Quotient rings of noncommutative rings

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Algebraische Theorie der Körper

E. Steinitz (1910)

Integral domains (§3)

- **Definition:** Commutative rings without zero-divisors (Krönecker)

- **Main result:** Every integral domain has a quotient ring.

- Gives the construction that has become familiar from basic algebra.

Moderne Algebra

B. van der Waerden (1930)

Chapter **III**, §12: Quotient construction

- Gives Steinitz's construction of quotient rings for commutative integral domains.

- Does not mention Steinitz.

- Sources for chapters **II** and **III** were lecture notes of Noether and Artin.

The question

The problem of embedding a noncommutative ring without zero-divisors in a noncommutative field is an unsolved problem, except in some special cases.

Terminology:

- domain = ring without zero-divisors.
- division ring = noncommutative field.

The Answers

A. Malcev (1933): example of a noncommutative domain whose multiplicative semigroup cannot be embedded in a group.

A. Malcev (1939): necessary and sufficient condition for a semigroup to be embedded in a group.

P. M. Cohn (1975): necessary and sufficient condition for a domain to be embedded in a division ring.

The key point

The quotient construction used in the commutative case breaks down if the domain is noncommutative.

Quotient ring

Let R be a (not necessarily commutative) domain.

The right quotient ring of R is a ring $Q(R)$ such that

1. $R \subseteq Q(R)$;
2. every $0 \neq c \in R$ is invertible in $Q(R)$;
3. every element of $Q(R)$ can be written in the form ac^{-1}, where $a, c \in R$ and $c \neq 0$.

The key argument

Let R be a domain with quotient ring $Q(R)$, and let $a, c \in R$, with $c \neq 0$.

By (2): $c^{-1} \in Q(R)$.

By (1): $a \in Q(R)$.

Since $Q(R)$ is a ring $c^{-1}a \in Q(R)$.

By (3): there exist $a_1, c_1 \in R$, with $c_1 \neq 0$ such that $c^{-1}a = a_1 c_1^{-1}$.

Therefore: $ac_1 = ca_1$ must hold in R.

The Quotient Problem

The result: If R has a quotient ring then, given $a, c \in R$, $c \neq 0$, there exist $a_1, c_1 \in R$ such that

$$ac_1 = ca_1 \text{ and } c_1 \neq 0.$$

Problem: Is this condition sufficient?

Answer: Yes, proved independently by:

- O. Ore (1931)
- D. E. Littlewood (1931)
- J. H. M. Wedderburn (1932)

O. Ore (1899-1968)

Studied at:
- The Cathedral School (Oslo)
- Oslo University
- Göttingen University

Worked at:
- Oslo University (1925-1927)
- Yale University (1927-1968)

O. Ore (1899-1968)

Mathematical interests:

- algebraic number theory (1923-1930)
- noncommutative rings and lattices (1930-1955)
- graph theory (1955-1968)

Books on the history of mathematics (Abel, Cardano).

Helped to edit Dedekind's complete works.

Linear equations over noncommutative fields

Annals of Mathematics (1931)

Aim: define determinants over noncommutative domains, by generalizing work of A. R. Richardson, A. Heyting and E. Study.

Van der Waerden's question is mentioned in a footnote at the introduction.

Linear equations over noncommutative fields

Key concept: A *regular ring* is a (not necessarily commutative) domain which satisfies

> M$_V$. *Existence of common multiplum* When $a \neq 0$, $b \neq 0$ are two elements of S, then it is always possible to determine two other elements $m \neq 0$, $n \neq 0$ such that
>
> $$an = bm. \qquad (1)$$

Nowadays:

- Regular ring = Ore domain.

- M$_V$. = Ore condition.

Relation between M_V. and determinant

$$x_1 a_{11} + x_2 a_{12} = b_1,$$
$$x_1 a_{21} + x_2 a_{22} = b_2$$

with coefficients in a regular ring S. Use M_V. to find A_{12} and A_{22} such that

$$a_{12} A_{22} = a_{22} A_{12}$$

Right multiply first equation by A_{22} and second by A_{12} and subtract them:

$$x_1 a_{11} A_{22} + x_2 a_{12} A_{22} = b_1 A_{22},$$
$$x_1 a_{21} A_{12} + x_2 a_{22} A_{12} = b_2 A_{12}$$

$$x_1 (a_{11} A_{22} - a_{21} A_{12}) = b_1 A_{22} - b_2 A_{21}$$

This is Cramer's rule!

Subsection 2

Theorem 1 *All regular rings can be considered as subrings* (more exactly: *are isomorphic to a subring*) *of a non-commutative field*.

Ore gives:

- (unmotivated) definitions for equality, addition and multiplication of "fractions".
- detailed proofs that all the required properties are satisfied.
- no interesting examples of noncommutative regular rings.

Common Denominators

Define $\left(\frac{a}{b}\right)$ to be ab^{-1}.

Question When is $\left(\frac{a}{b}\right)$ equal to $\left(\frac{\alpha}{\beta}\right)$?

Use M_V. to find b_1 and β_1 such that

$$b\beta_1 = \beta b_1 \; := c$$

Thus

$$ab^{-1} = a\beta_1 c^{-1} \text{ and } \alpha\beta^{-1} = \alpha b_1 c^{-1}.$$

Summing up:

$$\left(\frac{a}{b}\right) = \left(\frac{\alpha}{\beta}\right)$$

if and only if there exists b_1 and β_1 such that

$$b\beta_1 = \beta b_1 \text{ and } a\beta_1 = \alpha b_1.$$

Theory of noncommutative polynomials

Annals of Mathematics (1933)

Defines generalized polynomial ring $K[x]$ over a division ring K, such that

$$xa = \overline{a}x + a'.$$

where $a \mapsto \overline{a}$ is an endomorphism of K and $a \mapsto a'$ is a derivation of K.

Ore proves that

- there is a division algorithm in $K[x]$.
- there is a euclidean algorithm in $K[x]$.
- $K[x]$ is a regular ring.

This paper generalizes a previous one in Crelle (1931) on formal differential operators.

D. E. Littlewood (1903-1979)

Graduated from Trinity College (Cambridge) in 1925.

Worked at
University College Swansea (1930-1947) and
University College of North Wales, Bangor (1948-1970)

- Met A. R. Richardson at Swansea.
- Published 5 joint papers with Richardson.
- Best known for his work on groups.
- Littlewood-Richardson rule.

On the Classification of Algebras

Proceedings of the London Mathematical Society (1933).

Aim: study properties of algebras that are related to physics, specially Dirac's q-numbers, polynomials in x and p such that

$$px - xp = ih/2\pi \quad (= 1).$$

Pre *Moderne Algebra* style

- ring = linear algebra.

- ideal = modulus.

On the classification of Algebras

Theorem XIX. *If P and Q are polynomials in x and p, then non-zero polynomials R and S can be found such that*

$$RP = SQ.$$

Theorem XXI. *The algebra of rational expressions in p and x is a division algebra.*

Proof of Theorem XIX

Suppose that P and Q have degree at most r in x and in p.

Choose R and S of degree at most $3r$ in x and in p.

- Total number of coefficients of R and S is $2(3r+1)^2$.
- $\deg_x(RP - SQ)$ and $\deg_p(RP - SQ) \le 4r$.
- Total number of coefficients of $RP - SQ$ is $(4r+1)^2$.

But

$$2(3r+1)^2 > (4r+1)^2.$$

There are more variables than equations, hence the system must have a solution.

J. H. M. Wedderburn (1882-1948)

Noncommutative domains of integrity, Crelle, (1933)

- Results stated for *euclidean domains* (= domains with a euclidean algorithm).

- Gives a detailed proof that works in general.

- Was aware that his results applied to more general rings:

  H is a *Hamiltonian domain* if for all $a \in H$ there exists $\overline{a} \in H$ with

  $$a\overline{a} = \alpha \in Z(H).$$

  Hence,

  $$a(\overline{a}\alpha^{-1}) = \alpha\alpha^{-1} = 1.$$

  Enough to invert central elements.

O. Ore

Linear equations in non-commutative fields

Received (Annals): 8 December 1930
Published: 1931

D. E. Littlewood

On the classification of algebras

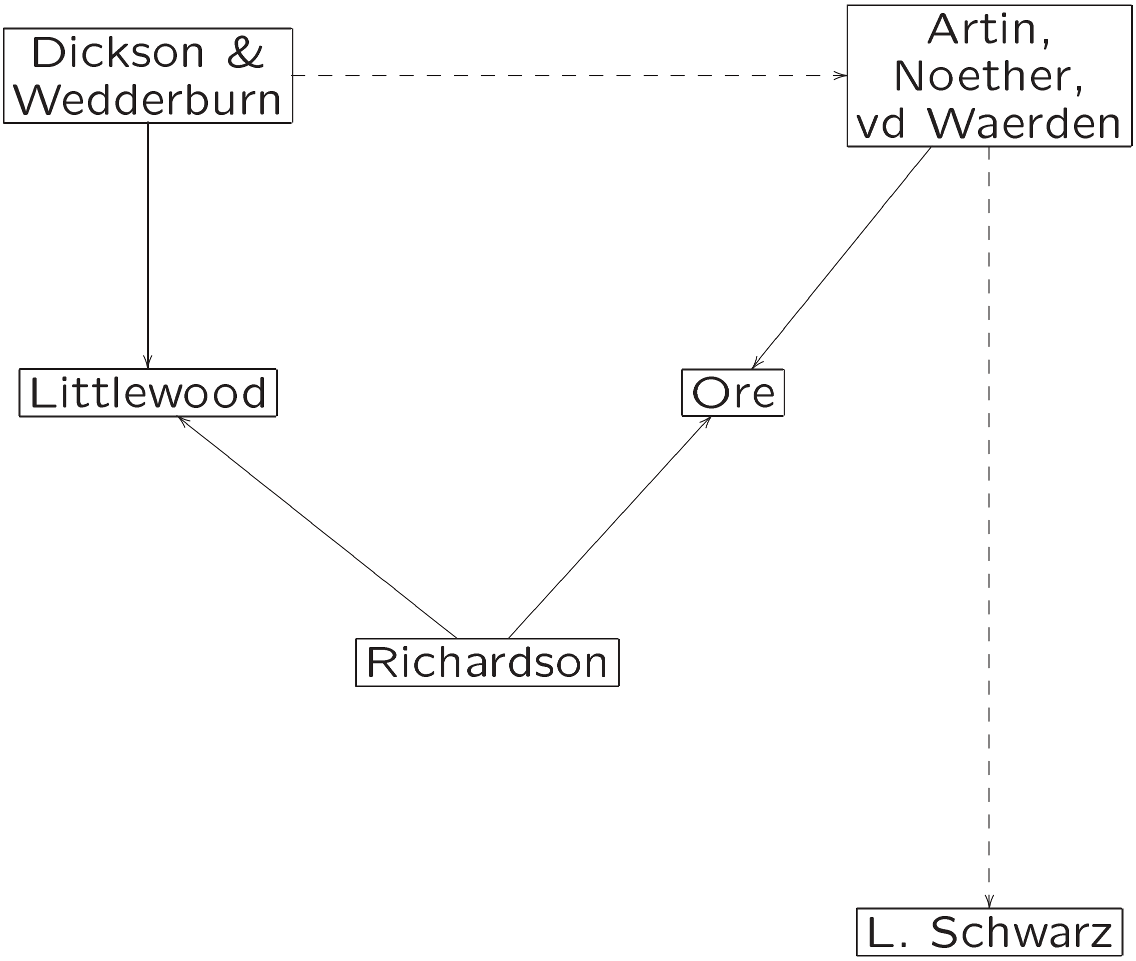
Read to LMS: 13 March 1930
Revised version published in PLMS: 1933

J. H. M. Wedderburn

Non commutative domains of integrity

Accepted (Crelle): 20 August 1931
Published: 1932.

The web of influences



The next twenty years

The 1940s: generalizations of the construction to rings with zero-divisors.

The 1950s: new examples of rings that are Ore domains.

Rings with zero-divisors

Paul Dubreil: *Algèbre* (1946).

- Studied under E. Noether and E. Artin.
- Direct generalization of Ore's approach to rings with zero-divisors.

K. Asano: *Über die Quotientenbildung von Schiefringen*

- Introduces a totally different approach to the construction of quotient rings.
- Avoids most of the complicated calculations required in Ore's direct approach.

Examples

Ring	**Author**	**Date**	**Argument**
Ore extension	Ore	1931	division algorithm
Weyl algebra	Littlewood	1931	counting argument
Enveloping algebra (positive characteristic)	N. Jacobson	1951	central polynomial
Enveloping algebra (all fields)	D. Tamari	1952	counting argument
PI domain	S. Amitsur	1955	minimal identity

Goldie's theorem

A. W. Goldie:

The structure of prime rings under ascending chain conditions (1958)

Semi-prime rings with maximal conditions (1960)

Goldie's theorem Every noetherian semiprime ring has a quotient ring.

All these examples are covered by Goldie's theorem and the special proofs have mostly been forgotten.

Today

- The solution of the quotient problem is usually attributed to Ore.
- No mention is made of the fact that Ore dealt only with domains.
- The work of Littlewood, Wedderburn and Dubreil is never mentioned after the 1960s.
- Asano's approach still survives because his construction is less ardous than Ore's direct approach.