Lecture Notes on Complex Geometry

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Lecture Notes on Complex Geometry

Lecture One: Preliminaries, 8/11/2003.

A complex manifold is a topological space such that one can define holomorphic functions on it without ambiguity

Let X be a topological space. Then a good way to study X is to study $C(X)$, the vector space of continuou functions of X. $C(X)$ is a better object than X because ~~that~~ at least $C(X)$ is a vector space. In this sense, X is linearized.

If X is a complex manifold, then we can define holomorphic functions over X. Let $f: X \to \mathbb{C}$ be a function over X. Let $U \subset X$ be a coordinate patch, with $\varphi_U : U \longrightarrow \mathbb{C}^n$ be the local coordinate. Then by definition, $f: U \to \mathbb{C}$ is holomorphic iff $f \circ \varphi_U^{-1} : \varphi_U(U) \longrightarrow \mathbb{C}$ is holomorphic. However, Let $x \in U$ and let (V, φ_V) be another local holomorphic coordinate system. If $f \circ \varphi_U^{-1}$ is holomorphic, is $f \circ \varphi_V^{-1}$ holomorphic? The answer is yes if $\varphi_U \circ \varphi_V^{-1}$ is holomorphic because

$$f \circ \varphi_V^{-1} = f \circ \varphi_U^{-1} \; \varphi_U \circ \varphi_V^{-1}$$

But $\varphi_U \circ \varphi_V^{-1}$ being holomorphic is just the ~~requirement of~~ complex structure. Thus we say that the complex structure is the one that one can define holomorphic functions without ambiguity.

In Calculus, once we have function, we can define the derivative of the function:

Let B be the unit ball of $\mathbb{C}^n$ and let $f: B \to \mathbb{C}$ be a holomorphic function. The differential df of f is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i$$

If we want to take the second derivative, we identify

$$df \sim \left(\frac{\partial f}{\partial z_1}, \cdots \frac{\partial f}{\partial z_n}\right)$$

so that df becomes a $\mathbb{C}^n$-valued function. The second derivative of f is thus defined to be the derivative of the function $\left(\frac{\partial f}{\partial z_1}, \cdots \frac{\partial f}{\partial z_n}\right)$.

We must modify the above method in order to take derivati of a function on a complex manifold. ~~Let~~ For the sake of simplicity, we let

$$X = U \cup V$$

where (U, φ_u) and (V, φ_v) are two local coordinate system. The first derivative, as above, can be defined by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i = \sum_{j=1}^{n} \frac{\partial f}{\partial w_j} dw_j$$

In order to take the second derivature, we must identify

$$\left(\frac{\partial f}{\partial z_i}\right) \quad \text{and} \quad \left(\frac{\partial f}{\partial w_j}\right)$$

on $U \cap V$.

The key observation here is the chain rule

$$\frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial w_j} \cdot \frac{\partial w_j}{\partial z_i}$$

where the matrix $\left(\frac{\partial w_j}{\partial z_i}\right)$ is independent of ~~the choice~~ ~~of~~ f !

A $\mathbb{C}^n$-valued function on X can be viewed as two functions

$$f_1 : U \to \mathbb{C}^n , \quad f_2 : V \to \mathbb{C}^n$$

such that

$$f_1 \equiv f_2 \quad \text{on} \quad U \cap V$$

In this sense, the set

$$\left(\frac{\partial f}{\partial z_i}\right) , \quad \left(\frac{\partial f}{\partial w_j}\right)$$

is not a function over $U \cup V$, it is a section of the vector bundle T^*X, the holomorphic cotangent bundle.

In view of the above discussion, we have the following definition of vector bundles:

Let X be a complex manifold and let

$$X = \cup U_\alpha$$

where $(U_\alpha, \varphi_\alpha)$ is a coordinate system. Given matrix valued-functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$$

We define a relation $\sim$ on the set

$$\bigcup_{\alpha} U_\alpha \times \mathbb{C}^r$$

by

$(x, V_i) \sim (x, W_j)$ for $(x, V_i) \in U_\alpha \times \mathbb{C}^r$ and $(x, W_j) \in U_\beta \times \mathbb{C}^r$ iff $V_i = \sum (g_{\alpha\beta})_{ji} W_j$, $x \in U_\alpha \cap U_\beta \neq \phi$.

In order to make $\sim$ an equivalence relation, we need to assume that

① $g_{\alpha\alpha} = I$ on U_α

② $g_{\alpha\beta} \cdot g_{\beta\alpha} = I$ on $U_\alpha \cap U_\beta \neq \phi$

③ $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = I$ on $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$

Definition $E = \bigcup_\alpha U_\alpha \times \mathbb{C}^r / \sim$ is the vector bundle defined by $(g_{\alpha\beta})$. The ~~set of~~ $(g_{\alpha\beta})$ ~~is~~ are called tranisition functions.

① If $g_{\alpha\beta} \in C^\infty$, then E is a complex vector bundle,

② If $g_{\alpha\beta}$ holomorphic, the E is a holomorphic vector bundle

③ if $r = 1$, the E is called a line bundle

④ If $g_{\alpha\beta} = I$, then E is a locally trivial bundle.

By definition of E, we see that there is a projection $\pi : E \to X$ and for each $x \in X$, $\pi^{-1}(x)$ is a linear space isomorphic to $\mathbb{C}^r$.

A section of E can be written as

$$S = (S_\alpha)$$

where S_α is a $\mathbb{C}^r$-valued function. Or more invariantly, Let $e_\alpha = (e_{\alpha,1}, \cdots, e_{\alpha,r})^T$ be a frame (that is $e_{\alpha,i}$ are smooth and at each point $x \in U_\alpha$, $(e_{\alpha,1}, \cdots e_{\alpha,r})$ forms a basis of the fiber). Then

$$S = S_\alpha e_\alpha$$

where S_α is considered as a row vector and e_α is a ~~et~~ column vector such that each entry is a smooth section. e_α are defined locally, so unlike the case of Eucliedean space, e_α is a moving frame. Thus in order to give the derivative of S, we first need to take the derivative of e_α.

Let's assume that

$$D e_\alpha = \theta_\alpha e_\alpha$$

where θ_α is a matrix-valued 1-form. ~~The~~ The matrix θ_α is called the connection matrix. Since

$$\cancel{e_\beta = e_\alpha g_{\alpha\beta}} \quad e_\alpha = g_{\alpha\beta} e_\beta$$

We have

$$\theta_\alpha g_{\alpha\beta} = g_{\alpha\beta} \theta_\beta + d g_{\alpha\beta} \qquad (*)$$

The operator D and the set of matrix valued forms θ_α satisfying $(*)$ are called ~~the~~ a connection of the vector bundle E.

Consider again ~~cla~~ Calculus. After defining df

We can extend d to 1-forms. If we do the same thing for D, the extension would ~~be~~ be.

$$D(\omega \wedge s) = d\omega \wedge s - \omega \wedge Ds$$

Using this we compute

$$D^2 e_\alpha = D(\theta_\alpha e_\alpha) = d\theta_\alpha e_\alpha - \theta_\alpha D e_\alpha$$
$$= (d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha) e_\alpha$$

Let

$$\Omega_\alpha = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha$$

Then formly

$$\Omega_\alpha e_\alpha = D^2 e_\alpha$$

Ω_α is called the curvature form of D. From the definition, it measures how far away D^2 from zero. In the Euclidean case, $D^2 = d^2 = 0$.

There are a lot of connections on a vector bundle E. The corresponding curvature is a matrix-valued 2-form. The invariants of the curvature tensor, surprisingly, is independent of the choice of the connection. So they are the properties of the vector bundle E itself. We will come back to this point in our third lecture.

In what follows we shall consider holomorphic vector bundle only. Let S be a section of $E \to X$, a holomorphic vector bundle of rank r. We know that

$$S = (S_\alpha)$$

where (S_α) are a set of $\mathbb{C}^r$-valued functions. Then

$$S_\alpha g_{\alpha\beta} = S_\beta$$

Since $g_{\alpha\beta}$ is holomorphic, we have

$$\bar\partial S_\alpha \, g_{\alpha\beta} = \bar\partial S_\beta$$

Thus $(\bar\partial S_\alpha)$ is a global section of $E \otimes \Lambda^{0,1}(X)$. Thus we have the following

Proposition: $\bar\partial$ is a global operator

$$\bar\partial : \Gamma(X, E) \to \Gamma(X, E \otimes \Lambda^{0,1}(X))$$

Before we define the Dolbeault cohomology, we take the following example:

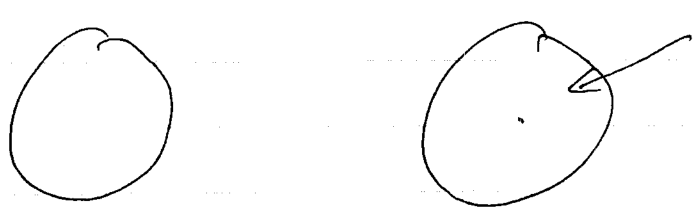
Example: The difference between the holomorphic function theories on Δ and Δ^*.

Let f be a holomorphic function on Δ, the close unit disk. Then the Stokes theorem gives

$$\oint_{\partial\Delta} f \, dz = 0.$$

On the other hand, Let $f = \frac{1}{z}$. Then f is holomorphic on Δ^* but

$$\oint_{\partial\Delta} f \, dz = 2\pi\sqrt{-1}$$



There is a hole there.

Furthermore, for any f holomorphic on Δ^*

$$\oint_{\partial\Delta} f\,dz = 2\pi\sqrt{-1}\,\operatorname*{Res}_{z=0} f$$

If we define

$$H(\Delta^*) = \{\text{all holomorphic functions on } \Delta^*\}$$

$$H_0(\Delta^*) = \{\text{all holomorphic functions with } \operatorname*{Res}_{z=0} f = 0\}$$

Then

$$H(\Delta^*)/H_0(\Delta^*) = \mathbb{C}.$$

On the other hand, we have the formula

$$\int z^{\mu} = \frac{1}{\mu+1} z^{\mu+1} + C$$

for $\mu \neq -1$. Thus if $f \in H_0(\Delta^*)$, $f = \frac{\partial g}{\partial z}$. ~~We have~~
or

$$f\,dz = dg = \partial g$$

Let

$$\tilde{H}(\Delta^*) = \{\bar{f}\,d\bar{z} \mid f \text{ holo on } \Delta^*\}$$

$$\tilde{H}_1(\Delta^*) = \{\bar{\partial}\bar{g} \mid g \text{ holo on } \Delta^*\}$$

Then

$$\tilde{H}(\Delta^*)/\tilde{H}_1(\Delta^*) = \mathbb{C}$$

And this is the anlytic way to detect if the space has a HOLE!

With the above example, we can define the Dolbecault cohomology:

Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold. Let

$$Z^{p,q}(E) = \{ \eta \mid \eta \text{ is a } E\text{-valued } (p,q) \text{ form}, \ \bar\partial \eta = 0 \}$$

$$B^{p,q}(E) = \{ \eta = \bar\partial \xi \mid \xi \text{ is a } E\text{-valued } (p,q-1) \text{ form} \}.$$

Then

$$H^q(X, \Omega^p(E)) = Z^{p,q}(E) / B^{p,q}(E)$$

Example: Let $X = T$ be the torus and let $E = \mathbb{C}$ be the trivial bundle. Let $p=0$, $q=1$. Then

$$Z^{0,1}(\mathbb{C}) = \{ \eta \mid \text{any } (0,1) \text{ form} \}$$

$$B^{0,1}(\mathbb{C}) = \cancel{\{ f d\bar z \mid \int_0^1 f(x+iy)\, dx = 0 \}}$$

$$= \{ \bar\partial f \mid f \in C^\infty(T) \}$$

It is easy to check that

$$Z^{0,1}(\mathbb{C}) / B^{0,1}(\mathbb{C}) = \mathbb{C}.$$

On a holomorphic vector bundle $E \to X$, we can define a Hermitian metric h_α such that

$$h_\alpha(s_\alpha, s_\alpha) = h_\beta(s_\alpha, s_\beta)$$

A connection D is called Hermitian connection, if

$$\begin{cases} D = D' + D'' & \text{with } D'' = \bar{\partial} \\ D h_\alpha \equiv 0 \end{cases}$$

Theorem: Let $E \to X$ be a holomorphic vector bundle with Hermitian metric. Then there is a unique Hermitian connection.

Proof: Let θ_α be the connection form defined from

$$\theta_\alpha = \partial h_\alpha h_\alpha^{-1}$$

Then it is easy to varify.

Let X be a complex manifold. Then we have a lot of functions, sections to consider. For example, we would like to define the ~~function~~ C^∞-functions, holomorphic functions, constant functions, continuous functions as well as C^∞, holomorphic, ~~constantus~~, locally constant sections of a vector bundle. A general treatment of the above concepts is the language of sheaf.

Definition A sheaf S of Abelian groups over X is a triple (S, π, X) satisfying

1. $\pi : S \to X$ is a local homemorphism of S onto X;
2. $\pi^{-1}(m)$ is an Abelian group for $m \in X$.
3. The composition laws are continuous in to topology of S.

We elaborate on (3). Let $S \circ S$ be the subspace of $S \times S$ such that $\pi(s_1) = \pi(s_2)$. Then (3). requies that the map $S \circ S \to S$, $(s_1, s_2) \mapsto (s_1 - s_2)$ be continuous.

Among the properties of sheaves, ① is the most important one because it gives restrictions of the possible sections.

Definition: A section $s : U \to S$ ~~is~~ for an open set $U \subset X$ is a continuous map such that $\pi \circ s = id$.

Example 1: Let $\mathbb{R}$ be the set of real numbers with the discrete topology. Then the triple $(\mathbb{R} \times X, \pi, X)$ is a sheaf, called the constant sheaf.

Let $U \subset X$ and let $s: U \to \mathbb{R} \times X$ be a section. Since $\mathbb{R}$ is given the discrete topology, Let V be any open set, then $V \times \{y\}$ is an open set of $\mathbb{R} \times X$. Thus

$$\{x \mid s(x) = y\}$$

is an open set. ~~it~~ ~~[illegible]~~ ~~it~~ it is obviously closed. So $s(x)$ is locally a constant.

Example 2: Sheaves of C^∞ functions. We assume that X is a C^∞ manifold. Let $p \in X$. Let $\mathcal{U}_p$ be the set of neighborhood of p. Define

$$F_p = \varinjlim_{\mathcal{U} \in \mathcal{U}_p} C^\infty(\mathcal{U})$$

The topology ~~and~~ between vector bundles and sheaves are totally different. We can think a sheaf as a book, whose open sets are made from open sets of each pages. In partic each page of the book is an open set.

Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X. We have the following sheaves

①. $\Omega^p(E)$ sheaf of holomorphic E-valued $(p,0)$ forms;
②. $\mathcal{A}^{p,q}(E)$ sheaf of C^∞ E-valued (p,q) forms.

Then we have

We define the Čech cohomology as follows: Let

$$\mathcal{U} = \bigcup U_\alpha$$

be a covering of U. Let

$$C^p(\mathcal{U}, S) = \prod \cancel{\Gamma(U_\alpha, S)}\, \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, S)$$

Then we have

$$\delta : C^p(\mathcal{U}, S) \longrightarrow C^{p+1}(\mathcal{U}, S)$$

by

$$(\delta\sigma)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0 \cdots \hat{i_j} \cdots i_{p+1}} \Big|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_{p+1}}}$$

It is easy to see that $\delta\delta = 0$. Thus the Čech cohomology ~~in to~~ with respect to the covering $\mathcal{U}$, is

$$\check{H}^p(\mathcal{U}, S) = \ker\delta \cap C^p(\mathcal{U}, S) \big/ \mathrm{Im}\,\delta$$

The Dolbeault Theorem is

Theorem Let $E \to X$ be a holo vector ~~of~~ bundle over X. Then

$$\check{H}^q(X, \Omega^p(E)) = H^q(X, \Omega^p(E))$$