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The general solution of first order functional equations in one variable

Problem I:

Let us consider the functional equation

$$\psi(x) = \psi(\alpha(x))$$

and let us assume that we know a particular solution of it:

$$f(x) = f(\alpha(x)).$$

Let φ be an arbitrary function, to create a general solution, set

$\psi := \varphi[f(x)]$

then,

$\varphi[f(x)] = \varphi[f(\alpha(x))]$.

solves the give equation.

<u>Example 1:</u>

$\psi(x) = \psi(-x)$

where $\alpha(x) = -x$. We take as a particular solution $f(x) = x^2$, then $f(x) = f(-x)$

Define $\psi(x) = \varphi(f(x)) = \varphi(x^2)$, i.e, the general solution is of the form $\varphi(x^2)$.

Problem II:

We now consider a slightly more general problem,

$$\psi(x) = \psi(\alpha(x))$$

with $\alpha : \alpha^2 = I$, i.e, $\alpha^2(x) = x$.

Assume we know a particular solution such that

$$f(x) = f(\alpha(x));$$

we construct the general solution in terms of a symmetric function

$\psi(x) := \varphi[f(x), f_l(x)]$

Since

$\psi(x) = \varphi[f(x), f_l(x)] = \varphi[f(\alpha(x)), f_l \alpha((x))]$,

it follows that $f(x) = f_l(\alpha(x))$, and $f_l(x)] = f(\alpha(x))$.

Set $x \Rightarrow \alpha(x)$ in the first of the last,

$f(\alpha(x)) = f_l(\alpha^2(x)) = f_l(x)$, which fixes f_l in terms of f. Then:

$\psi(x) = \varphi[f(x), f(\alpha(x))]$.

<u>Example 2:</u>

Consider $\psi(x) = \psi(x/(ax-1))$,

ehere $f(x) = x$ and $\alpha(x) = x/(ax-1)$,

then $\psi(x) = \varphi[x, x/(ax-1)]$.

In particular, if $a = 0$,

$\psi(x) = \psi(\alpha(x)) = \varphi[x, -x] = \varphi(x^2)$.

<u>Problem XXXIII</u>

$\psi(\alpha(x)) = d\psi(x)/dx$

with $\alpha : \alpha^2 = I$.

Replace $x \dashrightarrow \alpha(x)$:

$\psi(x) = d\psi(\alpha(x))/d\alpha(x)$.

Apply d/dx and take into account that:

$$d/dx\ \{d\psi\ (\alpha(x))/\ d\ \alpha(x)\} =$$
$$d/dx\{[d\psi\ (\alpha(x))/dx][d\ \alpha(x)/dx]^{-1}\},$$

or,

$$\psi(\alpha(x)) =$$
$$d/dx\ \{[d\psi\ (\alpha(x))/dx]\ [d\alpha(x)/dx]^{-1}\},$$

which I'll call(1)

<u>Example 3:</u>

$\psi(1/x) = d\psi(x)/dx$,

with $\alpha : \alpha^{2} = I$. Then, setting in (1)

$\psi(\alpha(x)) = z$:

$z = d/dx \{ [dz/dx] [d\, \alpha(x)/dx]^{-1} \}$,

as $d\, \alpha(x)/dx = -1/x^2$

we get

$x^2 \; z'' + 2x \; z' + z = 0$, which reduces to the form $y'' + y = 0$.

Therefore $y(x)$ is a sum of sines and cosines. Changing variables back, we get:

$\psi(x) =$
$A\, x^{1/2} \cos [(3^{1/2}/2) \log x - (\pi/3)]$,

which is the solution given by Ludwick Silberstein in

Philosophical Magazine (7) 30 (1940), p. 186, and quoted as the penultimate example on functional equations in Kamke's *Differentialgleichungen: Lösungsmethoden und Lösungen*, Leipzig, 1944, p. 660.