"Very Full of Symbols": Duncan F. Gregory, the Calculus of Operations, and the *Cambridge Mathematical Journal*

Sloan Evans Despeaux

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and Computer Science

THE

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MATHEMATICAL JOURNAL.

VOL. I.

Νήπιοι, οὐδε ἴσασιν ὅσῳ πλέον ἥμισυ παντός.

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1839.

"You should not forget the Cambridge 'Mathematical Journal'. It is done by the younger men… It is full of very original contributions. It is, as is natural in the doings of young mathematicians, very full of symbols. The late Dr F. Gregory, whom you must notice most honourably… gave his extensions of the Calculus of Operations… in it. He was the first editor. He was the most rising man among the juniors."

--Augustus De Morgan to
John Herschel, 1845

Augustus De Morgan (1806-1871)

7

PHILOSOPHICAL TRANSACTIONS:

GIVING SOME

ACCOMPT

OF THE PRESENT

Undertakings, Studies, and Labours

OF THE

INGENIOUS

IN MANY

CONSIDERABLE PARTS

OF THE

WORLD

Vol I.

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THE

PHILOSOPHICAL MAGAZINE,

OR

ANNALS

OF

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BY

RICHARD TAYLOR, F.S.A. L.S. G.S. M. Astr. S. &c.

AND

RICHARD PHILLIPS, F.R.S. L. & E. F.L.S. &c.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Monit. Polit.* lib. i. cap. 1.

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PART III.

MATHEMATICAL MEMOIRS,

Extracted from Works of Eminence.

THE

MATHEMATICAL REPOSITORY

MATHEMATICAL QUESTIONS.

To be answered in Number XXII.

I. QUESTION 491, *by the Rev. Mr.* E. C. TYSON.

A bent lever of which the arms are a and b and the angle θ makes small oscillations in its own plane; find the time of one vibration?

II. QUESTION 492, *by the Rev. Mr.* TYSON.

Sum n terms of the series

$$1^2 . 2^2 - 2^2 . 3^2 + 3^2 . 4^2 - 4^2 . 5^2 + \&c.$$

III. QUESTION 493, *by the Rev. Mr.* TYSON.

Having given the focal length of a convex lens, whereby a person may see distinctly at a given distance, to find the length of another lens, which will enable the same person to see distinctly at another given distance.

IV. QUESTION 494, *by Mr.* JAMES CUNLIFFE.

Suppose a body revolves in a given circle, the centre of force being a given point in the periphery; it is required to compare its velocity, ar any given point with that of a body revolving about the centre of the same, (or an equal) circle.

V. QUESTION 495, *by Mr.* CUNLIFFE.

Find the sum of the infinite series

$$\frac{3^2}{4^2 . 6} + \frac{3^2 . 5^2}{4^2 . 6^2 . 8} + \frac{3^2 . 5^2 . 7^2}{4^2 . 6^2 . 8^2 . 10} + \&c.$$

VI. QUESTION 496, *by Mr.* SAMUEL JONES, *Liverpool.*

Take any two poits in the opposite sides of a given trapezium, and from each of these draw two straight lines to the opposite angles; then the two points where these lines intersect and the point of intersection of the diagonals are in a straight line. Required the demonstration.

VII. QUESTION 497, *by* R. N.

Find the sum of the series

$$1 + 1 + 2 + 3 + 5 + 8 + \&c.$$

any term being the sum of the two preceding terms.

VIII. QUESTION 498, *by* R. N.

At page 84, Whewell's Dynamics, it is stated that, "The semicubical parabola with its axis vertical is a curve, the time of a body's falling down which, can be found." It is required to find the time, if possible, in finite terms.

IX. QUESTION 499, *by Mr.* CUNLIFFE.

Suppose a small body or pellet of lead at a given point, in a perfectly smooth narrow tube, of a given length, and suppose the tube to be whirled about one of its ends as a centre, with a given angular celerity, in a horizontal plane: It is required to determine the velocity and direction of the pellet when it leaves the tube; the motion being solely generated by the revolving of the tube.

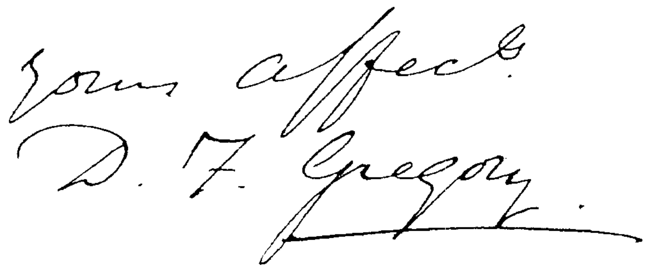
X. QUESTION 500, *by* R. N.

The plane of the paper representing the horizon of a place at any time, and the positions of the earth, sun, and moon being given, it is required to find the equation to the straight line which is the intersection to the two planes passing through the moon's centre perpendicular to the lines joining the sun and moon and the earth and moon.

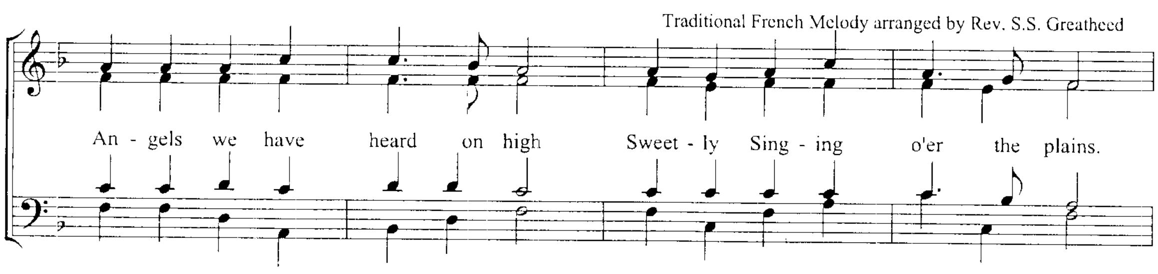
XI. QUESTION 501, *by Mr.* THOMAS STEPHENS DAVIES, *Bath.*

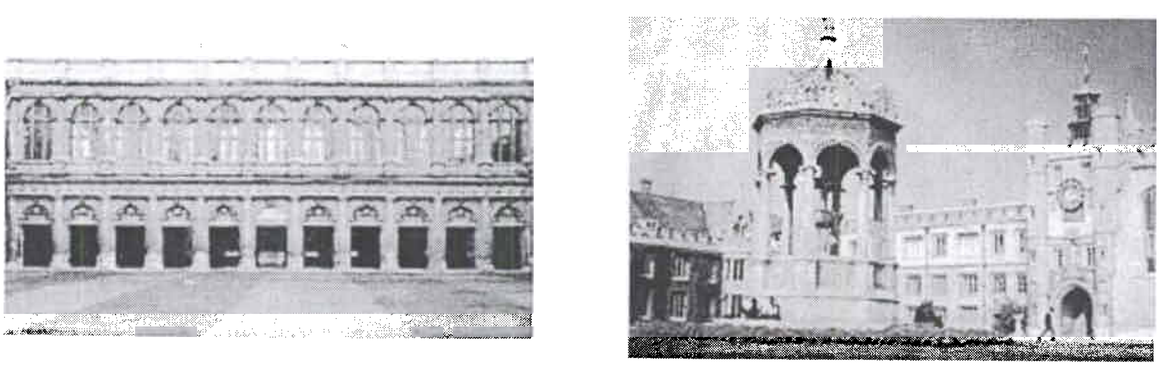
Lines drawn from the angles of a triangle to the points of contact of the inscribed circle meet in the same point (G), as is well known: lines drawn to th

Archibald Smith
1813-1872



Angels we have heard on high





Trinity College, Cambridge

PREFACE.

In bringing before the Public the First Number of *The Cambridge Mathematical Journal*, it will naturally be expected that we should say a few words, in the way of preface, on the objects we have in view, and the means we intend to adopt for carrying them into effect.

It has been a subject of regret with many persons, that no proper channel existed, either in this University or elsewhere in this country, for the publication of papers on Mathematical subjects, which did not appear to be of sufficient importance to be inserted in the Transactions of any of the Scientific Societies; the two Philosophical Journals which do exist having their pages generally devoted to physical subjects. In this place in particular, where the Mathematics are so generally cultivated, it might be expected that there would be an opening for a work exclusively devoted to that science, which does not command much interest in the world at large. We think that there can be no doubt that there are many persons here who are both able and willing to communicate much valuable matter to a Mathematical periodical, while the very existence of such a work is likely to draw out others, and make them direct their attention in some degree to original research. Our primary object, then, is to supply a means of publication for original papers.

But we conceive that our journal may likewise be rendered useful in another way—by publishing abstracts of important and interesting papers that have appeared in the memoirs of foreign academies, and in works not easily accessible to the generality of students. We hope in this way to keep our readers, as it were, on a level with the progressive state of Mathematical science, and so lead them to feel a greater interest in the study of it. For this purpose we

The Calculus of Operations

The "exponent" of $\frac{d}{dx}$, which records how many times differentiation occurs, and the "exponent" on Δ, which records the order of a finite difference, obey the same laws of "exponentiation" that govern a variable x or a function f:

$$\frac{d^a}{dx^a}\left(\frac{d^b y}{dx^b}\right) = \frac{d^{a+b} y}{dx^{a+b}}$$

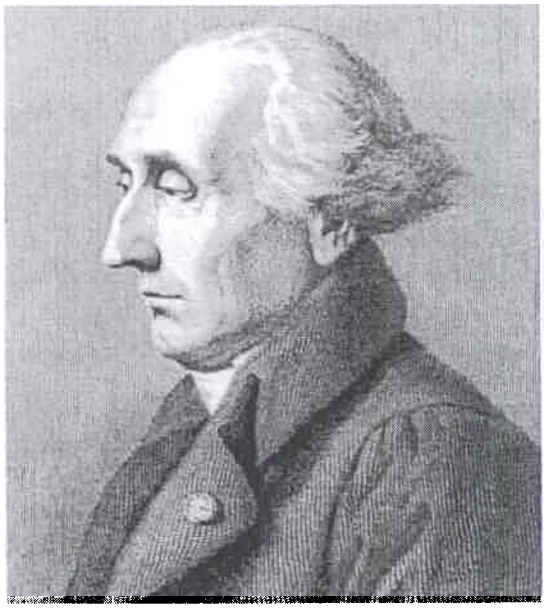
$$\Delta^n \Delta^m f(x) = \Delta^{n+m} f(x)$$

$$[\text{where } \Delta f(x) = f(x+h) - f(x)$$

$$\text{and } \Delta^m f(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(x+jh)]$$

$$x^n \cdot x^m = x^{n+m}$$

$$f^n(f^m(x)) = f^{n+m}(x)$$



Joseph-Louis Lagrange (1736-1813)

Joseph Liouville (1809-1882)

Simon Laplace 1749-1827

Jean Baptiste Joseph Fourier (1768-1830)

infiniment petites, qui s'ajoute à la somme d'une série ordonnée suivant les puissances ascendantes de l'accroissement de la variable. Or, pour obtenir cette formule, il suffit de recourir à l'équation symbolique qui existe entre les lettres caractéristiques Δ, D, dont l'une sert à indiquer une différence finie, l'autre une fonction dérivée; et de développer $\frac{1}{\Delta}$ suivant les puissances ascendantes de D. Il y a plus : on pourra développer pareillement, suivant les puissances ascendantes de la lettre D, une fonction rationnelle symbolique qui aurait pour numérateur l'unité et pour dénominateur une fonction entière de Δ. Enfin on pourra décomposer une fraction rationnelle de cette espèce en fractions simples dont chacune ait pour dénominateur une fonction linéaire de D. Les formules obtenues, comme on vient de le dire, pourront servir à développer l'intégrale d'une équation linéaire aux différences finies, qui aura pour second membre une fonction donnée de la variable, en une série dont chaque terme sera proportionnel, ou à l'une des dérivées de cette fonction, ou à l'intégrale d'une équation différentielle linéaire du premier ordre. Toutefois, ces formules, ainsi déduites d'une équation symbolique, ne pourront encore être considérées comme rigoureusement établies, la méthode qui les aura fait découvrir n'étant en réalité qu'une méthode d'induction, et l'on doit même observer que cette méthode ne parait nullement propre à faire connaître dans quel cas chaque série sera convergente, et sous quelles conditions chaque formule subsistera. Or ces dernières questions se résoudront assez facilement, dans beaucoup de cas, à l'aide des considérations suivantes.

D'abord, pour obtenir les règles de la convergence des séries, il suffira souvent de recourir à deux théorèmes que j'ai démontrés, l'un dans l'*Analyse algébrique*, page 143 [1], l'autre dans le Mémoire de 1831 sur la Mécanique céleste [2]. A l'aide de ces deux théorèmes, on prouvera aisément, par exemple, que la série donnée par Maclaurin

[1] *OEuvres de Cauchy*, S. II, T. III.
[2] *Ibid.*, S. II, T. XV.

Joseph-Louis Lagrange (1736-1813)

Analytical Society Members

John Frederick William Herschel
(1792-1871)

Charles Babbage
(1791-1871)

George Peacock
(1791-1858)

$$y = ax + \sqrt{a^2 a^2 - b^2},$$

and those of the parabola from the equation

$$y = ax + \frac{m}{a}.$$

But we need do no more than indicate this, as the method is the same as in the ellipse.

O.

III.—ON GENERAL DIFFERENTIATION.

The idea of differential coefficients with general indices is not modern, for it occurred to Leibnitz, who has expressed it in his correspondence with Jean Bernouilli. Euler has written a few pages on this subject, which Lacroix has copied into his large work on the differential calculus. Formulæ for expressing the general differential coefficients of functions by means of definite integrals, have been given by Laplace (*Théorie des Probabilités*, p. 85, 3rd. edit.), by Fourier (*Théorie de la Chaleur*, p. 561), and by Mr. Murphy (*Cambridge Phil. Trans.*, vol. 5.). But it appears that the only person who has attempted to reduce the subject to a system, is M. Joseph Liouville; three memoirs by whom,—one on the principles of the calculus, and two on applications of it,—are inserted in the 13th volume of the *Journal de l'Ecole Polytechnique*, for the year 1832. Professor Peacock, in his valuable and interesting Report on certain branches of Analysis, which forms a part of the *Report of the British Association for* 1833, has spoken of M. Liouville's system as erroneous in many essential points, and has given a sketch of one very different. But after referring to M. Liouville's memoirs, and bestowing considerable attention on the subject, we have come to a contrary opinion, at least with respect to his conclusions, which are the same for the most part as will be found in this article. Some points in his theory we admit to be objectionable, and these we have altered.

2. The transition from differential coefficients whose indices are positive integers, to those whose indices are any whatever, should be made in the same manner as the transition in algebra, from symbols of quantity with positive integral indices to those with general indices. Before we can prove any equations involving $\frac{d^\alpha u}{dx^\alpha}$, where α is general, we must affix a meaning to that expression, which can only be done by making some definition or assumption

Greatheed's Definitions for General Exponents of Differentiation *Cambridge Mathematical Journal* vol. 1 (1837):

$$\frac{d^\alpha(u+v)}{dx^\alpha} = \frac{d^\alpha u}{dx^\alpha} + \frac{d^\alpha v}{dx^\alpha}$$

$$\frac{d^\alpha}{dx^\alpha} \cdot \frac{d^\beta}{dx^\beta} u = \frac{d^{\alpha+\beta} u}{dx^{\alpha+\beta}}$$

$$\frac{d^\alpha}{dx^\alpha} \cdot \frac{d^\beta u}{dx^\beta} = \frac{d^\beta}{dx^\beta} \cdot \frac{d^\alpha u}{dx^\alpha}$$

If the point were outside of the triangle, we should have to subtract one of the terms, such as $c\,z$, so that the resulting equation would be

$$\frac{x}{p} + \frac{y}{q} - \frac{z}{r} = 1.$$

Now let ρ be the radius of the inscribed circle, then taking the centre of this circle as the given point, we have $x = y = z = \rho$, and consequently

$$\frac{1}{\rho} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Again, let ρ_1, ρ_2, ρ_3 be the radii of the circles which touch one of the sides of the triangle externally, and the other two internally; then we shall have, by similar reasoning,

$$\frac{1}{\rho_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r}$$

$$\frac{1}{\rho_2} = \frac{1}{p} - \frac{1}{q} + \frac{1}{r}$$

$$\frac{1}{\rho_3} = -\frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Adding these equations together, we get

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{\rho}.$$

It is obvious that similar theorems hold good for any tetrahedron, but it is needless to do more than indicate them.

W. W.

V.—ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

THE following method of integrating linear differential equations deserves attention, not only as leading readily to the solution of these equations, but also as placing their theory in a clear light, and pointing out the cause of the success of the method usually employed.

M. Brisson appears to have been the first person who applied the principle of the separation of the signs of operation from those of quantity to the solution of differential equations. This he did in two memoirs of the dates of 1821 and 1823, but we have not been fortunate enough to meet with them, (if indeed they have been published), and our knowledge of them is derived from a

Gregory's "On the Solution of Linear Differential Equations with Constant Coefficients"
***Cambridge Mathematical Journal* vol. 1 (1837):**

"If we take the general linear equation with constant coefficients

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + ... + R\frac{dy}{dx} + Sy = X$$

When X is any function of x, and separate the signs of operation from those of quantity, it becomes

$$(\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + ... + R\frac{d}{dx} + S)y = X$$

The quantity within the brackets involving only constants, and the signs of operation may be considered as one operation performed on y, and it may be represented by

$$f(\frac{d}{dx})y = X$$

Here y is given at once explicitly if we are able to perform the inverse operation of $f(\frac{d}{dx})$. For if we ... perform that operation on both sides, we get

$$\{f(\frac{d}{dx})\}^{-1} \cdot f(\frac{d}{dx})y = \{f(\frac{d}{dx})\}^{-1}X$$

$$\text{or,} \quad y = \{f(\frac{d}{dx})\}^{-1}X."$$

Gregory's "On the Solution of Linear Differential Equations with Constant Coefficients" *Cambridge Mathematical Journal* vol. 1 (1837):

"...at first sight it would appear as if the principles on which the method is founded, were drawn only from analogy. But a little consideration will show that this is not really the case, and that the reasoning on which we proceed is perfectly strict and logical...we might with propriety call ... [symbols of quantity] also symbols of operation. For instance, x is the operation designated by (x) performed on unity, x^n is the same operation performed n times in succession on unity, $a + x$ is the operation $(a + x)$ performed on unity, $(a + x)^n$ is the operation $(a + x)$ performed n times in succession on unity...

If then, we take this view of what are usually called symbols of quantity, we shall have little difficulty in seeing the correctness of the principle by which other operations, such as we represent by $(\frac{d}{dx})$, (Δ), &c., are treated in the same way as a, b, &c. For whatever is proved of the latter symbols, from the known laws of combination, must be equally true of all other symbols which are subject to the same laws of combination."

And, if f, f_1, &c. be any other general symbols of operation (f and f_1 being of the same kind) subject to the same laws of combination, so that

$$f^m . f^n (x) = f^{(m+n)} (x). \quad \ldots\ldots\ldots(1),$$
$$f \{f_1 (x)\} = f_1 \{f(x)\} \quad \ldots\ldots\ldots\ldots(2),$$
$$\text{and } f(x) + f(y) = f(x + y) \ldots\ldots\ldots(3).$$

Then, whatever we may have proved of a, b, &c. depending on these three laws, must necessarily be equally true of f, f_1, &c.

Now we know that the symbol (d) is subject to these laws for

$$d^m . d^n (x) = d^{(m+n)} (x)$$
$$\frac{d}{dx}\left(\frac{d}{dy}(z)\right) = \frac{d}{dy}\left(\frac{d}{dx}(z)\right) \ldots\ldots(2)$$
$$d(x) + d(y) = d(x + y),$$

and the same is true for the symbol Δ.

Hence the binomial theorem (to take a particular case) which has been proved for (a) and (b) is equally true for $\left(\frac{d}{dx}\right)$ and $\left(\frac{d}{dy}\right)$: so that we require no farther proof for the proposition, that when u is a function of two independent variables x and y,

$$d^n (u) = \left(\frac{d}{dx} dx + \frac{d}{dy} dy\right)^n u = \frac{d^n u}{dx^n} dx^n + n \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dy} u\, dx^{n-1} dy + \ldots$$

But this reasoning will not apply in the case of those functions where the same laws do not hold. For instance, if we take the function log, we have not the condition

$$\log (x) + \log (y) = \log (x + y).$$

But
$$\log (x) + \log (y) = \log (xy).$$

Consequently the binomial theorem will not hold for this function, though a binomial theorem might possibly be deduced for it, if the expressions did not become so complicated as to be unmanageable.

We have as yet only considered the combinations of operations of one kind, but in the preceding pages we frequently made use of operations of different kinds together, as in the expression $\left(\frac{d}{dx} - a\right)$. Now so long as each of the operations is subject to the same laws, and that they are independent, that is to say, that the one symbol is not supposed to act on the other, the same deductions will follow as when the operations are of the same kind. Hence we assumed that as the expression

$$x^n + Ax^{n-1} + Bx^{n-2} + \text{\&c.} + S$$

can be resolved into the factors

$$(x - a_1)(x - a_2)(x - a_3) \text{ \&c.}$$

The expression

$$\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + \&c. + S$$

can be resolved into the factors

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right) \ldots\ldots \left(\frac{d}{dx} - a_n\right),$$

which is the foundation of the method we have explained.

But if we have united together such symbols as $\left(\frac{d}{dx} + x\right)$, the same result will not hold. For though (x) is an operation of the same kind as (a), yet it bears a different relation to $\left(\frac{d}{dx}\right)$, as by the nature of this last operation it affects the operation (x), so that

$$x\left(\frac{d}{dx}(z)\right) \text{ is not equal to } \frac{d}{dx}\{x(z)\},$$

or the second law of combination does not hold with regard to these symbols of operation, and, consequently, theorems for other symbols deduced from this law are not true for such symbols as $\left(\frac{d}{dx}\right)$ and (x) together. It is this peculiarity with regard to the combinations of the symbols (x) and $\frac{d}{dx}$ which gives rise to the difficulty in the solution of linear equations with variable coefficients.

Since this article was written, we have learnt that a report by *Cauchy* on *Brisson's* Memoirs, which appears to have been favourable, was rejected by the Academy of Sciences. We know not for what reason.

Γ.

VI.—SOLUTION OF TWO PROBLEMS IN ANALYTICAL GEOMETRY.

In one of the Problem papers for 1836 there is given the following problem: To draw a tangent to a curve of the second order from a point P without it. From P draw any two lines, each cutting the curve in two points. Join the points of intersection two and two, and let the points in which the joining lines (produced if necessary) cross each other be joined by a line which will in general cut the curve in two points A, B. PA, PB are tangents at A and B. This and the sum of these two digits is

$$a_{x+1} - b_{x+1} + a_x - b_x + r - 1;$$

hence $S(M - N)$ may be greater, but cannot be less, than

$$S(M) - S(N).$$

PROP. 3. The product of any m consecutive integers is divisible by $1.2.3..\, m$.

By Prop. 1, the index of p in $N!$ is $\dfrac{N - S(N)}{p-1}$, and in $(N+m)!$, $\dfrac{N+m-S(N+m)}{p-1}$; hence the index of p in

$$(N+1)(N+2)\ldots(N+m) \text{ or } \frac{(N+m)!}{N!},$$

is the difference of these quantities, or

$$\frac{m - S(N+m) + S(N)}{p-1}.$$

The index of p in $m!$ is $\dfrac{m - S(m)}{p}$; and, by Prop. 2, making

$$M = N + m, \;\; S(m) \text{ is not } < S(N+m) - S(N),$$

$$\text{therefore } m - S(m) \text{ is not } > m - S(N+m) + S(N);$$

and the index of p in $m!$ is not greater than the index of p in $(N+1)(N+2)\ldots(N+m)$. Hence, if

$$(N+1)(N+2)\ldots(N+m) = 2^a.3^b.5^c\ldots,$$

$$\text{and } 1.2.3\ldots m = 2^\alpha.3^\beta.5^\gamma\ldots,$$

α is not $> a$, β not $> b$, γ not $> c$, &c. Consequently

$$(N+1)(N+2)\ldots(N+m) \text{ is divisible by } 1.2.3\ldots m.$$

S. S. G.

IV.—NOTES ON FOURIER'S HEAT.

THE method employed by Fourier to integrate the partial differential equations which occur in the Theory of Heat, is to assume some simple form of a singular solution, and afterwards to extend it so as to include all the circumstances of the problem. It is in effecting this that he has displayed the great resources of his analysis, and imparted so great an interest to his work by the variety and ingenuity of his methods. Indeed there is a freshness and originality in the writings of Fourier which make them in no ordinary degree arrest the attention of the reader. But however much we may admire the means by which Fourier has overcome the difficulties of the problems he had to deal with, yet it seems more

VIII.—ON THE INTEGRATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.

In the present article we shall apply to Simultaneous Linear Differential Equations the principles, the applications of which we have developed in the preceding Numbers of this Journal. The usual method for solving these equations was first given by D'Alembert, and has received but little improvement since his time, although it is so long and tedious, that some change was highly desirable. The process which we shall give here is at once simple and direct, and shows the advantage of recurring frequently to the principles on which our calculations are founded. The theory of the method is sufficiently simple. Since we have shown that the symbols of differentiation are subject to the same laws of combination as those of number, they may be always treated in the same manner if the coefficients be all constants, which is the only case we shall consider. We have therefore only to separate the symbol of differentiation from its subject, and then proceed to eliminate one of the variables between the given equations, exactly as if the symbol of differentiation were an ordinary coefficient. Thus the difficulty of

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"We have... only to separate the symbol of differentiation from its subject, and then proceed to eliminate one of the variables between the given equations, exactly as if the symbol of differentiation were an ordinary coefficient. Thus the difficulty of elimination becomes reduced to that between ordinary and algebraical equations."

. . .

"We have now given a sufficient number of examples to enable the student to understand thoroughly the method, and we think that they show clearly the advantages of a process, which, to some persons, might appear to carry out to a startling extent the principles on which it is founded."

VII.—ON THE SYMPATHY OF PENDULUMS.

By the Sympathy of Pendulums is meant the effect on the motions of different pendulums produced by their mutual action, when their points of suspension have any elastic or moveable connexion.

• • •

The most general case of the influence of one pendulum on another, when the motions as we have supposed are all in the same horizontal direction and infinitesimal, will be when, calling A and B the points of support, each of these when disturbed performs vibrations in known times; and a disturbance given to A communicates a known motion to B, and *vice versâ*.

To investigate the motion in this case, let u, v be the coordinates of A and B, $u + x$, $v + y$ of the balls suspended to them; then we have the equations

• • •

If three of the quantities K are equal to zero, the vibrations of the two pendulums are isochronous, and there are therefore four modes of this isochronous vibration. In all cases the pendulums affect each other, so that none of the points oscillates in its natural time. The extreme generality of the equations we have assumed, and consequently of the solution derived from them, prevents us from interpreting our result in a more precise manner. For this purpose it would be necessary to assign some relations between the constants in the equations, but it would lead us too far if we were to attempt any such investigation; and we may add, that any particular case will in general be more easily solved by a direct reference to its own circumstances than by a reduction of the general solution.

D. G. S.

III.—ON THE MOTION OF A PENDULUM WHEN ITS POINT OF SUSPENSION IS DISTURBED.

IN a former article* we investigated the nature of the mutual action of two pendulums united by any elastic or moveable connexion; we shall here consider more particularly the effect produced on the motion of a simple pendulum by a disturbance of its point of support. As before, we shall suppose the motions to be infinitesimal, in order that the equations may be at all manageable, and also for the sake of simplicity we shall assume the disturbances of the point of suspension to be rectilinear.

I. Let the point of suspension have a horizontal motion parallel to that of the pendulum.

The pendulum being a simple one, let u be the horizontal coordinate of the point of suspension, $u + x$ of the ball; let l be the length of the pendulum, and $n^2 = \frac{g}{l}$. Then the equation for the motion of the pendulum is

$$\frac{d^2(u + x)}{dt^2} + n^2x = 0 \ldots\ldots \ldots (1).$$

Two suppositions may be made regarding the nature of the disturbances of the point of suspension, that is, regarding the motion of u: either it has a vibratory motion independent of the motion of the pendulum or depending on it.

In the first case, when the motion of u is independent of that of the pendulum, u is given simply in terms of t, or

$$u = c\cos(at + \beta) \ldots\ldots (2);$$

whence equation (1) becomes

$$\frac{a^2x}{dt^2} + n^2x - ca^2\cos(at + \beta) = 0 \ldots\ldots\ldots (3).$$

The solution of this is

$$x = A\cos(nt + B) - \frac{ca^2}{a^2 - n^2}\cos(at + \beta) \ldots (4).$$

The most important conclusion from this result is, that if $A = 0$, or the ball be originally at rest, it will have no motion communicated to it by the vertical motion of the point of suspension: and that if it has an oscillatory motion in any direction, this will not be permanently altered unless $a = 2n$, or the period of oscillation of the pendulum be double of that of the point of suspension. In this case the integral becomes infinite; and if, as before, we put it into another shape, we find that x increases continually with the time. This accords with experiment, which shows that the arc of vibration of a pendulum may be increased indefinitely by giving the point of suspension a vertical motion of oscillation, the period of which is half of that of the pendulum.

G. S.

VI.—ON THE IMPOSSIBLE LOGARITHMS OF QUANTITIES.

(*By* D. F. Gregory, B.A. *Trinity College.*)

In a Paper printed in the fourteenth volume of the *Transactions of the Royal Society of Edinburgh*, I gave a short sketch of what I conceive to be the true nature of Algebra, considered in its greatest generality; that it is the science of symbols, defined not by their nature, but by the laws of combination to which they are subject. In that paper I limited myself to a statement of the general view, without pretending to follow out all the conclusions to which such views would lead us: such an undertaking would be too extended for the limits of a memoir, and would involve a complete treatise on Algebra. It will not, however, be attempting too much to trace out, in one or two cases, some of the more important elucidations which this theory affords of several disputed and obscure points in Algebra, and therefore in the following pages I shall endeavour to point out the deductions which may be derived from the definition of the operation $+$, given in the paper above alluded to. I there stated, that we must not consider it merely as an affection of other symbols, which we call symbols of quantity, but as a distinct operation possessing certain properties peculiar to itself, and subject, like the more ordinary symbols, to be acted on by any other operations, such as the raising to powers,

IV.—DEMONSTRATIONS OF THEOREMS IN THE DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

I PROPOSE in this Article to bring together the more important of the theorems in the Differential Calculus and in the Calculus of Finite Differences, which, depending on one common principle, can be proved by the method of the separation of symbols. These theorems are usually demonstrated by induction in each particular case, which, although a method satisfactory so far as it goes, wants that generality which is desirable in Analytical Demonstrations. As the ordinary Binomial Theorem is the basis on which these theorems are founded, it will be not amiss to say a few words by way of preface regarding the extent of its application, which being said once for all, will prevent useless repetition when we treat of each particular case.

The theorem that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \&c.$$

is originally proved when a and b are numbers, and $(a+b)^n$ represents the repetition of the operation n times, implying that n is an integer number. Having the form of the expansion once suggested, it can be shown, by the method of Euler, that the same form is true when n is a fraction or negative number; in which case the left-hand side of the equation acquires different meanings. Moreover, it will be found, on examining Euler's demonstration, that it includes not only these cases, but also all those in which a, b, and n are operations subject to certain laws; for it may be seen, that in the proof no other properties are presumed than that a, b, and n are distributive and commutative functions, and that a^n, b^n are subject to the laws of index functions. These laws are,

(1) The commutative, $ab = ba$,

(2) The distributive, $c(a+b) = ca + cb$,

(3) The index law, $a^m.(a^n) = a^{m+n}$.

Now, since it can be shown that the operations both in the Differential Calculus and the Calculus of Finite Differences are subject to these laws, the Binomial Theorem may be at once assumed as true with respect to them, so that it is not necessary to repeat the demonstration of it for each case.* This being premised, I proceed to consider the particular cases of the applications of these theorems.

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* It is scarcely necessary to add, that those theorems which depend on the binomial, as the polynomial and exponential, are equally extensive, so that they too may be applied to the Differential Calculus and Calculus of Finite Differences.

Number of this Journal. I there stated that, so far as I knew, Brisson was the first person who had applied the method of the separation of symbols to the solution of differential equations. I have since found that Sir John Herschel was really the first person who did so, in a paper published in the *Philosophical Transactions* for 1816, five years before the date of Brisson's Memoir. It is much to be regretted, that neither Sir John Herschel himself, nor any other person, followed up this method, which is calculated to be of so much use in the higher analysis. Perhaps this may have arisen from the theory of the method not having been properly laid down, so that a certain degree of doubt existed as to the correctness of the principle. I trust, however, that the various developments which I have given in several articles in this Journal, of the principles of the method as well as the proofs of its utility, are sufficient for removing all doubts on this head, and that it will now be regarded as a powerful instrument in the hands of mathematicians.

D. F. G.

$$\therefore \Sigma \frac{1}{r^2} = \frac{n}{b^2}\left(1 - e^2 \cos^2 \frac{\pi}{4}\right),$$

therefore &c.

Q. E. D.

It is to be regretted that we have hardly any idea by what considerations Dr. Stewart was led to the curious theorems which bear his name. It is said, indeed, that he was engaged on geometrical porisms when he discovered them, and we are told that he would have published them under the title of porisms, but for his unwillingness to interfere with a subject which the researches of his friend, Dr. Simson, seemed to have appropriated. Whether they are in reality porismatic, is a question on which it would not be worth while to enter.

The fundamental formula of our analysis is perhaps not new; the geometrical applications which we have made of it appear to be original.

IX.—ON A METHOD OF ALGEBRAIC ELIMINATION.*

It is the object of this paper to explain a method of elimination common to algebraic equations, and also to differential equations of all orders and degrees. When applied to the former class, it will always, I believe, lead to calculations which do not differ much from those required by the method in common use; but in principle it appears to me much more simple and satisfactory than that method.

Let it be required to eliminate x from the equations

$$x^2 + px + q = 0,$$
$$x^2 + p'x + q' = 0.$$

Multiply each of the proposed equations by x, and you obtain

$$x^3 + px^2 + qx = 0,$$
$$x^3 + p'x^2 + q'x = 0.$$

These two combined with the two given equations make a system of four equations containing three quantities to be eliminated, viz. x, x^2, x^3; and they are of the first degree with respect to each of these quantities. We may therefore eliminate x, x^2, and x^3, by the rules for equations of the first degree. The result is

$$pq' - p'q + \frac{(q - q')^2}{p - p'} = 0.$$

The same result may be obtained rather more simply thus.

* From a Correspondent.

In the preceding propositions the problem of elimination seems to me to be placed in a simple and luminous point of view. The different cases treated are referred to one common principle, which seems obvious as soon as it is enunciated, and which will probably be found applicable to other classes of equations.

I shall be very glad if this contribution is seen in as favourable a light by the readers of this Journal, and is found practically useful.

A. Q. G. C.

Feb. 8th, 1841.

April 2nd, 1841.

P.S.—The preceding system of elimination was suggested to me by reading an article on Simultaneous Differential Equations, in the fourth number of this Journal. I observed that the method of separating the symbols of operation from those of quantity, employed in that article, is, so far as elimination is concerned, the same in fact as the method given by La Croix. (The method of the separation of the symbols being however applicable to none but equations of the first degree; but having the advantage, where it is applicable, of indicating at once what differentiations are necessary.) This reflection led me to observe the principle on which elimination between two differential equations depends, viz. that whereas the proposed equations contain several functions of the quantity to be eliminated, this difficulty is evaded by *forming new equations*.

The process by which these equations are formed introduces new functions; nevertheless it answers the purpose for which it is employed, because it increases the number of equations still more than that of functions.

This method seemed natural, and properly applicable to the problem; whereas the method employed in treating algebraic equations of the higher degrees had always appeared to me very unsatisfactory, and obscure in principle. The question was thus raised, whether a method similar to that used in treating differential equations might not be discovered for algebraic. This question being once asked, the answer to it was soon found; especially as the paper which had suggested these reflections pointed out an analogy between the processes of Differentiation and Multiplication.

At the time when this occurred to me, and when I sent my article to you, I was entirely ignorant that any other mathematician had been occupied with the subject, and was not aware that there was any known method of elimination between algebraic equations, except that which makes the problem depend on the method of finding the greatest common measure. From Professor Sylvester's interesting paper in your last number, and from his paper in the *Philosophical Magazine* to which you referred me, I find that that gentleman has not only anticipated me in the fundamental idea, but has likewise devised some very ingenious rules for the more expeditious, and even merely mechanical, application of it. Should you, however, think my article worth inserting,

Ever affectionately yrs
R. L. Ellis

Published by Deighton, Bell & Co. Cambridge.

Robert Leslie Ellis, "Remarks on the Distinction Between Algebraical and Functional Equations"
Cambridge Mathematical Journal vol. 3

$$x^2 + ax + b = 0 \quad \ldots\ldots (1)$$

$$\frac{dy}{dx} - x = 0 \quad \ldots\ldots (2)$$

$$\phi(mx) + x = 0 \quad \ldots\ldots (3)$$

$$\phi\phi x - x = 0 \quad \ldots\ldots (4)$$

$$\phi\frac{dy}{dx}\phi x + x = 0 \quad \ldots\ldots (5)$$

. . . .

$$\{\phi(ab)\}^2 + a\phi(ab) + b = 0 \quad \ldots\ldots (1)$$

$$\frac{d}{dx}\phi(x) - x = 0 \quad \ldots\ldots (2)$$

"The name of functional equation is not happy; it refers to the notation, and not to the essence of the thing."

From this we have

$$dr\left\{1 - \frac{r}{\sqrt{(c^2 - r^2)}}\right\} = 0.$$

This equation is satisfied either by

$$r = \sqrt{(c^2 - r^2)}, \quad \text{i.e. by } r = \frac{c}{\sqrt{2}} = r_1,$$

$$\text{or by } dr = 0, \text{ which involves } dr_1 = 0.$$

The former of these results gives the equal conjugate diameters, the sum of which is, as we know, a maximum. The latter result implies that both r and r_1 are maxima or minima, or that they are the principal axes, the sum of which is a minimum. By a different method we might have obtained the minimum instead of the maximum value of $r + r_1$, by the usual process for determining maxima and minima. For since $r^2 + r_1^2 = a^2 + b^2$ and $rr_1 \sin\theta = ab$, θ being the angle between the axes, we have

$$(r + r_1)^2 = a^2 + b^2 + \frac{2ab}{\sin\theta},$$

and hence $$\frac{d}{d\theta}(r + r_1)^2 = -\frac{2ab\cos\theta}{(\sin\theta)^2} = 0.$$

This is satisfied by $\cos\theta = 0$ or $\theta = \frac{1}{2}\pi$, implying that r and r_1 are the principal axes. In this case the maximum value of $r + r_1$ is given by $d\theta = 0$, since the equal conjugate diameters are those which make the greatest angle with each other.

G.

X.—ON THE SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS.

By D. F. GREGORY, M.A. Fellow of Trinity College.

In the fourteenth number of this Journal Mr. Leslie Ellis pointed out what appeared to him to be the essential difference between Functional Equations and those which are usually met with in the various branches of analysis. His idea is, that these classes of equations are distinguished by the *order* in which the operations are performed, so that, whereas in our ordinary equations the known operations are performed on those which are unknown, in functional equations the converse is the case, the unknown operations being performed on those which are known. As this view appears to me to be not only correct, but of very great importance for the proper understanding of the higher departments of analysis, I shall endeavour in the following pages to enforce and illustrate it.

Duncan Gregory, "On the Solution of Certain Functional Equations"
Cambridge Mathematical Journal vol. 3

"...[In the case of the solution of an ordinary differential equation] the whole difficulty lies in performing the inverse operation; and although practically the difficulty of doing so may be very great, yet it is a difficulty wholly different in *kind* from that which we meet with in trying to solve an equation in which the unknown operation is performed on that which is known [i.e., a functional equation]. Thus in the equation

$$\phi(ax) - \phi(x) = 0,$$

where the object is to determine the form of ϕ, we cannot...write

$$\phi(ax - x) = 0,$$

since the form of ϕ is unknown, and we therefore cannot assume it to be subject to the distributive law; neither can we we write

$$a\phi(x) - \phi(x) = 0,$$

since we cannot assume that ϕ and a are commutative operations."

Friday Dover

Do you know the integral $\int_0^\infty dx\, e^{-ax}\frac{\sin^2 rx}{x} = \log\left\{\frac{a^2+4r^2}{a^2}\right\}^{\frac{1}{4}}$?
It is very simply proved.

My dear Gregory,

I have to thank you both for your letter, & for a newspaper with the papers. I did not mean to say that I had no idea how ten of the morning ~~[illegible]~~ problems were to be solved, but only number (10).

I can imagine the Johnians read the journal diligently; it must have done them good. But I was pleased to see how fairly in your problem paper you kept clear of the parts of analyses to which your own taste inclines you the most, especially as I had heard it surmised that you would not be able to avoid giving them an undue preponderance.

Since I sent off my paper on the

George Boole

(1815-1864)

Duncan Gregory to George Boole,
16 February 1840:

"Your method of simplifying the solution of Linear Differential Equations with constant coefficients is exceedingly ingenious, and, I think, reduces the problem to the greatest degree of simplicity of which it admits...I do not think that the non-insertion of your paper in the Phil. Mag. was due to any other cause than this: that the editor is ignorant of mathematics, and is very unwilling to risk the publication of any mathematical communication, unless a previous knowledge of the author gives him some security for the correctness of the paper...."

Duncan Gregory to George Boole, 16 February 1840, continued:

"...I shall be very happy to get your article inserted in the journal, but I have some doubts whether the paper, as you have sent it to me, is in the best form...If it be agreeable to you I will draw up the paper in the way which I think is best fitted for publication, and will transmit [it] to you for your inspection. I shall be glad to hear that you have made progress in the solution of equations, with variable co-efficients. The question is a very difficult one, and of the highest importance, as it is in that direction that we must look for some extension of our means of analysis."

V.—ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

[By G. Boole.]

In an article in the first number of this Journal, (Vol. I. p. 22.) Mr. Gregory has applied the method of the separation of symbols to the Integration of Linear Differential Equations with Constant Coefficients. The greater part of the process is at once simple and direct, which gives this method an advantage over that of the Variation of Parameters; but the reduction of the complex inverse operations to a sum of similar simple terms by means of integration by parts, is laborious and tedious, and may be very greatly abbreviated by a method which I propose here to exhibit.

If we represent the general equation

$$\left(\frac{d^n}{dx^n} + A_1 \frac{d^{n-1}}{dx^{n-1}} + A_2 \frac{d^{n-2}}{dx^{n-2}} + \&c. + A_n\right) y = X \ldots (1),$$

by $$f\left(\frac{d}{dx}\right) y = X,$$

we deduce, as is done in the paper referred to,

$$y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X \ldots\ldots\ldots\ldots (2).$$

Now, instead of splitting the operating factors into simple binomial factors, and operating with them in succession, which renders it necessary to simplify the result by integration by parts, we may at once resolve the general operating factors into the sum of a number of simple binomial factors, exactly as in ordinary Algebra a rational fraction is decomposed into the sum of a number of simple fractions. The expression

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1},$$

is the same in form as the rational fraction

$$\{f(z)\}^{-1} = \frac{1}{z^n + A_1 z^{n-1} + A_2 z^{n-2} + \&c. + A_n}.$$

Now the method of the resolution of this into a sum of partial fractions, is independent of any properties of the variable, except the three which have been shown by Mr. Gregory (Vol. I. p. 31.) to be common to the symbol $\frac{d}{dx}$, and to the algebraical symbols generally supposed to represent numbers. Consequently the same means which enable us to determine the form of the partial fractions in ordinary Algebra, may be applied to the circumstances of the case now under consideration. This, it will be seen, is nothing more than a farther extension of the application of the principles on which the whole method of the separation of symbols is founded. It is not necessary therefore to repeat the process of reasoning by which we arrive at the conclusion, that $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ may be expanded into a sum of partial operations, the same in form as those into which $\{f(z)\}^{-1}$ may be resolved. Still less necessary is it to work out the actual result by employing the symbol $\frac{d}{dx}$ in place

Duncan Gregory to George Boole, 19 June 1843:

"I have been prevented from answering your letter by a severe attack of illness, from which I have not yet recovered. My advice certainly is, that you should endeavour to get your paper printed by the Royal Society, both because you will thereby avoid a considerable expense, and, because a paper in the 'Philosophical Transactions' is more likely to be known and read than one printed separately."

VIII. *On a General Method in Analysis.* *By* George Boole, *Esq.* *Communicated by* S. Hunter Christie, *Esq., Sec. R.S. &c.*

Received January 12th,—Read January 18th.

MUCH attention has of late been paid to a method in analysis known as the calculus of operations, or as the method of the separation of symbols. Mr. Gregory, in his Examples of the Differential and Integral Calculus, and in various papers published in the Cambridge Mathematical Journal, vols. i. and ii., has both clearly stated the principles on which the method is founded, and shown its utility by many ingenious and valuable applications. The names of M. Servois (Annales des Mathématiques, vol. v. p. 93), Mr. R. Murphy (Philosophical Transactions for 1837), Professor De Morgan, &c., should also be noticed in connection with the history of this branch of analysis. As I shall assume for granted the principles of the method, and shall have occasion to refer to various theorems established by their aid, it may be proper to make some general remarks on the subject by way of introduction.

Mr. Gregory lays down the fundamental principle of the method in these words: "There are a number of theorems in ordinary algebra, which, though apparently proved to be true only for symbols representing numbers, admit of a much more extended application. Such theorems depend only on the laws of combination to which the symbols are subject, and are therefore true for all symbols, whatever their nature may be, which are subject to the same laws of combination." The laws of combination which have hitherto been recognised are the following, π and ϱ being symbols of operation, u and v subjects.

1. The commutative law, whose expression is

$$\pi\varrho u=\varrho\pi u.$$

2. The distributive law,

$$\pi(u+v)=\pi u+\pi v.$$

3. The index law,

$$\pi^m\pi^n u=\pi^{m+n}u.$$

Perhaps it might be worth while to consider whether the third law does not rather express a necessity of notation, arising from the use of general indices, than any property of the symbol π.

The above laws are obviously satisfied when π and ϱ are symbols of quantity. They are also satisfied when π and ϱ represent such symbols as $\frac{d}{dx}$, Δ, &c., in combination with each other, or with *constant* quantities. Thus,

Yours affect.
D. F. Gregory.

THE CAMBRIDGE

MATHEMATICAL JOURNAL.

VOL. IV.] NOVEMBER, 1844. [No. XXII.

I.—MEMOIR OF THE LATE D. F. GREGORY, M.A., FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

By R. LESLIE ELLIS, Esq., Fellow of Trinity College, Cambridge.

THE subject of the following memoir died in his thirty-first year. He had, nevertheless, accomplished enough not only to justify high expectations of his future progress in the science to which he had principally devoted himself, but also to entitle his name to a place in some permanent record.

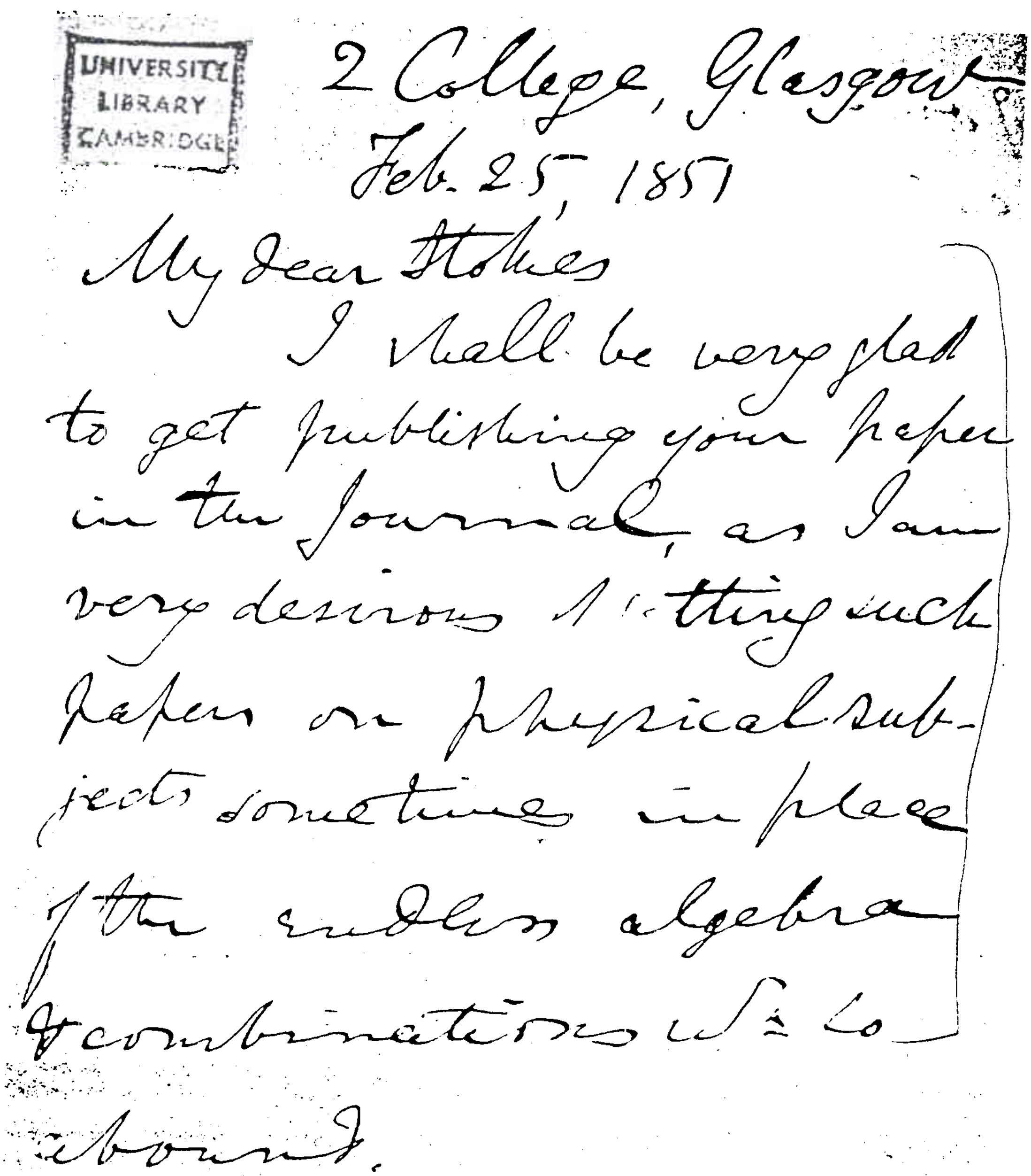
Duncan Farquharson Gregory was born at Edinburgh in April 1813. He was the youngest son of Dr. James Gregory, the distinguished professor of Medicine, and was thus of the same family as the two celebrated mathematicians James and David Gregory. The former of these, his direct ancestor, is familiarly remembered as the inventor of the telescope which bears his name; he lived in an age of great mathematicians, and was not unworthy to be their contemporary.

Of the early years of Mr. Gregory's life but little need be said. The peculiar bent of his mind towards mathematical speculations does not appear to have been perceived during his childhood; but, in the usual course of education, he shewed much facility in the acquisition of knowledge, a remarkably active and inquiring mind, and a very retentive memory. It may, perhaps, be mentioned here, that his father, whom he lost before he was seven years old, used to predict distinction for him; and was so struck with his accurate information and clear memory, that he had pleasure in conversing with him, as with an equal, on subjects of history and geography. In his case, as in many others, ingenuity in little mechanical contrivances seems to have preceded, and indicated the developement of a taste for abstract science.

Two years of his life were passed at the Edinburgh Academy; when he left it, being considered too young for the University, he went abroad and spent a winter at a pri-

"I shall be very glad to get publishing your paper in the Journal, as I am very desirous of getting such papers on physical subjects sometimes in place of the endless algebra & combinations wh.[ich] so abound."

– William Thomson to G.G. Stokes, 25 February 1851



2 College, Glasgow
Feb. 25, 1851

My dear Stokes

I shall be very glad to get publishing your paper in the Journal, as I am very desirous of getting such papers on physical subjects sometimes in place of the endless algebra & combinations wh. so abound.