from the geometrical center of the array, where $D = (M-1)\lambda/2$ is the length of array aperture.
- The performance of the methods tested versus the number of training snapshots $N$ for the fixed $\mathrm{SNR} = -10$ dB is shown in Fig. (c). Fig. (d) shows the performance of these techniques versus SNR for the fixed training data size $N = 30$.

# Performance Comparison




(c) output SINR v.s. sample size $N$; 2nd example




(d) output SINR versus SNR; 2nd example

# Concluding Remarks

Our work is on-going in two fronts: theoretical and application, both are nontrivial.

1. We provided a summary of our recent progress on the theoretical front:
   - Various extensions of the well known S-Procedure to the matrix setting
   - Characterization of robust QMI over the Stiefel manifold
2. In terms of applications:
   - Robust linear programming with data and rounding errors
   - Robust beamforming: seemingly nonconvex and difficult $\Rightarrow$ convex SOC
   - Other applications we have successfully pursued:
     channel equalization, transmitter-receiver design, robust Kalman filtering, ...

# Concluding Remarks

- Good news: Recent advances in convex optimization research (LMI, SDP, SOC, interior point methods, robust optimization,...) are beginning to to find exciting applications in digital signal processing and communications, giving powerful new modeling and computational tools to solve previously considered intractable problems. Previously used tools in these engineering disciplines are Linear Least Squares, Gradient Descent.

# Thank You!

`http://www.ece.mcmaster.ca/~luozq`