Lecture two, Hodge Theory 8/12/2003

Let $E \to X$ be a Hermitian bundle over a compact Hermitian manifold. Then we have the following Hodge theorem.

Theorem (complex Hodge Theorem) Define

$$H^{p,q}(X,E) = \{ \varphi \mid \Delta_{\bar\partial} \varphi = 0 \}$$

Then

① $\dim H^{p,q}(X,E) < +\infty$

② $H^q(X, \Omega^p(E)) = H^{p,q}(X,E)$

We also have the real version of Hodge Theorem. In particular, we can define the real Laplacian Δ_d.

Now assume that X is a compact Kähler manifold such that the Kähler form ω is the curvature of some line bundle L over X. The pair (X, ω) is called a polarized ~~Hodge metric~~. kähler manifold. ~~Let~~ We wish to split the cohomology group $H^*(X, C)$ into pieces. There are two ways to split the group:

①. The Hodge decomposition. On a kähler manifold

$$\Delta_d = 2\Delta_{\bar\partial}$$

Thus we have the following

Theorem (Hodge)

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

2 The Lefschetz decomposition. On a polarized Kähler manifold, we can define the map

$$L : H^k(X, \mathbb{C}) \longrightarrow H^{k+2}(X, \mathbb{C}) \quad , \quad [\alpha] \longmapsto [\alpha \wedge \omega]$$

Since $H^k(X, \mathbb{C})$, $H^{k+2}(X, \mathbb{C})$ can be identified as harmonic k, $(k+2)$-forms, respectively. We can define the dual operator $\Lambda : H^k(X, \mathbb{C}) \longrightarrow H^{k-2}(\mathbb{C})$. We ~~leat~~ let

$$B = [\Lambda, L]$$

The actually B is a number operator. In fact

$$B\varphi^{p,q} = (n-p-q)\varphi^{p,q}$$

Proposition. We have

$$\begin{cases} [B, \Lambda] = 2\Lambda \\ [B, L] = -2L \\ [\Lambda, L] = B \end{cases}$$

Thus B, L, Λ spanned the Lie algebra $sl_2(2, \mathbb{C})$. In this way, $H^*(X, \mathbb{C})$ becomes a $sl_2(2, \mathbb{C})$-module.

By purely algebraic consideration, we have the following

Theorem (Lefschetz decomposition) On a polarized algebraic manifold (X, ω), we have the following

$$H^m(X, \mathbb{C}) = \bigoplus_{k=0} L^k P^{m-2k}(X, \mathbb{C})$$

where

$$P^k(X, \mathbb{C}) = \{ [\alpha] \mid \Lambda[\alpha] = 0 \}$$

The Hodge decomposition is compatible with the ~~Lefsch~~ Lefschetz decomposition in the sense that

$$P^k(X,\mathbb{C}) = \bigoplus_{p+q=k} P^{p,q}(X,\mathbb{C})$$

Now let

$$H_{\mathbb{Z}} = P^n(X,\mathbb{C}) \cap H^n(X,\mathbb{Z})$$

Let

$$H^{p,q} = P^n(X,\mathbb{C}) \cap H^{p,q}(X,\mathbb{C}).$$

Then we have

$$\bigoplus H^{p,q} = H_{\mathbb{Z}} \otimes \mathbb{C}$$

We have another way to represent the decomposition of $H_{\mathbb{Z}} \otimes \mathbb{C}$. We can define a filtration

$$F^k = H^{n,0} \oplus H^{n-1,1} \oplus \cdots \oplus H^{k,n-k}$$

Then

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = H$$

and we have

$$H^{p,q} = F^p \cap \bar{F}^q, \quad F^p \oplus \bar{F}^{n-p+1} = H_{\mathbb{Z}} \otimes \mathbb{C}$$

The set $\{H^{p,q}\}$ and $\{F^p\}$, in determine the decomposition of $H_{\mathbb{Z}} \otimes \mathbb{C}$, are equivalent.

We define a quadratic form Q on $H_{\mathbb{Z}}$ as follows

$$Q(\varphi,\psi) = (-1)^{n(n-1)/2} \int_X \varphi \wedge \psi$$

for $\varphi, \psi \in H_{\mathbb{Z}} \otimes \mathbb{C}$. Q is a nondegenerate form.

Furthermore, Q satisfies the following two Hodge-Riemann relations:

① $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = n-p$, $q' = n-q$

② $(\sqrt{-1})^{p-q} Q(\varphi, \overline{\varphi}) > 0$ for $\varphi \in H^{p,q}$, $\varphi \neq 0$.

Definition A polarized Hodge structure of weight n, is given by $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ satisfying the above relations.

Next we turn to the deformation of complex structures.

We say that the triple $(\mathcal{X}, \pi, B)$ is a family of compact complex manifolds, if

① $\mathcal{X}$, B are complex manifolds and $\pi: \mathcal{X} \to B$ is a surjective holomorphic map;

② For arbitrary $p \in B$, $\pi^{-1}(p)$ is a compact complex manifold and $d\pi$ has full rank.

The implication of deformation of complex structures is that the differentiable structure wont change. Or in other word, for $p_0 \in B$ and $p \in B$ closed to p_0 enough, $\pi^{-1}(p)$ and $\pi^{-1}(p_0)$ are differmorphic.

To see this, we consider a vector field X near the point p_0 on B. If we can find an $\widetilde{X}$ on $\mathcal{X}$ such that

$$\pi_* \widetilde{X} = X$$

Then the flow of $\tilde{X}$ gives the differmorphism between fibers. In the C^∞ catagory, it is not difficult to do so using partition of unity.

We assume the base space $B = \Delta$, the unit disk. Since all fibers are differmorphic, the different complex structures can be characterized by the $\bar\partial$-operator. Assume that for $t \in B$ small, $\bar\partial_t$ is the $\bar\partial$-operator for the complex manifold $\pi^{-1}(t)$. After a differmorphism, we may assume that

$$\bar\partial_t = \bar\partial + \eta$$

$\eta = \eta_j^i(z, t) \frac{\partial}{\partial z_i} d\bar z_j$. The integrabity condition $\bar\partial_t^2 = 0$ gives the equation

$$\bar\partial \eta + \frac{1}{2}[\eta, \eta] = 0$$

We assume that

$$\eta = t\eta_1 + t^2\eta_2 + \cdots$$

Then

$$\bar\partial \eta_1 = 0$$

Thus $[\eta_1] \in H^1(X, TX)$. $[\eta_1]$ defines an infinitesimal deformation of complex structure.

Kodaira-Spencer map

Example Let $\mathcal{X} \to B$ be a family of Kähler-Einstein manifold. That is, for each t, there is the unique $g(t)$ such that

$$Ric(g(t)) = k\,\omega_{g(t)}$$

where $k \neq 0$ is a constant and $\omega_{g(t)}$ is the Kähler form of $g(t)$. The following result is observed by Schmucher

Theorem: The form

$$\xi_t = -\bar\partial g^{i\bar j} \frac{\partial^2}{\partial t\,\partial \bar z_j} \log \det g(t) \frac{\partial}{\partial z_i}$$

is a harmonic form in $H^1(\pi^{-1}(0), T\pi^{-1}(0))$. Furthermore, the Kodaira - Spencer map is given by

$$\frac{\partial}{\partial t} \mapsto \xi_t$$

Remark: Let $\varphi_t : \pi^{-1}(0) \to \pi^{-1}(t)$ be a harmonic map. Then the infinitesmal of φ_t is the above harmonic $T\pi^{-1}(0)$ valued form.

Of course, the biggest problem ~~in~~ deformation theorem is the inverse problem: For any infintesmal deformation $[\eta] \in H^1(X, TX)$, can it be realized as a "real" deformation

The above problem, in terms of differential equations, is

$$\begin{cases} \bar\partial \eta + \frac{1}{2}[\eta, \eta] = 0 \\ \eta(0) = 0 \\ \eta'(0) = \eta_1 \quad \text{given} \end{cases}$$

Can we solve η as

$$\eta = t\eta_1 + t^2\eta_2 + t^3\eta_3 + \cdots$$

Theorem (Kodaira - ~~Nirenberg~~ Spencer) Yes if $H^2(X, TX)=0$

Example: If X is Calabi-Yau 3-fold. That is K_X is trivial. Then

$$H^2(X, TX) = H^1(X, \Omega^1) \neq 0$$

because $[\omega] \in H^1(X, \Omega^1)$

Theorem (Tian) The deformation of X is unobstructed if X is Calabi-Yau.

Proof: We assume that

$$\eta = t\eta_1 + t^2\eta_2 + \cdots$$

Then the equation becomes

$$\bar\partial \eta_N + \frac{1}{2}\sum_{i=1}^{N-1}[\eta_i, \eta_{N-i}] = 0 \qquad N \geq 2$$

Since $K_X = 0$, ~~Tian observed~~ Tian observed the following identity

$$[\omega_1, \omega_2] = \partial(\omega_1 \boxtimes \omega_2) - \partial\omega_1 \Delta \omega_2 + \omega_1 \Delta \partial\omega_2$$

for $\omega_1, \omega_2 \in \Gamma(X, \Omega^{n-1,1})$

The following result is from Mumford

Theorem (Mumford) The coarse moduli space of polarized Calabi-Yau manifolds exist.

Theorem The moduli space of polarized Calabi-Yau manifold is a complex orbifold or smooth Deligne-Mumford stack.

Moduli spaces of algebraic curves and Calabi-Yau moduli are smooth enough such that we can use differential geometry to study them.

A brief introduction of variation of Hodge structure.

Definition The classifying space D for the polarized Hodge structure is the set of all filtrations

$$0 \subset F^n \subset \cdots \subset F^1 \subset F^0 = H$$

with $F^p \oplus \bar{F}^{n-p+1} = H$ such that they satisfy the Hodge-Riemann Relation.

Remark: D is a homogeneous complex manifold, but the invariant Hermitian metric is in general not Kähler.

Griffith defined the following period map

$$f: \mathcal{M} \longrightarrow \Gamma \backslash D$$

where $\mathcal{M}$ is the ~~mo~~ Calabi-Yau moduli, and Γ is the monodromy group. The map essentially send a Calabi-Yau manifold to its Hodge structure.

We have the following property

Proposition

①. If $p \in \mathcal{M}$ is a smooth point, then f is an immersion

②. f is holomorphic

We have Hodge bundles F^k over the classifying space. The pull back of these Hodge bundles give a set of Hodge bundles over M. Let $C_1(F^n)$ be the curvature of the first Hodge bundle. Then it was observed by Tian that the classical Weil-Petersson metric can be defined via the period map:

$$\omega_{WP} = p^* C_1(F^n)$$

For Calabi-Yau 3-fold moduli space, the curvature of ω_{WP} is

$$R_{i\bar{j}k\bar{\ell}} = \delta_{i\bar{j}}\delta_{k\bar{\ell}} + \delta_{i\bar{\ell}}\delta_{k\bar{j}} - F * F$$

where F is the Yukawa coupling.

In order to seek a better metric on $\mathcal{M}$. we pull back the invariant Hermitian metric of D to $\mathcal{M}$. Define $\omega_H = p^* \omega_D$. Then

Theorem (Lu). ω_H is a Kähler metric. Furthermore, The bisectional curvature of ω_H is non-positive and the holomorphic sectional curvature and Ricci curvature are all bounded away from zero.

The most important ~~pro~~ global property of $\mathcal{M}$ is that it is quasi-projective. Viehweg was the first one who proved this fact. Recently, Donaldson proved the following result

Theorem (Donaldson) If ~~(M, L)~~ Aut(M, L) is discrete. Then M has a metric of constant scalar curvature $\Rightarrow$ M is Hilbert-Mumford stable.

To see Donaldson's result implies the second proof of the quasi-projectivity of $\mathcal{M}$, we need the following lemma

Lemma: If M is Calabi-Yau, then Aut(M) is discrete.

Proof: Aut(M) is a Lie group. Let X be a holomorphic vector field. Then

$$\Delta |X|^2 \geq 0$$

Thus $|X|^2 =$ const. In fact, we can prove that $X \equiv 0$. Thus Aut(M) is discrete.

Using the above lemma ~~and GIT~~, (M, L) is always stable. Thus using GIT, the moduli space is quasi-projective.