MANIFOLDS WITH SPECIAL HOLONOMY
LECTURE 1:
BACKGROUND MATERIAL

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1. Bundles: Let G be a Lie group. A (smooth principal right) *G-bundle* over a manifold M is a manifold B equipped with

(1) a smooth, free right action $\rho : B \times G \to B$ and
(2) a smooth submersion $\pi : B \to M$

such that the orbits of ρ are the fibers of π. A *section* of B over M is a smooth map $\sigma : M \to B$ such that $\pi \circ \sigma = \mathrm{id}_M$.

Two G-bundles (B, π_i, ρ_i) for $i = 1, 2$ are *equivalent* if there is a diffeomorphism $f : B_1 \to B_2$ that commutes with the structure maps:

$$\begin{tikzcd} B_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & B_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd} \qquad \begin{tikzcd} B_1 \times G \arrow[r, "f \times \mathrm{id}_G"] \arrow[d, "\rho_1"'] & B_2 \times G \arrow[d, "\rho_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

We often let ρ be understood, writing $\rho(b, g) \in B$ as $b \cdot g$ and $\pi : B \to M$ to denote the bundle.

Basic Examples I:

(1) *The trivial bundle:* $B = M \times G$ with the obvious projection to M and right multiplication by G.

(2) *Coset spaces:* B is Lie group and G is a closed subgroup. Then the left coset space $M = B/G$ has the structure of a smooth manifold so that the coset projection $\pi : B \to M$ defined by $\pi(b) = bG \in M$ is a smooth submersion.

(3) *Homogeneous bundles:* $H_1 \subset H_2 \subset H_3$ are Lie groups with H_i closed in H_{i+1} and H_1 normal in H_2. Then $G = H_2/H_1$ is a Lie group and $B = H_3/H_1$ and $M = H_3/H_2$ are smooth manifolds so that the natural projection

$$\pi : H_3/H_1 \to H_3/H_2 \qquad \pi(hH_1) = hH_2$$

and the right action

$$\rho : H_3/H_1 \times H_2/H_1 \to H_3/H_1 \qquad \rho(hH_1, gH_1) = hgH_1$$

define B as a G-bundle over M.

Associated Bundles: Given a (pr. right) G-bundle $\pi : B \to M$ and a smooth left action $\lambda : G \times F \to F$, define the *associated bundle*

$$B \times_G F = \left\{ \, [b, f] \,\middle|\, (b, f) \in B \times F \, \right\} = \big(B \times F\big) / \sim$$

where $(b, f) \sim (b \cdot g, g^{-1} \cdot f)$.

The quotient space $B \times_G F$ has a natural smooth manifold structure such that the projection $B \times F \to B \times_G F$ defined by $(b, f) \mapsto [b, f]$ is a smooth submersion and the induced mapping $B \times_G F \to M$ defined by $[b, f] \mapsto \pi(b)$ is also a smooth submersion, with each fiber diffeomorphic to F.

In this way, principal bundles give rise to a large class of bundles. For example, if F is a vector space and $\lambda : G \times F \to F$ is a linear representation of G into $\mathrm{GL}(F)$, then $B \times_G F$ is a vector bundle over M.

Extension and Reduction: Let $H \subset G$ be a subgroup.

If $\pi : B \to M$ is an H-bundle over M, then

$$B \times_H G = \{\ [b, g] \,\big|\, (b, g) \in B \times G\ \} = \big(B \times G\big)/\sim$$

(where $(b, g) \sim (b{\cdot}h, h^{-1}g)$) is a smooth manifold that is naturally a G-bundle over M and called the G-*extension* of B, sometimes written $B{\cdot}G$.

Conversely, if $\pi : B \to M$ is a G-bundle over M and there is a submanifold $B' \subset B$ so that $\rho(B' \times H) = B'$ and the restriction of ρ gives B' the structure of an H-bundle over M, we say that B' is an H-*reduction* of B.

Remarks:

(1) A G-bundle $\pi : B \to M$ is trivial iff it has an $\{e\}$-reduction.

(2) By the Implicit Function Theorem, every G-bundle is locally trivial, i.e., M can be covered by open sets $U \subset M$ such that $B_U = \pi^{-1}(U)$ is trivial over U.

Example: Reductions of Sphere Bundles.

(1) $\mathrm{SO}(3) \to \mathrm{SO}(3)/\mathrm{SO}(2) \simeq S^2$ has no nontrivial reductions.
(2) $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ defines an $\{e\}$-reduction of

$$\mathrm{SO}(4) \to \mathrm{SO}(4)/\mathrm{SO}(3) \simeq S^3,$$

so it's trivial.
(3) $\mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{SO}(4) \simeq S^4$ has no nontrivial reductions.
(4) $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ defines an $\mathrm{SU}(2)$-reduction of

$$\mathrm{SO}(6) \to \mathrm{SO}(6)/\mathrm{SO}(5) \simeq S^5$$

that has no further reductions.
(5) $G_2 \subset \mathrm{SO}(7)$ defines an $\mathrm{SU}(3)$-reduction of

$$\mathrm{SO}(7) \to \mathrm{SO}(7)/\mathrm{SO}(6) \simeq S^6$$

that has no further reductions.
(6) $\mathrm{SO}(8) \to \mathrm{SO}(8)/\mathrm{SO}(7) \simeq S^7$ has an $\{e\}$-reduction defined by octonion multiplication.
(7) $\mathrm{SO}(9) \to \mathrm{SO}(9)/\mathrm{SO}(8) \simeq S^8$ has no nontrivial reductions.

Basic Examples II: Let V be a vector space of dimension n. (Often, $V = \mathbb{R}^n$, so that $\mathrm{GL}(V) = \mathrm{GL}(n, \mathbb{R})$.)

Coframe Bundles. If $E \to M$ is a vector bundle of rank n, let $F(E, V)$ be the set of isomorphisms $u : E_x \to V$ for $x \in M$. There is a unique smooth structure on $F(E, V)$ such that setting

$$\pi(u) = x \qquad \text{and} \qquad \rho(u, g) = g^{-1} \circ u$$

for $u \in F(E, V)$ and $g \in \mathrm{GL}(V)$ gives $F(E, V)$ the structure of a $G = \mathrm{GL}(V)$-bundle over M.

Reductions. When E has extra structure, we can often use it to reduce $F(E, V)$. For example, fix a positive definite inner product on V and let $\mathrm{O}(V) \subset \mathrm{GL}(V)$ denote its isometries. Now suppose that E is an Euclidean bundle. Then $F'(E, V) \subset F(E, V)$, consisting of the $u : E_x \to V$ that are isometries, is an $\mathrm{O}(V)$-bundle.

Similar considerations apply when V also has a compatible complex structure with unitary automorphism group $\mathrm{U}(V) \subset \mathrm{O}(V)$ and E is an Hermitian vector bundle, etc.

Tangent Coframe Bundles and G-structures. An important special case: If $E = TM$, then $F = F(TM, V)$ has some extra structure: A canonical V-valued 1-form ω. *The canonical form.* Define $\omega_u : T_uF \to V$ by

$$\omega(v) = u\big(\pi'(v)\big) \qquad \text{for} \qquad v \in T_uF \xrightarrow{\pi'} T_xM \xrightarrow{u} V.$$

This form ω has some remarkable properties, as will be seen.

G-structures. If $G \subset \mathrm{GL}(V)$ is a subgroup, then a G-reduction of $F = F(TM, V)$ is said to be *G-structure* on M.

For example, if V has an Hermitian structure, with automorphism group $\mathrm{U}(V) \subset \mathrm{GL}(V)$, then a $\mathrm{U}(V)$-structure on M is equivalent to specifying a Riemannian metric and a compatible almost complex structure.

We are going to be interested in finding G-structures on Riemannian manifolds that satisfy an extra 'flatness' condition and in studying what geometric properties those G-structures induce on the manifold.

Connections: Let $\pi : B \to M$ be a (principal right) G-bundle over M. A *connection* on B is a splitting

$$TB = \ker \pi' \oplus L$$

of the tangent bundle TB for which $L \subset TB$ is G-invariant, i.e., $R'_g(L) = L$ for all $g \in G$.

In particular, $\pi'(b) : L_b \to T_{\pi(b)}M$ is an isomorphism $\forall\, b \in B$. For any differentiable curve $c : [\alpha, \beta] \to M$ and any $a \in \pi^{-1}\big(c(\alpha)\big)$, there is a unique differentiable curve $\tilde{c} : [\alpha, \beta] \to B$ such that

$$\pi\big(\tilde{c}(t)\big) = c(t), \quad \tilde{c}(\alpha) = a, \quad \text{and} \quad \tilde{c}'(t) \in L_{\tilde{c}(t)}\,, \quad \forall t \in [\alpha, \beta].$$

A curve $\tilde{c}$ with this last property is said to be *L-horizontal.* The G-invariance of L implies that, if $\tilde{c}$ is L-horizontal, then $R_g \circ \tilde{c}$ is also L-horizontal for all $g \in G$. It then also satisfies

$$\pi\big(R_g \circ \tilde{c}(t)\big) = c(t), \qquad \text{and} \qquad R_g \circ \tilde{c}(\alpha) = a \cdot g.$$

Thus, a connection on B defines a trivialization of B along each differentiable curve $c : [\alpha, \beta] \to M$, once a trivialization of the fiber over $c(\alpha)$ is chosen.

Example: Homogeneous Connections. Suppose that B is a Lie group with Lie algebra $\mathfrak{b} = T_e B$ and that $G \subset B$ is a closed subgroup with corresponding subalgebra $\mathfrak{g} \subset \mathfrak{b}$.

Generally $\pi : B \to B/G$ does not have a B-invariant connection.

However, if there is a complement $\mathfrak{l} \subset \mathfrak{b}$ to $\mathfrak{g}$ that is $\mathrm{Ad}(G)$-invariant, then the left-invariant subbundle $L \subset TB$ that satisfies $L_e = \mathfrak{l}$ defines a connection on $\pi : B \to B/G$ that is B-invariant. (Such $\mathfrak{l}$ always exists, for example, if G is compact or, more generally, reductive.)

Induced Connections. If $L \subset TB$ is a connection on $\pi : B \to M$, then for any associated bundle $B \times_G F$, there is a corresponding subbundle of $T(B \times_G F)$, defined as the image of L under the submersion $B \times F \to B \times_G F$. (It is well-defined because of the G-invariance of L.) This is known as the induced connection.

Holonomy: Let $\pi : B \to M$ be a (p.r.) G-bundle with connection $L \subset TB$. Assume that M is connected. Let $b \in B$ be fixed and set

$$\mathcal{L}(b) = \left\{ c : [0,1] \to B \,\middle|\, c(0) = b \text{ and } c \text{ is pcwise diff. and } L\text{-hor.} \right\}.$$

and

$$B_L(b) = \left\{ c(1) \in B \,\middle|\, c \in \mathcal{L}(b) \right\}.$$

Theorem: (Borel-Lichnerowicz) There is a Lie subgroup $H_L(b) \subset G$ such that $B_L(b)$ is an $H_L(b)$-reduction of B. Moreover, L is tangent to the submanifold $B_L(b) \subset B$ and this reduction is the smallest reduction of B with this property that passes through b.

Remark: $B_L(b)$ is called the L-holonomy bundle through b and $H_L(b)$ is called the *holonomy* of L through b. We have

$$B_L(b \cdot g) = B_L(b) \cdot g \qquad \text{and} \qquad H_L(b \cdot g) = g^{-1} H_L(b) g$$

We often call $H_L(b)$ 'the holonomy of L', even though this is somewhat ambiguous.

The connection form. Let $\pi : B \to M$ be a (p.r.) G-bundle. By definition, the linear map $\iota_b : \mathfrak{g} \to T_b B$ defined by

$$\iota_b(v) = \frac{d}{dt}(b{\cdot}e^{tv})\Big|_{t=0}$$

is an isomorphism $\iota_b : \mathfrak{g} \xrightarrow{\sim} \ker \pi'(b)$.

Given a connection L on B, there is a unique $\mathfrak{g}$-valued 1-form θ on B that satisfies

(1) $\theta\big(\iota_b(v)\big) = v$ for all $v \in \mathfrak{g}$ and $b \in B$
(2) $\theta(w) = 0$ for all $w \in L$.

This θ $(= \theta^L)$ is the *connection form* associated to L. It satisfies

$$R_g^*(\theta) = \mathrm{Ad}\big(g^{-1}\big)(\theta)$$

(a consequence of the G-invariance of L).

Curvature. The 2-form

$$\Theta = \mathrm{d}\theta + \tfrac{1}{2}[\theta, \theta]$$

is known as the *curvature form* of the connection.

Curvature and Holonomy. Let $\pi : B \to M$ have a connection L and associated connection form θ with curvature form Θ.

For each vector field X on M, there is a unique vector field X^L on B that satisfies $X^L(b) \in L_b$ and $\pi'\bigl(X^L(b)\bigr) = X\bigl(\pi(b)\bigr)$ for $b \in B$. These satisfy the identity

$$[X^L, Y^L](b) = -\iota_b\bigl(\Theta(X^L, Y^L)\bigr) + [X, Y]^L(b).$$

This motivates defining the subset

$$\mathfrak{k}^L(b) = \left\{ \ \Theta(v, w) \,\middle|\, v, w \in L_b \ \right\} \subset \mathfrak{g}\,.$$

Theorem: (Ambrose-Singer) The Lie algebra of $H_L(b) \subset G$ is the subspace generated by the subsets $\left\{ \ \mathfrak{k}_L(b) \,\middle|\, b \in B_L(b) \ \right\}$.

'Squeeze Play':

(1) An H-reduction $B' \subset B$ is a union of L-holonomy bundles (so that $H_L(b) \subseteq H$ for all $b \in B'$) iff the pullback of θ to B' takes values in $\mathfrak{h}$.
(2) If $\mathfrak{k}_L(b)$ generates $\mathfrak{h} \subset \mathfrak{g}$, then $H \subseteq H_L(b)$.

Example: $\mathrm{SO}(n{+}1) \longrightarrow \mathrm{SO}(n{+}1)/\,\mathrm{SO}(n) \simeq S^n$.

Let $g : \mathrm{SO}(n{+}1) \hookrightarrow \mathrm{GL}(n{+}1,\mathbb{R})$ be inclusion and write the left-invariant form as

$$g^{-1}\mathrm{d}g = \begin{pmatrix} 0 & -{}^t\omega \\ \omega & \theta \end{pmatrix} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\omega &= -\theta \wedge \omega, \\ \mathrm{d}\theta &= -\theta \wedge \theta + \omega \wedge {}^t\omega \end{aligned}$$

where ω has values in $\mathbb{R}^n$ and $\theta = -{}^t\theta$ has values in $\mathfrak{so}(n) = \mathfrak{g}$ and is the unique $\mathrm{SO}(n{+}1)$-invariant connection for this bundle over S^n. Its curvature is

$$\Theta = \mathrm{d}\theta + \tfrac{1}{2}[\theta,\theta] = \mathrm{d}\theta + \theta \wedge \theta = \omega \wedge {}^t\omega.$$

Thus, for any $b \in B = \mathrm{SO}(n{+}1)$,

$$\mathfrak{k}_L(b) = \left\{\; x\,{}^ty - y\,{}^tx \;\middle|\; x, y \in \mathbb{R}^n \;\right\},$$

and the span of these rank one matrices in $\mathfrak{so}(n)$ is all of $\mathfrak{so}(n)$. Thus, the holonomy group of this connection is all of $\mathrm{SO}(n)$.

Exercise: Try showing directly that any two points of $\mathrm{SO}(n{+}1)$ can be joined by a curve $c : [0,1] \to \mathrm{SO}(n{+}1)$ that satisfies $c^*(\theta) = 0$.

Riemannian Holonomy. Let (M, g) be a Riemannian manifold. For simplicity, assume that M is connected and that $\pi_1(M) = 0$. In particular, can assume that M is oriented.

Let $\pi : F \to M$ denote the bundle of oriented, g-orthonormal coframes $u : T_xM \to \mathbb{R}^n$, a p.r. $\mathrm{SO}(n)$-bundle over M. As usual, let ω be the canonical $\mathbb{R}^n$-valued 1-form on F, and let $\theta = -{}^t\theta$ be the $\mathfrak{so}(n)$-valued connection form associated to the Levi-Civita connection of g.

The first structure equations of É. Cartan:

$$\mathrm{d}\omega = -\,\theta\wedge\omega, \qquad\qquad \pi^* g = \omega\circ{}^t\omega = {\omega_1}^2 + \cdots + {\omega_n}^2.$$

Local splitting. Cartan observed that, if the holonomy of θ through $u \in F$ preserves a splitting, i.e., $H_L(u) \subset \mathrm{SO}(p)\times\mathrm{SO}(n{-}p)$, then every point of M lies in an open set U with a coordinate system $x = (x', x'') : U \to \mathbb{R}^p\times\mathbb{R}^{n-p}$ such that

$$U^*(g) = (x')^*(g') + (x'')^*(g'')$$

for metrics g' on $\mathbb{R}^p$ and g'' on $\mathbb{R}^{n-p}$.

Theorem (de Rham) If M is 1-connected and g is a complete metric on M, then M can be written as a Riemannian product

$$M = (\mathbb{R}^p, g_{\mathrm{can}}) \times (M_1, g_1) \times \cdots \times (M_k, g_k)$$

where each M_i is simply-connected and (M_i, g_i) is complete and holonomy-irreducible. This factorization is unique up to order.

Remark: Even without the completeness hypothesis, if the holonomy of (M, g) through $u \in F$ satisfies $H(b) \subseteq \mathrm{SO}(p) \times \mathrm{SO}(n-p)$, then

$$H(b) = \big(H(b) \cap \mathrm{SO}(p)\big) \times \big(H(b) \cap \mathrm{SO}(n-p)\big),$$

that g has a corresponding global splitting $g = g' + g''$ on M, and that there exist submanifolds $M', M'' \subset M$ such that (M', g') and (M'', g'') have holonomy

$$H'(b') = H(b) \cap \mathrm{SO}(p) \quad \text{and} \quad H''(b'') = H(b) \cap \mathrm{SO}(n-p).$$

Fundamental question: What are the possible holonomy groups of holonomy-irreducible manifolds (M, g)?

An example. SU(2) has one irreducible (real) representation in each dimension $n \geq 3$ odd or divisible by 4, say $H_n \subset \mathrm{SO}(d)$:

$$H_{2k+1} \simeq \mathrm{SU}(2)/\{\pm I_2\} \simeq \mathrm{SO}(3) \qquad \text{and} \qquad H_{4k} \simeq \mathrm{SU}(2).$$

(1) $n = 3$. Then $H_3 = \mathrm{SO}(3)$ and this is the holonomy of any metric in dimension 3 that is not a product metric.
(2) $n = 4$. Then $H_4 = \mathrm{SU}(2) \subset \mathrm{SO}(4)$. Cartan showed that this does occur as the holonomy of a Riemannian 4-manifold and in infinitely many non-isometric ways. (More on this below.)
(3) $n = 5$. Then $H_5 \subset \mathrm{SO}(5)$ and there are two known examples, the invariant metrics on the homogeneous spaces $\mathrm{SU}(3)/\mathrm{SO}(3)$ and $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. Every (M^5, g) whose holonomy is conjugate to H_5 is locally isometric to a constant multiple of one of these two metrics. (More on this below.)
(4) $n > 5$. There is no (M^n, g) whose holonomy is conjugate to H_n.

Berger's Analysis. To get some insight, look at Cartan's first structure equation

$$\mathrm{d}\omega = -\theta \wedge \omega$$

Setting $\Theta = \mathrm{d}\theta + \theta\wedge\theta$ as usual, its exterior derivative gives

$$\Theta \wedge \omega = 0$$

Writing $\Theta = R(\omega, \omega)$ (since Θ must be quadratic in ω), this equation says, for every $u \in F$

$$R_u(x, y)z + R_u(y, z)x + R_u(z, x)y = 0 \qquad \text{(first Bianchi identity).}$$

On the other hand, if θ has holonomy bundle $B(u) \subset F$ with holonomy group $H(u) = H \subset \mathrm{SO}(n)$, then $R_v(x, y) = -R_v(y, x)$ lies in $\mathfrak{h} \subset \mathfrak{so}(n)$ for all $v \in B(u)$.

For any Lie algebra $\mathfrak{h} \subset \mathfrak{so}(n)$, define

$$K(\mathfrak{h}) = \left\{ R : \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{h} \;\middle|\; \begin{array}{r} R(x, y) + R(y, x) = 0, \\ R(x, y)z + R(y, z)x + R(z, x)y = 0 \end{array} \right\}$$

This is a vector space canonically associated to $\mathfrak{h} \subset \mathfrak{so}(n)$.

Example (continued): Back to the examples of $H_n \subset \mathrm{SO}(n)$, irreducibly acting and isomorphic to either $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$, we can compute

Dimension n	$K(\mathfrak{h}_n)$
3	$\mathbb{R}^1 \oplus \mathbb{R}^5$
4	$\mathbb{R}^5$
5	$\mathbb{R}^1$
> 5	0

Thus, for example, for $n > 5$, the group $H_n \subset \mathrm{SO}(n)$ cannot be a holonomy group because such a connection would have to have vanishing curvature and this would violate Ambrose-Singer. When $n = 5$, the fact that $K(\mathfrak{h}_5) \simeq \mathbb{R}^1$ implies that the curvature of θ is determined up to a multiple at every point and, in particular, this implies Einstein, so that this curvature is constant.

Obviously, if $K(\mathfrak{h}) = 0$, then $H \subset \mathrm{SO}(n)$ cannot be the holonomy of any (M^n, g). More generally, if $K(\mathfrak{h}) \subseteq K(\mathfrak{h}')$ for any proper subalgebra $\mathfrak{h}' \subset \mathfrak{h}$, then H cannot be the holonomy of any (M^n, g).

It remains to examine the other cases.

Theorem: (Berger-Kostant) Assume that $H \subset \mathrm{SO}(n)$ acts irreducibly on $\mathbb{R}^n$, that $n > 2$, and that $K(\mathfrak{h}) \simeq \mathbb{R}^1$. Then there is an n-dimensional irreducible Riemannian symmetric space $M = G/H$ of compact type whose holonomy is conjugate to H.

Moreover, any (M^n, g) whose holonomy is conjugate to H is, up to a constant scale, locally isometric to either G/H or its noncompact dual G^*/H.

Remark: The irreducible Riemannian symmetric spaces were classified by É. Cartan, who remarked that each such G/H gives an example of a manifold with holonomy H. The ones of type A:

$$\mathrm{SU}(n)/\,\mathrm{SO}(n) \qquad \mathrm{SU}(2n)/\mathrm{Sp}(n) \qquad \mathrm{SU}(n)/\mathsf{S}\big(\mathrm{U}(p)\times\mathrm{U}(n-p)\big)$$

Theorem: (Berger) Assume that $H \subset \mathrm{SO}(n)$ acts irreducibly on $\mathbb{R}^n$, that $\dim K(\mathfrak{h}) > 1$, and that $K(\mathfrak{h})$ is not contained in $K(\mathfrak{h}')$ for any proper subalgebra $\mathfrak{h}' \subset \mathfrak{h}$. Then $\mathfrak{h}$ is conjugate to one of the subalgebras of $\mathfrak{so}(n)$ listed in the following table:

n	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$-module
n	$\mathfrak{so}(n)$	$\mathbb{R} \oplus \mathsf{S}^2_0(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus \mathsf{S}^{1,1}_0(\mathbb{C}^m)^{\mathbb{R}} \oplus \mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$\mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus \mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$\mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$\mathsf{V}^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$\mathsf{V}^{0,2,0} \simeq \mathbb{R}^{168}$
$n = 16$	$\mathfrak{spin}(9)$	$\mathbb{R}^1$

The Holonomy Principle: By the transformation rule

$$H(u\cdot g) = g^{-1}H(u)g,$$

for each $x \in M$, there is a subgroup $H(x) \subset \mathrm{SO}(T_xM)$ such that

$$H(u) = uH(x)u^{-1} \qquad \text{for all } u \in F_x = \mathrm{Iso}^+(T_xM, \mathbb{R}^n).$$

As has already been remarked, the Levi-Civita connection on F induces a connection on each tensor bundle over M

$$T^{(r,s)}M = F \times_{\mathrm{SO}(n)} \big(\otimes^r(\mathbb{R}^n) \otimes \otimes^s((\mathbb{R}^n)^*)\big).$$

Essentially by construction, a tensor field $\sigma \in \Gamma\big(T^{(r,s)}M\big)$ that is parallel with respect to this connection must have the property that $\sigma_x \in T_x^{(r,s)}M$ is invariant under the action of $H(x)$ on $T_x^{(r,s)}M$.

Conversely, if $s \in T_x^{(r,s)}M$ is invariant under the action of $H(x)$, then there is unique a well-defined tensor field $\sigma \in \Gamma\big(T^{(r,s)}M\big)$ that satisfies $\sigma_x = s$ and is parallel along curves in M.

This correspondance is the *holonomy principle*: The holonomy of a Riemannian metric determines its parallel tensor fields.

Example: Differential Forms. A particularly interesting example is that of differential forms. Because the composition

$$\Omega^p(M) \xrightarrow{\nabla} \Omega^1(M) \otimes \Omega^p(M) \xrightarrow{\wedge} \Omega^{p+1}(M)$$

is the exterior derivative, it follows that parallel p-forms are closed. Similarly, since the Hodge star operation is parallel, it follows that parallel differential forms are co-closed as well.

By the Hodge Theorem, this gives a 'lower bound' on the cohomology of a manifold with reduced holonomy:

Theorem: If (M^n, g) is a compact, oriented Riemannian manifold whose holonomy is conjugate to a subgroup $H \subset \mathrm{SO}(n)$, then there is an injection

$$\Lambda^*(\mathbb{R}^n)^H \hookrightarrow H^*_{\mathrm{dR}}(M)$$

of the ring of H-invariant exterior forms on $\mathbb{R}^n$ into the deRham cohomology of M.

Invariant forms and Irreducible Holonomy: Now, it turns out that for each of the groups H on Berger's list (other than $\mathrm{SO}(n)$ itself), the ring of H-invariant exterior forms is nontrivial:

Dimension	Group	Invariant forms (generators)
n	$\mathrm{SO}(n)$	$1 \in \Lambda^0,\ *1 \in \Lambda^n$
$n = 2m$	$\mathrm{U}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2$
$n = 2m$	$\mathrm{SU}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2,\ \phi, \psi \in \Lambda^m$
$n = 4m$	$\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$
$n = 4m$	$\mathrm{Sp}(m)$	$1 \in \Lambda^0,\ \omega_1, \omega_2, \omega_3 \in \Lambda^2$
$n = 7$	G_2	$1 \in \Lambda^0,\ \phi \in \Lambda^3,\ *\phi \in \Lambda^4$
$n = 8$	$\mathrm{Spin}(7)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$

(More about these later.)

Application: (Kostant) Any Riemannian metric on S^n has holonomy $H = \mathrm{SO}(n)$.

Proof: Let g be a metric on S^n. Since S^n is simply connected and not a product of lower dimensional manifolds, the holonomy of g must act irreducibly on each tangent space.

Going through the list of compact Riemannian symmetric spaces, one sees that only $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ is homemorphic to S^n, and the holonomy of this metric (the standard constant curvature one) is $\mathrm{SO}(n)$, as has already been seen.

The only remaining metrics to examine are the ones that are not locally symmetric, which implies that the holonomy must be on Berger's list. However, each of these groups other than $H = \mathrm{SO}(n)$ leaves invariant a p-form for some p in the range $1 < p < n$, which would force S^n to have nontrivial cohomology in dimension p, which it does not. Thus, the holonomy must be $\mathrm{SO}(n)$.

Remark: 'Most' manifolds have no metric with reduced holonomy.