Gauge theory after Donaldson

$$\underset{M}{\text{Manifold}} \xrightarrow{\text{PDE}} \underset{\mathcal{M}}{\text{Moduli space}} \xrightarrow[\text{`counting'}]{H^*} \text{Invariants}$$

The hard parts include:

- compactification of $\mathcal{M}$,
- perturbation of the equations and transversality to make $\mathcal{M}$ smooth,
- infinite dimensional symplectic geometry (moment maps, etc.) to reinterpret the solutions of the equations in terms of purely geometric data (e.g. algebraic geometry),
- nonlinear analysis to solve the equations.

The art to doing this is to find the right PDE:

- geometric PDE, ideally first order,
- elliptic (so $\mathcal{M}$ finite dimensional),
- nonlinear with nonlinearities giving negative feedback so that solution space is compact (or compactifiable naturally so that cohomology classes extend over the compactification). (Topological bounds.)

Such equations usually come from physics as minimisers (or critical points) of some "action" functional (which then gives L^2 "energy" bounds).

E.g. the Yang-Mills functional for unitary connections on a Riemannian manifold.

$$YM(A) = \int_M |F_A|^2 = \int_M \operatorname{tr} F_A \wedge *F_A.$$

Critical points: to first order in a,

$$\begin{aligned} YM(A+a) &= YM(A) + \int_M \operatorname{tr} d_A a \wedge *F_A \\ &= YM(A) + \langle a, d_A^* F_A \rangle. \end{aligned}$$

So critical points are connections A satisfying the *Yang-Mills* equations:

$$\begin{aligned} d_A F_A &= 0 \\ d_A^* F_A &= 0. \end{aligned}$$

In some special geometries there are some connections which attain a *topological* absolute minimum of YM, giving a first order equation in A instead of second order. (Solutions often called *instantons*.)

("Supersymmetry/BPS solutions". Compare "maps in" equations, minimal submanifolds etc, and their first order versions – calibrated submanifolds. These are very often dual to gauge equations under string dualities, mirror symmetry, etc. and will arise via "bubbling" later.)

E.g. flat connections $F_A = 0$. Locally trivial connections described globally by their parallel transport (holonomy) round loops, so

$$\mathcal{M} \cong \text{Rep}\,(\pi_1(M), U(r))/\,\text{conjugation}.$$

E.g. On a Riemannian 4-manifold M, $\Lambda^2 \cong \Lambda^+ \oplus \Lambda^-$ the orthogonal ± 1 eigenspaces of $*$.

So $F_A = F^+ \oplus F^-$ with

$$\begin{aligned} YM(A) &= \int_M |F^+|^2 + |F^-|^2 \\ &= \int_M \operatorname{tr} F^+ \wedge F^+ - \operatorname{tr} F^- \wedge F^-, \end{aligned}$$

$$\begin{aligned} -4\pi^2 c_2(E) &= \int_M \operatorname{tr} F_A \wedge F_A \\ &= \int_M \operatorname{tr} F^+ \wedge F^+ + \operatorname{tr} F^- \wedge F^-. \end{aligned}$$

So

$$YM(A) = 4\pi^2 c_2(E) + 2 \int_M |F_A^+|^2$$

and *anti-self-dual* (asd) connections are absolute minima:

$$F_A^+ = 0.$$

(These can exist only if $c_2(E) \geq 0$, and are *flat* $F_A = 0$ if $c_2 = 0$. Of course asd connections are Yang-Mills:

$$d_A^* F_A = - * d_A * F_A = * d_A F_A = 0.)$$

E.g. On a Kähler manifold,

$$\Lambda^2 \otimes \mathbb{C} \cong \Lambda^{0,2} \oplus \Lambda_0^{1,1} \oplus \mathbb{C}.\omega \oplus \Lambda^{2,0}$$

gives a corresponding decomposition of F_A, with

$$YM(A) = \int_M 2|F^{0,2}|^2 + |\omega.F^{1,1}|^2 + |F_0^{1,1}|^2$$

and

$$-4\pi^2 c_2(E).\omega^{n-2} = \int_M 2|F^{0,2}|^2 + |\omega.F^{1,1}|^2 - |F_0^{1,1}|^2.$$

Therefore the Yang-Mills functional of A equals

$$4\pi^2 c_2(E) \cup \omega^{n-2} + \int_M 4|F^{0,2}|^2 + 2|\omega.F^{1,1}|^2$$

with absolute minima given by *Hermitian-Yang-Mills* connections

$$\begin{aligned} F_A^{0,2} &= 0, \\ \omega.F_A^{1,1} &= 0. \end{aligned}$$

(Exercise: these satisfy $d_A^* F_A = 0$ by using the Kähler identities.)

On a Kähler surface the self-dual 2-forms are

$$\{\sigma + \bar{\sigma} \,:\, \sigma \in \Lambda^{0,2}\} \oplus \langle \omega \rangle$$

so the asd equations are the same as the HYM equations.

These define holomorphic structures on E ($\bar{\partial}_A^2 = 0$) with the metric fixed (see below) by the HYM condition $\omega.F_A = 0$.

For E to carry a HYM connection, $\deg(E) = c_1.\omega^{n-1}$ must equal

$$\int_M \mathrm{tr}\, F_A \wedge \omega^{n-1} = \frac{1}{n} \int_M \mathrm{tr}(\omega.F_A)\omega^n = 0.$$

We can generalise to deg $\neq 0$ by splitting the ω component of F_A into a constant piece and an L^2-orthogonal piece; i.e.

$$F_A = F^{2,0} \oplus F^{0,2} \oplus F_0^{1,1} \oplus \lambda\omega.\,\mathrm{id} \oplus \sigma,$$

where λ is a (topological) constant $\deg E / \int_M \omega^n$, and $\int_M \mathrm{tr}\, \sigma \wedge \omega^{n-1} = 0$.

Then the same analysis shows that connections with $\sigma = 0$ are also absolute minima, giving the general HYM equations

$$\begin{aligned} F_A^{0,2} &= 0, \\ \omega.F_A^{1,1} &= \mathrm{const.\,id}\,. \end{aligned}$$

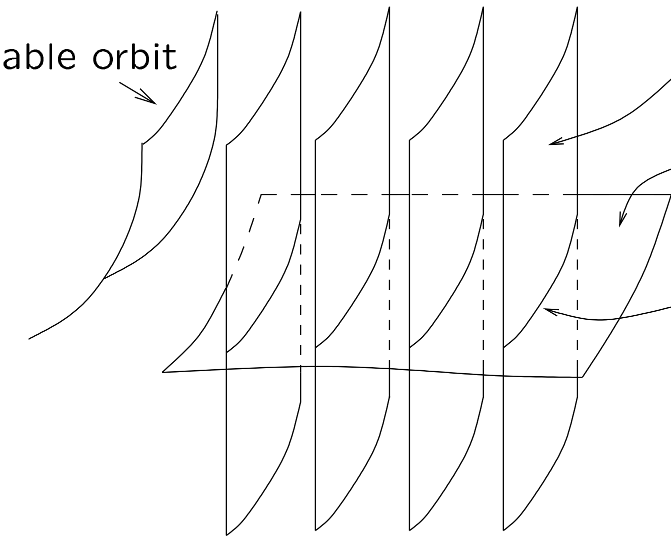
These have a purely holomorphic description since the second equation more-or-less fixes the metric, so that solutions are roughly equivalent to holomorphic structures (compare uniformisation for Riemann surfaces):

Theorem 6 [Donaldson-Uhlenbeck-Yau]
E admits a compatible Hermitian-Yang-Mills metric/connection iff it is polystable. This HYM connection is then unique.

E is stable iff for all coherent subsheaves $F \subset E$ we have $\mu(F) < \mu(E)$, where

$$\mu(E) = \frac{\int_X c_1(E) \cup \omega^{n-1}}{\text{rank}\, E} = \frac{\deg E}{\text{rk}\, E}.$$

This is the generic situation. Polystable = direct sum of stables of same slope.



(Moment map formalism is a nonlinear generalisation of $V/W \cong W^{\perp}$. Linearised gauge action on $A^{0,1} = \bar{\partial}_A$ is by adding $a^{0,1} = \bar{\partial}_A g$ for $g \in \Gamma(\mathrm{End}_0\, E)$; unitary action has $g \in \Gamma(\mathfrak{su}(E))$ skew-adjoint. Linearised HYM condition is

$$\omega.\partial_A a^{0,1} + (\omega.\partial_A a^{0,1})^* = 0 = \bar{\partial}_A^* a^{0,1} + (\bar{\partial}_A^* a^{0,1})^*.$$

So at level of tangent spaces moment map gives orthogonal slice to self adjoint part of gauge action; then divide by remaining skew adjoint part.)

Tian observed that these and many other gauge equations can be fit into a common framework. Suppose M is a Riemannian n-manifold with a distinguished closed $(n-4)$-form Ω (in examples it will be a *calibration*). Then we can consider Ω-asd connections A satisfying

$$*(F_A \wedge \Omega) = -F_A.$$

These satisfy the YM equations $d_A^* F_A = 0$, and are absolute minimisers of $YM(A)$ if Ω is a calibration.

$$\begin{aligned}
\Omega = 1 \quad &\Rightarrow \quad \text{asd equations} \\
\Omega = \omega^{n/2-2} \quad &\Rightarrow \quad \text{HYM equations} \\
\Omega_{\mathrm{Spin}(7)} \text{ or } \Omega_{G_2} \quad &\Rightarrow \quad \mathrm{Spin}(7)\,/G_2\text{-instantons} \\
\Omega = 4\,\mathrm{Re}\,\theta + \frac{\omega^2}{2} \quad &\Rightarrow \quad \text{complex asd equations} \\
\text{on CY 4-fold} \quad & \qquad\;\; F_A^{0,+} = 0 \quad (\Lambda^{0,2} = \Lambda^{0,+} \oplus \Lambda^{0,-})
\end{aligned}$$

We can think of gauge theory as nonlinear Hodge theory. (YM eqns $d_A F_A = 0 = d_A^* F_A \Rightarrow F_A$ harmonic...) E.g. linear case of flat $U(1)$ connections on topologically trivial bundle:

$$\begin{aligned} d_A &= d + A, \qquad A \in \Omega^1(M, i\mathbb{R}) \\ d_A^* F_A = 0 &\Leftrightarrow d^* dA = 0 \\ F_A = 0 &\Leftrightarrow dA = 0 \\ A \sim \exp(g)^*(A) &\Leftrightarrow A \sim A + dg. \end{aligned}$$

Instead of dividing by gauge transformations $\exp(g)$ connected to the identity we can impose the Coulomb gauge condition

$$d^* A = 0,$$

(i.e. A perpendicular to gauge orbits dg) giving us the usual Hodge theory $dA = 0 = d^* A$ (or YM $d_A^* F_A = \Delta A = 0$) with solutions $H^1(M, \mathbb{R})$. Further dividing out by gauge transformations not connected to the identity ($[M, U(1)] = H^1(M, \mathbb{Z})$!) makes this the torus

$$\mathcal{M} \cong \frac{H^1(M, \mathbb{R})}{H^1(M, \mathbb{Z})}.$$

Linearisation

More generally the linearisation of YM equations about a solution A have the same properties. For illustration we concentrate on the asd equation on *simply-connected* M^4, for unitary connections with *fixed determinant* and $\operatorname{rk} E = 2$, i.e. $SU(2)$ connections. Then $\mathcal{G} = SU(E)$ is connected.

If $F_A^+ = 0$ then $F_{A+a}^+ = d_A^+ a + (a \wedge a)^+$ is zero, to first order in small a, iff $d_A^+ a = 0$. We wish to divide by (small) gauge transformations: $\exp(g)^*(A) = A + d_A g$, so we are led to the linearisation

$$\Omega^0(\mathfrak{su}(E)) \xrightarrow{d_A} \Omega^1(\mathfrak{su}(E)) \xrightarrow{d_A^+} \Omega^+(\mathfrak{su}(E)).$$

This is an *elliptic* (the symbol sequence is exact) *complex* (as $d_A^+ d_A = 0$) describing the infinitesimal equation and gauge action; its first cohomology can be considered to be the "Zariski tangent space"

$$T_A\mathcal{M} = \frac{\ker d_A^+}{\operatorname{im} d_A} = H_A^1(\mathfrak{su}(E)).$$

(Ellipticity $\Rightarrow$ finite dimensional and, often, Sobolev solutions smooth. For what follows should work with L_k^2-Sobolev connections, $k > 1$, for completeness; Sobolev multiplication then allows us to form $a \wedge a \in L_k^2$ in 4 dimensions.)

Similarly $H_A^0(\mathfrak{su}(E))$ is the infinitesimal automorphisms (stabiliser) of A, and $H_A^+(\mathfrak{su}(E))$ is the "obstruction space" – the nonlinear piece $(a \wedge a)^+$ of the equations maps to here giving the obstruction to extending a (linearised) first order deformation in $H_A^1(\mathfrak{su}(E))$ to second order.

By Hodge theory theory $H^1_A(\mathfrak{su}(E))$ ker $d^+_A \cap$ker d^*_A; by the following theorem we can work locally about A we work with connections $A+a$ in Coulomb gauge: $d^*_A a = 0$.

Theorem 7 [Uhlenbeck]
$\mathcal{M}$ is locally homeomorphic about $A \in \mathcal{M}$ to

$$\{a\ :\ F^+_{A+a} = 0,\ d^*_A a = 0\}\Big/\Gamma_A,$$

and we have the estimate $||a||_{L^2_{k+1}} \leq c||a||_{L^2_k}$.
(Ellipticity of $d_A \oplus d^*_A$.)

Here Γ_A is the stabiliser of A

$$\{g \in \mathcal{G}\ :\ g^*(A) = A\} = \{g \in \mathcal{G}\ :\ d_A g = 0\}$$

of parallel automorphisms, with Lie algebra

$$\ker d_A \subset \Omega^0(\mathfrak{su}(E)).$$

As we perturb A these may no longer stabilise $A+a$, so we need to divide by them in our local model.

Usually try to avoid connections with Γ_A larger than $C(SU(2)) = \{\pm 1\}$, then the Theorem gives local model of $\mathcal{M}$: in the good case of $H^+_A(\mathrm{End}_0\, E) = 0$, $\mathcal{M}$ is smooth of dimension $h^1_A(\mathrm{End}_0\, E)$.

Γ_A is centraliser of the holonomy group of A in $SU(2)$:

$$U(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ or } \{1\} \text{ or } \{\pm 1\}$$

with $\Gamma_A = U(1),\ SU(2),\ SU(2)$ respectively. Correspond to (E, A) splitting as

$$L \oplus L^{-1} \quad \text{or} \quad L \oplus L \ (L \cong L^{-1}) \quad \text{or} \quad \underline{\mathbb{C}}^2,$$

and so are called *reducible* connections. (Reserve reducible for first case; other two are flat and won't feature.)

If (E, A) is reducible and $H^+_A(\mathrm{End}_0\, E) = 0$, then $h^0_A(\mathrm{End}_0\, E) = 1$ but vanishes for nearby A, so $h^1_A(\mathrm{End}_0\, E)$ also jumps by 1 at the reducible connection. We will use a case below where generically $h^1 = 5$, jumping to 6 at the reducible; locally $\mathcal{M}$ looks then like

$$\mathbb{R}^6/U(1) \cong \mathbb{C}^3/S^1 \cong \text{cone on } \mathbb{P}^2.$$

Reducibles in the asd moduli space correspond to line bundles with *abelian* asd connections, i.e. to asd harmonic 2-forms in

$$H^2(M; 2\pi i\mathbb{Z}) \cap \mathcal{H}^- \subset H^2(M, \mathbb{R})$$

(as $c_1(L)$ is integral). For $b^+ = \dim H^+(M) > 0$ could hope that generically this is empty and there are *no reducibles*.

Theorem 8 *If $b^+ > 0$ then $\mathcal{H}^- \subset H^2(M, \mathbb{R})$ misses the lattice $H^2(M; 2\pi i\mathbb{Z})$ for the generic choice of Riemannian metric on M.*

If $b^+ > 1$ then the same is true for the generic 1-parameter family of metrics.

Proof. The metric defines harmonic forms and so a point in

$$\mathrm{Gr}(H^2(M,\mathbb{R}), b^-).$$

The conformal structure of the metric is equivalent to the splitting of Λ^2, and so is a section of the

$$\mathrm{Gr}(\Lambda^2, 3)$$

bundle. Moving this section by a section $\phi \in \mathcal{H}om(\Lambda^-, \Lambda^+)$ we find that the projection of $\sigma \in \mathcal{H}^-$ to the new $\mathcal{H}^+$ is, to first order,

$$\pi_{\mathcal{H}} \circ \phi(\sigma) \in \mathcal{H}^+,$$

[DK Lemma 4.3.24]. Since ϕ can be arbitrary this map is *onto* and we can move any integral class away from $\mathcal{H}^-$ if $\mathcal{H}^+ \neq 0$. $\square$

Transversality

We can now deal with reducibles and so the quotient by the gauge group $\mathcal{G}$. Turn now to controlling the set of asd connections.

Zero set of map of Banach spaces

$$F_A^+ \colon \mathcal{A} \to \Omega^+(\mathfrak{su}(E)).$$

(Or, gauge fixing locally, $F_A^+ \oplus d_A^* \colon \mathcal{A} \to \Omega^+ \oplus \Omega^0$. Or, dividing by gauge and working on (the irreducibles in) $\mathcal{B} := \mathcal{A}/\mathcal{G}$, a section of $\Omega^+(\mathfrak{su}(E))$-Banach-bundle over $\mathcal{B}$.)

If this has surjective derivative at $0 \in \Omega^+(\mathfrak{su}(E))$ then by the Banach space Implicit Function Theorem $\mathcal{M}$ will be (away from reducibles) a smooth manifold of dimension $h_A^1(\mathfrak{su}(E))$.

Definition 9 *The virtual dimension of $\mathcal{M}$ is the index of the $\Omega^\bullet_A(\mathrm{End}_0\, E)$ complex*

$$d = -(h^0 - h^1 + h^+) = 8c_2(E) - 3(1 - b_1 + b^+)$$

by the Atiyah-Singer index theorem.

(A irreducible $\Leftrightarrow$ $h^0 = 0$; surjective derivative $\Leftrightarrow h^+ = 0$.)

To achieve this vanishing of $H^+_A(\mathfrak{su}(E))$ we have to again allow perturbations of the metric on M. We consider the above map to be defined on a bigger space

$$F^+_A \colon \mathcal{A} \times \mathrm{Conf} \to \Omega^+(\mathfrak{su}(E));$$

Conf$=$ { conformal classes of metrics on M }.

By the Smale-Sard theorem all we need is the surjectivity of the derivative of this map at $0 \in \Omega^+(\mathfrak{su}(E))$ to conclude that for a dense set of nearby metrics $\mathcal{M}$ is smooth.

Theorem 10 *The derivative of the above map is onto. So, for a generic metric,* $H_A^+(\mathfrak{su}(E)) = 0$ *for all* non-flat *asd connections* A. *If* $b^+(M) > 0$ *then* $\mathcal{M}$ *smooth.*

If $b^+(M) > 1$ *then any two such generic metrics can be joined by a path over which the universal moduli space is smooth.*

Last statement gives a cobordism between moduli spaces for different metrics, so the cobordism class of $\mathcal{M}$ is well defined in terms of only the C^∞ structure of M if $b^+(M) > 1$.

$\mathcal{M}$ is also oriented, but not necessarily compact.