F.S. Macaulay – an English schoolmaster and abstract algebra

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Francis Sowerby Macaulay

11 February 1862, near Oxford,

Kingswood School, Bath,

April 1879, St John's Cambridge

June 1882, 8th – Tripos

Jan. 1883, 7th (advanced exam)

1885 St Paul's School, London.

1891 BSc in mathematics,

University of London

1897 DSc. (by exam)

1895 research career

Algebraic curves in the projective plane,

after Brill and Noether.

The Riemann-Roch Theorem:

A plane curve – the ground curve – of degree n and genus p.

Adjoint curve – of degree $n-3$, $k-1$ times through every k-fold point of the ground curve.

What is a k-fold point?

What is a k-fold intersection of two curves at a common point?

Given two plane curves

$F[x, y, z] = 0$ **and** $G[x, y, z] = 0$,

when does $h = AF + BG$ **?**

Example

$F = x^2 + yz = 0$ **and** $G = y = 0$

at $[0,0,1]$**:**

the line $x = 0$ **cannot be written**

as $AF + BG = 0$

but passes through the origin.

Brill and Noether test:

enough to look at the common

intersection points of the curves

$F[x, y, z] = 0$ **and** $G[x, y, z] = 0$**.**

Macaulay

Given: $h = AF + BG$,

given coefficients of h, F, G.

To find:

the coefficients of A **and** B.

Not enough to count the number of these coefficients and the number of equations!

Cayley-Bacharach

Cramer's paradox.

By 1904, Macaulay:

singular points with all their tangent directions distinct, then when this was not the case, generalisation of the Riemann-Roch Theorem for adjoint curves not necessarily of degree $n-3$.

A singular point 'is' a set of distinct points clustered infinitely close together.

CA Scott

'invaluable help from Miss Scott, D.Sc., Professor of Mathematics at Bryn Mawr College, Pa., U.S.A.'

Charlotte Angas Scott

8th Wrangler Cambridge 1880.

D.Sc.University of London 1885, University of Bryn Mawr, Pennsylvania in September 1885.

Simple point at the origin:

$$Ax + By = 0$$

Double point:

$A_1x^2 + A_2xy + A_3y^2 = 0$ **,**

Not of the form $AF + BG = 0$ **for suitable** F **and** G**.**

Example: $x^2 = 0$ **and** $y^2 = 0$**.**

$xy = 0$ **is not of the form**

$$Ax^2 + By^2 = 0.$$

Macaulay thanked Scott for "regarding the one-set theorem as the fundamental one, from which the whole subject is most naturally developed, . . .".

Proofs in full generality, but very long.

Noether's Theorem when $F = 0$ and $G = 0$ "have multiple points of any order and complexity, and contact of any kind."

3rd International Congress of Mathematicians, 1904 Heidelberg.

Macaulay 'The intersections of plane curves, with extensions to *n*-dimensional algebraic manifolds'.

Met Brill and Noether for the first time,

Julius König's book [1905].

Kronecker's *Grundzüge*, 1882.

Given polynomials $F_1, F_2, \ldots, F_n$ **in any number of variables, when is**

$$f = A_1F_1 + A_2F_2 + \ldots + A_nF_n \text{ ?}$$

A_i 's **polynomials.**

$$f \equiv 0(F_1, F_2, \ldots, F_n),$$

a *modular equation*.

The F_i 's **form a module.**

Modern terminology – ideal

König [1905, § 12] Noether's Theorem for *m* polynomials $F_1, \ldots, F_m$ in *m* variables, Macaulay claimed Congress [9, 1905] that he could extend König's theorem to *k* non-homogeneous polynomials in *n* independent variables.

Brill and Noether – caution.

Macaulay [10, 1913] withdrew to space curves of the principal class (complete intersections).

The failure of Noether's Theorem

Let C defined parametrically by $\left[s^4, s^3t, st^3, t^4\right]$ be a rational quartic curve in projective 3-space with coordinates $[x, y, z, w]$.

Let H be a hyperplane meeting it in four distinct points, with equation $H = 0$.

C is cut out by the quadric

$$Q(x, y, z, w) = xw - yz = 0$$

and numerous cubics,

but no other second-degree surface.

If the Brill and Noether theorem is true,

any quadric through the four points must have an equation of the form $\alpha Q + uH = 0$**, where** $u = 0$ **is a linear expression in** x, y, z, w**.**

This is a 4-dimensional family.

Quadrics in projective 3-space form a 9-dimensional family.

Quadrics through the 4 points form a 5-dimensional family So some quadrics through the 4 points are not of the form required by the putative Brill and Noether theorem, which is therefore refuted.

Lasker's paper,

***Mathematische Annalen* [1905]**

Homogeneous polynomials in

$\mathbb{C}[x_1, x_2, \ldots, x_m]$

geometrically $\mathbb{C}P^{m-1}$.

An ideal *P* is *prime* if $p.q \in P$

$\Rightarrow p \in P$ **or** $q \in P$.

***P* prime, *V* its variety**

Ideal *Q* is a *primary* ideal corr. to

***P* if** $V(Q) \subset V$ **and** $p.q \in Q$ **and**

$p \notin P \Rightarrow q \in Q$.

Satz VII: Every ideal M is an intersection of primary ideals

$M = Q_1 \cap Q_2 \cap \ldots \cap Q_k \cap R$,

where $Q_1, Q_2, \ldots, Q_k$ **are primary ideals and** $V(R) = \varnothing$.

Satz XI: If $M = (u_1, u_2, \ldots, u_h)$,

$h \leq m - 1$, **codim** $V(M) \geq h$

then $M = M_{C_1} \cap M_{C_2} \cap \ldots \cap M_{C_j}$,

where $C_i = V(u_i = 0)$.

Littlewood was introduced to the Euler product formula

$$\sum n^{-s} = \prod \left(1 - p^{-s}\right)^{-1}$$

'as a *joke* (rightly enough, and in excellent taste).'

***Mathematical Gazette* since 1896.**

Resigned and moved to Cambridge, on the school's pension of £175 p.a.

1911 at St Paul's School for 26 years, age 49.

former pupils:

- **four Senior Wranglers,**
- **six Smith's Prize winners,**
- **four fellows of Trinity College**
- **one of St John's,**
- **two F.R.S (J.E. Littlewood, G.N. Watson).**

After the War, Baker's Saturday afternoon tea parties.

Baker's students

Smith's Prizes six years out of eight, from 1927 until 1935.

P.A.M. Dirac, St. John's 1923.

'Dirac's ideas of *q*-numbers and his axioms for them most probably originated at Baker's tea parties',

Macaulay's Cambridge Tract, *The algebraic theory of modular systems*, 1916.

Theory of resolvents and resultants (Sylvester, Brill and Noether, König, and Lasker) prime and primary ideals, Hilbert basis theorem following König, Lasker's primary decomposition theorem.

Any primary ideal contained in the intersection of all the rest is *irrelevant* – omit it.

Relevant* primary ideals – *relevant varieties.

A relevant variety not contained in another of higher dimension – *isolated*,

others (and their associated ideals) – *imbedded*.

***Unmixed* ideal:**

all relevant varieties, isolated and embedded, have same dimension.

Theorem: a module M of rank $r < n$ is unmixed if and only if none of the modules

$$(M, x_n - a_n),\ (M, x_{n-1} - a_{n-1}, x_n - a_n),$$

$$\ldots, (M, x_{r+2} - a_{r+2}, \ldots, x_{n-1} - a_{n-1}, x_n - a_n)$$

contains a relevant simple module.

‘Modern Algebra and polynomial Ideals’,

***Proc. Cam. Phil. Soc.* 1934,**

The basic features of the new theory (the first time in English)

Macaulay was 72.

van der Waerden’s *Moderne Algebra* – the standard work on the theory of ideals.

“There could be no better example of method than Krull’s

proof of Lasker's extension of M. Noether's "fundamental theorem", which has not been published as far as I am aware",

a result about principal systems "This theorem, if I am not mistaken, is one of Noether's in general ideal theory. Gröbner has used it for proving some important properties of ideals."

Emmy Noether's famous paper [1921] – abstract methods. Lasker and Macaulay (*Mathematische Annalen* paper) polynomial ideals from the standpoint of elimination theory

She reviewed the later Macaulay papers for *Fortschritte*, Ostrowski, Grete Hermann, van der Waerden, Krull.

Krull often referred to Macaulay in his *Idealtheorie* [1935] unmixedness, perfectness, and the idea of inverse systems.

Gröbner, in his [1937], Macaulay's inverse systems not 'artful, obscure and of little value'
use theory of linear ordinary differential equations with constant coefficients.

Macaulay died 1938

Obituary by H.F. Baker (72).

Generous reminiscence by J.E. Littlewood

Bibliography H.F. White, editor of the *Proc. Cam. Phil. Soc.*

Omits mention of Macaulay's obituary of Max Noether, and Macaulay's last paper.

If *R* is a Cohen-Macaulay ring and $I = (x_1, \ldots, x_r)$ is an ideal of codimension *r*, then the quotient R/I is a Cohen-Macaulay ring.

Polynomial rings – Macaulay

Regular local rings – Cohen, [1946].

The name Cohen-Macaulay – Samuel – [1951].

Macaulay – theory.

As a computer package in commutative algebra.

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F. S. MACAULAY.

(1862-1937)