Currents

X man.

$\mathcal{E}^p(X) = \Gamma(\Lambda^p T^* X)$ C^∞ top diff p-forms

$d : \mathcal{E}^p(X) \to \mathcal{E}^{p+1}(X)$

$\mathcal{E}'_p(X)$ = top dual of $\mathcal{E}^p(X)$ currents of dim. p.

$d : \mathcal{E}'_{p+1} \to \mathcal{E}'_p$. adjoint

$(dT)(\varphi) \equiv T(d\varphi)$

---

top: x

top

$T \in \mathcal{E}_p'$

$$\text{Mass}(T) \equiv \sup\{T\varphi : \sup_x \|\varphi_x\| = 1\}$$

If $\text{Mass}(T) < \infty$, then

$\exists$ Radon measure $\|T\|$

and $\vec{T}_x \in \Lambda^p T_x X$ $\qquad \|T\|$ a.a. x

$\|\vec{T}_x\| \equiv 1$

s.t.

$$T(\varphi) = \int \varphi(\vec{T}_x)\, d\|T\|(x)$$

Riesz Rep
R–N.

Ex. $M \subset X$ or p-dim'l sub of finite vol.

$$[M](\varphi) \equiv \int_M \varphi = \int_M \varphi(\vec{T}_x)\, d\mathcal{H}^p(x)$$

$\text{Mass}([M]) = \text{vol}(M)$

$d[M] = [\partial M]$ $\qquad$ Stokes Thm.

$\underline{\underline{Ex.2}}$ $E \subset M$ = as above

$$[E](\varphi) \equiv \int_E \varphi = \int_E \varphi(\vec{T_x})\, d\mathcal{H}^p$$

Rect. current of dim p

$$T = \sum_{j=1}^{\infty} n_j E_j$$

E_j as in Ex. 2. Mutually disjoint.

$\operatorname{supp} T \equiv \bigcup E_j \subset\subset X$

$$\text{Mass}(T) = \sum n_j \mathcal{H}^p(E_j) < \infty.$$

In this case $\vec{T_x}$ is a simple vector ~~$\mathcal{H}$~~ $\|T\|$ a.e.

$$\|T\| = \sum n_j \mathcal{H}^p \big|_{E_j}.$$

$\mathcal{R}_p(X)$ = set of rect. p-currents

$$\mathcal{I}_p = \{T \in \mathcal{R}_p : dT \in \mathcal{R}_{p-1}\}.$$

Thm $H_*(\mathcal{I}_*(X)) \cong H_*(X\,\mathbb{Z})$.

Thm Fix $K^{cpt} \subset X$ and $c > 0$

$\mathcal{I}_{p,c,K} = \{T \in \mathcal{I}_p : \text{Mass}\,T \leq c,\ \text{supp}\,T \subseteq K\}$

is compact in weak top

Def φ a cal. on X.

$T \in \mathcal{R}_p$ is a φ-current

if $\vec{T}_x \in \mathcal{G}(\varphi)$ $\|T\|$ a.a. x.

Thm φ-currents are H.M.M.

$\text{Mass}(T) = T(\varphi) = T'(\varphi) = \int \varphi(\vec{S}')\,d\mu$

-- etc

<u>Thm</u> (Almgren) T a φ-current

$\operatorname{supp} T =$ is a C^ω subman. (with $\mathbb{Z}$-mult.) with sing of codim-2

<u>Thm</u> (King) T a $\frac{\omega^p}{p!}$-current

Then $T =$ <u>holom. chain</u>

$= \sum n_i [W_i]$.

<u>Thm</u> Fix $K^{cpt} \subset X$ and $c > 0$. Then

$$\left\{ T : \begin{array}{l} T \text{ a } \varphi\text{-current} \\ \operatorname{supp} T \subset K \\ \text{Mass } T \le c \\ dT = 0 \end{array} \right\}$$

<u>weakly</u> <u>compact</u>.

Calibrations II

Special Lagrangian Geom.

$W = (W_{\mathbb{R}}, J)$ $\mathbb{C}$-vector space $\dim_{\mathbb{C}} W = n$.

$$J : W_{\mathbb{R}} \longrightarrow W_{\mathbb{R}}$$
$$J^2 = -Id.$$

$\langle \cdot , \cdot \rangle$ = pos. def. real inner prod. on $W_{\mathbb{R}}$

$$\langle Jv, Jw \rangle = \langle v, w \rangle \quad \forall v, w$$

Def A real subspace $V \subset W$ with $\dim_{\mathbb{R}} V = n$ is Lagrangian if $JV \perp V$

i.e. $\langle Ju, w \rangle = 0 \quad \forall\, u, w \in V$

Kähler form: $\omega(u, v) \equiv \langle Ju, v \rangle$

$$V \text{ is Lagrangian} \iff \omega\big|_V = 0$$

$\mathrm{Lag} \subset G_n(W)$ set of oriented Lagrangian n-planes

Lemma The unitary group U_n acts transitively on Lag. Fixing one plane $V_0 \in$ Lag gives a diffeomorphism

$$\text{Lag} \cong U_n / SO_n.$$

Pf Fix an o.o.n basis $e_1, \dots, e_n$ of V_0

Choose any $V \in$ Lag and any oon basis $e_1', \dots, e_n'$ of V.

$\exists!\ A \in U_n$ s.t.

$$A(e_i) = e_i' \quad i = 1, \dots, n.$$

$$(A(Je_i) = Je_i')$$

$\therefore A(V_0) = V$.

For $V = V_0$ we get $A \in SO_n$. qed

Def Fix $V_0 \in \mathrm{Lag}$ and set

$$\Phi \equiv (e_1^* + iJe_1^*) \wedge \cdots \wedge (e_n^* + iJe_n^*)$$

for an oon basis $e_1, \ldots, e_n$ of V_0.

- Φ is <u>independent</u> of choice.

$\tilde{e}_1, \ldots, \tilde{e}_n$ another choice

$$\tilde{e}_k = \sum_{\ell=1}^{n} g_{k\ell} e_\ell$$

$$J\tilde{e}_k = \sum g_{k\ell} Je_\ell$$

$$\tilde{\Phi} = \det((g_{k\ell}))\,\Phi = \Phi$$

- $V_0 = \mathbb{R}^n \subset \mathbb{C}^n = W$

$$\Phi = dz_1 \wedge \cdots \wedge dz_n.$$

Lemma For $\xi \in G_n(W)$

$$|\Phi(\xi)|^2 = |\xi \wedge J\xi|^* \leq 1$$

with equality $\iff \xi \in \mathrm{Lag}$.

Proof Let $\varepsilon_1, \dots, \varepsilon_n$ be an oon basis of ξ. Let $A: W \to W$ be the $\mathbb{C}$-linear map:

$$\left.\begin{aligned} A(e_i) &= \varepsilon_i \\ A(Je_i) &= J\varepsilon_i \end{aligned}\right\} \quad i = 1, \dots, n.$$

$$\varepsilon_1 \wedge \dots \wedge \varepsilon_n = A(e_1 \wedge \dots \wedge e_n)$$

(1). $$\Phi(\varepsilon_1 \wedge \dots \wedge \varepsilon_n) = (e_1^* + iJe_1^*) \wedge \dots \wedge (e_n^* + iJe_n^*)(A(e_1 \wedge \dots \wedge e_n))$$

$$= \det_{\mathbb{C}}(A)$$

$$|\Phi(\varepsilon_1 \wedge \dots \wedge \varepsilon_n)|^2 = |\det_{\mathbb{C}} A|^2$$

$$= |\det_{\mathbb{R}} A|$$

$$= |A(e_1 \wedge \dots \wedge e_n \wedge Je_1 \wedge \dots \wedge Je_n)|$$

$$= |\varepsilon_1 \wedge \dots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \dots \wedge J\varepsilon_n|$$

$$= |\xi \wedge J\xi|^*$$

$$\leq 1$$

"$= 1$" $\iff$ $\varepsilon_1, \dots, \varepsilon_n, J\varepsilon_1, \dots, J\varepsilon_n$ are orthogonal.

$$\boxed{\Phi : \mathrm{Lag} \longrightarrow S^1}$$

$$\mathrm{Lag} \simeq U_n / SO_n \xrightarrow[\det_{\mathbb{C}}]{} S^1 .$$

Def $\xi \in \mathrm{Lag}$ is Special Lagrangian if $\Phi(\xi) = 1$.

$\mathrm{SLag} \subset \mathrm{Lag}$. The set of those.

Write

$$\Phi = \varphi + i\psi$$

$$\begin{cases} \varphi = \mathrm{Re}\{\Phi\} \\ \psi = \mathrm{Im}\{\Phi\}. \end{cases}$$

Prop $\varphi(\xi) \leq 1 \quad \forall\, \xi \in G_n(W)$

and

$$G(\varphi) = \mathrm{SLag}.$$

Note $\xi \in G_n(W)$

$\varphi(\xi) = \pm 1 \iff \xi$ is Lag. and $\psi(\xi) = 0$

$\iff \omega\big|_{\xi} = 0$ and $\psi(\xi) = 0$.

Lemma $\xi \in$ Lag and $\varepsilon_1, \ldots, \varepsilon_n$ any oriented basis of ξ.

$A = \mathbb{C}$ $n \times n$-matrix

$A(e_k) = \varepsilon_k \qquad k = 1, \ldots, n.$
$(A(Je_k) = J\varepsilon_k)$

Then $\xi \in$ Slag $\iff$

$$\begin{cases} \operatorname{Re}\{\det_{\mathbb{C}} A\} > 0 \\ \operatorname{Im}\{\det_{\mathbb{C}} A\} = 0 \end{cases}$$

Pf $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n = \lambda \xi$ (some $\lambda > 0$)

$\Phi(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) = \det_{\mathbb{C}}(A).$

Note $\exists$ fam. of calibrations $\operatorname{Re}\{e^{i\theta}\Phi\}$.

$$\boxed{X = \mathbb{C}^n}$$

$$\varphi \equiv \mathrm{Re}\{dz_1 \wedge \cdots \wedge dz_n\}.$$

constant coeff form.
<u>Sp. Lag. Calibration</u>

$$G(\varphi) = \mathbb{C}^1 \times \mathrm{SLag}.$$

Lemma 1. $\Omega^{\text{open}} \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$
$\pi_1 \Omega = \{1\}$ C^1-map.

$$M \equiv \mathrm{graph}(f) \equiv \{(x, f(x)) \in \mathbb{R}^n \oplus \mathbb{R}^n : x \in \Omega\}.$$

The M is Lagrangian $\iff \dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i} \;\forall i,j.$

$\iff \exists\, F \in C^2(\Omega)$ s.t.

$$f = \nabla F$$

Pf Suffices to consider f linear.

Graph(f) is Lag. $\iff (v, f(v)) \perp J(w, f(w))$
$\forall\, v, w$

$\iff (v, f(v)) \perp (-f(w), w)$
$\forall\, v, w$

$\iff -\langle v, f(w)\rangle + \langle f(v), w\rangle = 0$

$\Longleftrightarrow \quad \langle v, f(w) \rangle = \langle f(v), w \rangle \quad \forall v, w \in \mathbb{R}^n$

This gives first claim: $f = f_*$ is symm.

This $\Longleftrightarrow$ $d(\sum f_i \, dx_i) = 0$

$\Longleftrightarrow$ $\sum f_i \, dx_i = dF$ some F. qed.

Prop $f : \Omega \to \mathbb{R}^n$ as above

Suppose graph(f) is Lagrangian and

$$f = dF$$

for a Potential fn $F : \Omega \to \mathbb{R}$.

Then graph(f) is Spec. Lag. (for approp. orientation)

$\Longleftrightarrow$

$$\boxed{\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1} (\operatorname{Hess} F) = 0}$$

Proof $e_1, \ldots, e_n$ standard basis of $\mathbb{R}^n$.

$\varepsilon_k = \cancel{(e_k, f} \; e_k + i f_*(e_k) \quad k = 1, \ldots, n,$

an oriented basis of $T(\text{grf})$.

$$\varepsilon_k = (I + i f_*)(e_k) \qquad k = 1, \dots, n$$

o-basis of tang space to the graph.

Apply Lemma:

$$SL \iff \operatorname{Im}\{\det(I + i f_*)\} = 0$$

$$\iff \operatorname{Im}\{\det(I + i \operatorname{Hess} F)\} = 0$$

$$\left(\begin{array}{l} f_* = \operatorname{Hess} F \\ f = \nabla F \end{array} \right)$$

qed.

Case $n = 3$

$$\boxed{\Delta F = \det(\operatorname{Hess} F)}$$

<u>Thm</u> $\Omega^{\text{open}} \subset \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}$ class C^2. If

$$\sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \sigma_{2k+1}(\operatorname{Hess} F) = 0 \quad \text{in } \Omega$$

Then the graph of ∇F is an absolutely volume minimizing submanifold of $\mathbb{R}^{2n}$.

Do solutions exist?

Yes!

I. Implicit Function Techniques

II. Cartan-Kahler Techniques.

III. Special Symmetric Sol^{ns}

IV. Normal bundles to austere sub'ns

I. $f = \nabla F : \Omega \to \mathbb{R}^n$ a sol^n.

$f_* = F_{**} = \mathrm{Hess}(F)$ satisfies

$$\begin{cases} \varphi((e_1 + i f_* e_1) \wedge \cdots \wedge (e_n + i f_* e_n)) > 0 \\ \psi(\text{ — } \quad \text{ — }) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \mathrm{Re}\{\det_{\mathbb{C}}(I + i F_{**})\} > 0 \\ \mathrm{Im}\{\det_{\mathbb{C}}(I + i F_{**})\} = 0. \end{cases}$$

Linearization at F

$$U : \Omega \to \mathbb{R} \qquad \text{class } C^2$$

Consider

$$\mathrm{Im}\{\det_{\mathbb{C}}(I + i(F + tU)_{xx})\}$$

$$= \mathrm{Im}\{\det_{\mathbb{C}}(A + itU_{xx})\}$$

$$A \equiv I + iF_{xx}$$

$$= \mathrm{Im}\{\det_{\mathbb{C}} A \, \det_{\mathbb{C}}(I + itA^{-1}U_{xx})\}$$

$$= \det_{\mathbb{C}} A \; \mathrm{Im}\{\det_{\mathbb{C}}(I + itA^{-1}U_{xx})\}.$$

Take $d/dt \cdot |_{t=0}$

$$\left[\frac{d}{dt} \det_{\mathbb{C}}(g_t)|_{t=0} = \mathrm{tr}\left(\frac{dg}{dt}|_{t=0}\right) \right]$$

if $g_0 = Id$.

Get

$$\mathrm{Im}\{\mathrm{tr}(iA^{-1}U_{xx})\}$$

$$\mathcal{L}_p(U) = \mathrm{Im}\{\det_{\mathbb{C}} A \cdot \mathrm{tr}(iA^{-1}U_{xx})\} = 0$$

$$= \operatorname{Im}\{ i \operatorname{tr}(A^* \cdot U_{**}) \}$$
$$= -\operatorname{Re}\{ \operatorname{tr}(A^* \cdot U_{**}) \}$$

$$= -\operatorname{tr}\{ \operatorname{Re}(A^* U_{**}) \}$$
$$= -\operatorname{tr}\{ (\operatorname{Re} A^*) U_{**} \}$$

$A^* \equiv$ transposed matrix of cofactors

So: F is a sol^n $\iff$ $\det_{\mathbb{C}} A > 0$

$A = I + i F_{**}$

and

$$\mathcal{L}_F(U) = -\operatorname{tr}\{ (\operatorname{Re} A^*) U_{**} \}.$$

This linearization is elliptic $\iff$

$\operatorname{Re} A^*$ is pos. def

F_{**} is symmetric $\therefore$

$\exists$ o o n basis s.t.

$$A = I + i F_{**} = \begin{pmatrix} 1+i\lambda_1 & & & \\ & 1+i\lambda_2 & & \\ & & \ddots & \\ & & & 1+i\lambda_n \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{1+i\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1+i\lambda_n} \end{pmatrix}$$

$$A^* = \det_{\mathbb{C}} A \cdot A^{-1}$$

$$\operatorname{Re} A^* = (\det_{\mathbb{C}} A) \operatorname{Re} A^{-1}$$

$$= (\det_{\mathbb{C}} A) \begin{pmatrix} \frac{1}{1+\lambda_1^2} & & \\ & \ddots & \\ & & \frac{1}{1+\lambda_n^2} \end{pmatrix}$$

$$> 0 .$$

ie

$$\mathcal{L}_F(U) = \sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2 U}{\partial x_i \partial x_j}$$

where $((g^{ij})) > 0$.

pd symm.

Thm Ω bounded strictly convex domain $\subset \mathbb{R}^n$.

smooth boundary. $F \in C^2(\bar{\Omega})$ a sol$^{\underline{n}}$ of the Special Lagrangian eq$^{\underline{n}}$.

Set $\mathcal{F} = F|_{\partial\Omega}$.

Then $\exists$ ~~a~~ nbhd $\mathcal{U}$ of $\mathcal{F}$ in $C^2(\partial\Omega)$

s.t. $\forall\ \mathcal{F}' \in \mathcal{U}$

$\exists\ F' \in C^2(\bar{\Omega}) \cap C^\omega(\Omega)$ with

① $F'|_{\partial\Omega} = \mathcal{F}'$

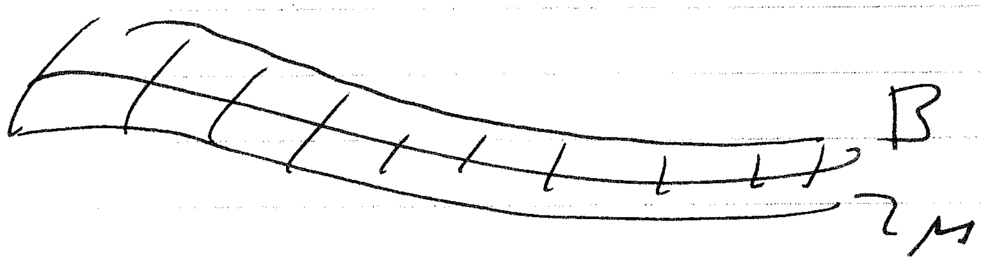
and

② F' is a sol^n of SL eq^n in Ω.

Thm Let $B^{n-1} \subset \mathbb{C}^n$ be a C^ω submanifold of $\dim_{\mathbb{R}} n-1$. Assume

$$\omega|_B \equiv 0 \quad (\text{"isotropic"})$$

Then $\exists$ a SLagrangean manifold $M^n \subset \mathbb{C}^n$ containing B^{n-1}



Pf ~~Cauchy-Kow.~~ Cartan-Kähler

Thm $U^{\text{open}} \subset \mathbb{C}^n$

$$H = (H_1, \dots, H_n) : U \longrightarrow \mathbb{R}^n \quad \text{smooth}$$

0 = reg-value so

$$M^n \equiv \{ z \in U : H(z) = 0 \} \quad \text{subman.}$$

Then M is Lag $\Longleftrightarrow$

$$\{h_k, h_\ell\} = 2i \sum_{j=1}^n \left(\frac{\partial h_\ell}{\partial \bar{z}_j} \frac{\partial h_k}{\partial z_j} - \frac{\partial h_k}{\partial \bar{z}_j} \frac{\partial h_\ell}{\partial z_j} \right)$$

$$\equiv 0 \quad \text{on } M$$

M is SLag if above and

$$\boxed{\operatorname{Im}\left\{ \det_{\mathbb{C}} \left(i \frac{\partial h_j}{\partial \bar{z}_k} \right) \right\} \equiv 0}$$

Ex Fix $c = (c_1, \dots, c_n) \in \mathbb{R}^n$

$$|z_1|^2 - |z_j|^2 = c_j \quad j = 2, \dots, n$$

$$\begin{cases} \operatorname{Re}(z_1 \cdots z_n) = c_1 & n \text{ even} \\ \operatorname{Im}(\text{---}) = c_1 & n \text{ odd.} \end{cases}$$

<u>Thm</u> $M^p \subset \mathbb{R}^n$

$N^n = \{(x,v) \in \mathbb{R}^n \oplus \mathbb{R}^n : x \in M,\ v \in N_x(M)\}$

(normal bundle in $T\mathbb{R}^n$)

N is $SL \iff M$ is "austere"

all odd symm. f$^{\underline{n}}$ of

2^{nd} FF vanish

When $p = 2$, <u>minimal</u> suffices.

Supplement to: Calibrated Geometries

by Reese Harvey and H. Blaine Lawson, Jr.

The Cartan-Kähler Theorem Applied to Boundaries of Minimal Surfaces in Calibrated Geometries

This supplement is included in preprint form for those not familiar with the Cartan-Kähler Theorem. First, a brief statement of the theorem is given and then six examples are discussed in detail. At the time this was written the authors were unaware of the beautiful notes of Bryant-Chern-Griffiths [1]. The reader should consult these notes for more information concerning differential systems.

The Cartan-Kähler Theorem

Suppose I is an ideal of differential forms. Let $I_k \equiv I \cap \Lambda^k$. The ideal I is said to be homogeneous if $I = \Sigma I_k$ (i.e. a form belongs to I if and only if each component of degree k belongs to I). The ideal I is said to be a differential system if $dI \subset I$. We shall assume that I is a homogeneous differential system which contains no functions (i.e. $I_0 = 0$).

Let $\mathcal{L}_k$ denote the space of k-dimensional <u>integral elements</u> W for $\mathcal{I}$ (at a fixed point x_0). A k-dimensional subspace W of T_{x_0} is an <u>integral element</u> for $\mathcal{I}$ if

$$i_W^* \alpha = 0 \quad \forall \alpha \in \mathcal{I}$$

Here $i_W : W \hookrightarrow T_{x_0}$ denotes inclusion. Equivalently, W is an <u>integral element</u> for $\mathcal{I}$ if

$$i_W^* \alpha = 0 \quad \forall \alpha \in \mathcal{I}_k .$$

This equivalence follows since $i_W^*(\beta \wedge \gamma) = i_W^*(\beta) \wedge i_W^*(\gamma)$ and $\mathcal{I}$ is an ideal.

Given an integral element $W \in \mathcal{L}_k$, let

$$E_k(W) \equiv \{u \in T_{x_0} : W + [u] \text{ is an integral element}\}.$$

denote the <u>polar space</u> of W or the <u>space of enlargement possibilities</u>. Note that $W \subset E_k(W)$ since W is integral. Consequently, dim $E_k(W) = k+1+r_k(W)$ where $0 \le r_k(W)$ measures the excess of the possibilities for enlarging W by one more dimension. The space $E_k(W)$ can be looked at dually as follows. Choose $\xi = W_1 \wedge \quad \wedge W_k$ a k-vector with $W \equiv \text{span}\ \xi$. The space $E_k(W)$ can be computed as the annihilator or polar of $\{\xi \lrcorner \alpha : \alpha \in \mathcal{I}_{k+1}\}$, a subspace of $T^*_{x_0}$. If $W \in \mathcal{L}_k$ is chosen so that dim $\{\xi \lrcorner \alpha : \alpha \in \mathcal{I}_{k+1}\}$ has the maximum possible value, then the dimension remains the maximum possible value for nearby $W \in \mathcal{L}_k$.

Equivalently (since $k+1+r_k(W)$ is the codimension) if $r_k(W)$ has the minimum possible value, then $r_k(\ )$ has the same value for nearby $W \in L_k$. Let $r_k \equiv \min \{r_k(W) : W \in L_k\}$ denote the minimum excess.

<u>Definition</u> 1): If $W \in L_1$ and $e_1(W) = e_1$ then W is called a <u>regular</u> integral element.

2) If $W \in L_k$, $e_k(W) = e_k$ <u>and</u> W contains a regular integral element of dimension $k-1$ then W is called a <u>regular</u> integral element.

It is easy to see that if (and only if) $W_k \in L_k$ is a regular integral element then there exists a flag of vector spaces $W_1 \subset \ldots \subset W_k$ with $W_j = \text{span}\{w_1, \ldots, w_j\}$ an integral element of dimension j and $r_j(W_j) = r_j$ for $j = 1, \ldots, k$.

Now suppose U, V are integral elements with U a co-dimension 1 subspace of V (such as $W_j \subset W_{j+1}$ above). Then $E(V) \subset E(U)$. To prove this suppose $u \in E(V)$. That is $V+[u]$. is an integral element. Note $U+[u]$ is a subspace of $V+[u]$ Since a subspace of an integral element is always an integral element this proves $u \in E(U)$. In particular, $\dim E(V) \le \dim E(U)$. If $j \equiv \dim U$ then $\dim E(V) = \dim V+1 + r_{j+1}(V) = j+2 + r_{j+1}(V)$ is less than or equal to $\dim E(U) = j+1 + r_j(U)$. Therefore $r_{j+1}(V) < r_j(U)$. Consequently the absolute excesses are strictly decreasing: $r_1 > r_2 > \ldots r_g = -1$.

The genus is the smallest g with $r_g = -1$. Everything in the following theorem is assumed real-analytic, and all statements are local.

Theorem. (Cartan-Kähler) Suppose I is a homogeneous differential system with $I_0 = 0$. Given a k-dimensional integral submanifold N, assume that $T_{x_0}N$ is regular. Next prescribe a "constraining submanifold" P of codimension r_k (note $r_k = r_k(T_{x_0}N)$) with $P \supset N$ and $T_{x_0}P$ transverse to $E_k(T_{x_0}N)$. (Then, in particular, $\dim E_k(T_xN) \cap T_xP = k+1$ for x near x_0 -- that is P determines an integral $k+1$ plane field along N and tangent to N.) Then there exists a unique integral submanifold M of dimension $k+1$ with $N \subset M \subset P$.

Example I. (Foliations) Suppose I is algebraically generated by a Pfaffian system, i.e., I is generated by I_1 (1-forms). Then $E = \{u \in T_{x_0} : [u]$ is integral, or $\alpha(u) = 0 \ \forall \alpha \in I_1\}$ is just the polar of I_1. Note that $W \in L_k$ is an integral element if and only if $W \subset E_1$. Moreover, $E_k(W) \equiv E_1$. Hence if $\rho \equiv \dim E_1$ and W is an integral element then $r_k(W) = \rho - k$. In particular r_k does not depend on $W \in L_k$ and $r_k(W)$ is locally a minimum (equivalently locally constant) if and only if the rank ρ is locally a minimum (equivalently, locally constant). Therefore, we must assume that the rank ρ of E_1 is constant or equivalently, the rank $n-\rho$ of I_1 is constant. Then each $W \in L_k$ is regular and $r_k = \rho - k = 1$. Consequently the genus is ρ. There are no integral elements of dim $\rho+1$ but there are regular integral elements of dim ρ.)

If N^k, an integral submanifold, is prescribed then since $r_k = \rho-k-1$ a "constraining submanifold" P of codimension $\rho-k-1$ transverse to the ρ-plane field E_1 must be prescribed to insure uniqueness. Of course if N is of dimension $k = \rho-1$ then no choice of P is required, we must take P equal to the ambient manifold. The Cartan-Kähler theorem concludes that there exists a unique integral submanifold M of dimension $k+1$ with $N \subset M \subset P$. If M is of maximal dimension ρ, then M is called a _leaf_ of the _foliation_. Repeated application shows that each k-dimensional integral submanifold N is contained in a leaf, say $\tilde{N}$. Moreover $\tilde{N}$ is unique. Since $E(W_k) = E_1 = I_1^{\perp}$ is the same for all k, each extension M given by (repeated) application of the Cartan-Kähler theorem must coincide with $P \cap \tilde{N}$ by the uniqueness part of the Cartan-Kähler Theorem. Thus each M must be contained in $\tilde{N}$, and hence at the last stage $M = \tilde{N}$, proving that $\tilde{N}$ is unique. The Cartan-Kähler Theorem implies that:

Each integral submanifold N _is contained in a unique leaf of the foliation_.

Conversely, this result implies the Cartan-Kähler Theorem generated by 1-forms; take $M \equiv P \cap \tilde{N}$!

Remark: This Example I is, of course, simply meant to be instructive in understanding the Cartan-Kähler Theorem. Using results about solving ordinary differential equations, (in the real-analytic category) the above result is obtained more easily. Moreover, the standard C^∞ version is also seen to be valid.

<u>Example II</u>. (<u>Complex Geometry</u>) Suppose we are given a complex manifold of dimension n.

Consider the ideal I generated by

$\{\mathrm{Re}\,dz^I, \mathrm{Im}\,dz^I : |I| = p+1\}$. The complexification $I \otimes_{\mathbb{R}} \mathbb{C}$ of I consists of all forms of bidegree r,s with either $r > p$ or $s > p$.

Before discussing the integral elements of I, we briefly mention some standard concepts concerning real k-dimensional linear subspaces W_k of a complex vector space $\mathbb{C}^n$ (later to be taken as the tangent space T_{z_0} to the ambient complex manifold with complex structure J).

First, the <u>holomorphic part of</u> W, denoted $H(W)$ is defined by

$$H(W) \equiv W \cap JW.$$

Note the $H(W)$ is the largest real linear subspace of W which is also complex linear; or equivalently $H(W)$ is the union of all the complex lines in W. The complex dimension of $H(W)$ is called <u>holomorphic dimension</u> of W and will be denoted h or $h(W)$. A real linear subspace W is said to be <u>totally real</u> if $H(W) = \{0\}$, or $h = 0$. Let R denote any complementary subspace to $H(W)$ in W. Then R

is totally real. The dimension of R will be referred to as the <u>totally real dimension of</u> W and denoted by ρ or $\rho(W)$. Thus by a complex linear coordinate change

$$W \cong \mathbb{C}^h \times \mathbb{R}^\rho, \quad \text{with} \quad H(W) \cong \mathbb{C}^h \quad \text{and} \quad R \cong \mathbb{R}^\rho.$$

In particular,

$$k = 2h(W) = \rho(W).$$

The complex subspace of $\mathbb{C}^n$,

$$\tilde{E}(W) = W + JW,$$

is called the complex envelope of W. Note that $\tilde{E}(W)$ is the smallest real linear subspace of $\mathbb{C}^n$ containing W which is also complex linear. The complex dimension of $\tilde{E}(W)$ is called the <u>complex envelope dimension of</u> W and denoted ε or $\varepsilon(W)$. Note that

$$\tilde{E}(W) = H(W) \oplus R \oplus JR.$$

In particular,

$$\varepsilon(W) = h(W) + \rho(W),$$

and $H(W)\oplus R\oplus JR$ is independent of the choice of the totally real complement R to $H(W)$ in W. Also note that W_k is generic or minimally complex in $\tilde{E}(W)$.

We may choose a complex basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$ (i.e., $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ is a real basis for $\mathbb{C}^n$) so that

$e_1, \ldots, e_h$ is a complex basis for $H(W)$, and $e_1, \ldots, e_h$, $Je_1, \ldots, Je_h, e_{h+1}, \ldots, e_{h+\rho}$ is a real basis for W. Then $e_1, \ldots, e_\varepsilon$ is a complex basis for $\tilde{E}(W)$ and R can be taken to be the real span of $e_{h+1}, \ldots, e_{h+\rho}$. Consider the real k-vector

$$\xi = e_1 \wedge Je_1 \wedge \ldots \wedge e_h \wedge Je_h \wedge e_{h+1} \wedge \ldots \wedge e_{h+\rho}$$

corresponding to the real k-dimensional linear subspace W. Let $\xi_{r,s}$ denote the component of ξ of bidimension r,s. The totally real dimension $\rho(W)$ can be characterized as the smallest integer ρ such that:

$$\xi = \sum_{\substack{|r-s| \le \rho \\ r+s=k}} \xi_{r,s} .$$

The complex envelope dimension $\varepsilon(W)$ can be characterized as the smallest integer ε such that:

$$\xi = \sum_{\substack{r,s \le \varepsilon \\ r+s=k}} \xi_{r,s}$$

Both can be seen as follows. As is standard,

let

$$\partial/\partial z_j = \tfrac{1}{2}(e_j - iJe_j) \quad \text{and} \quad \partial/\partial z_j = \tfrac{1}{2}(e_j + iJe_j) \quad j = 1, \ldots, n.$$

Then $e_j = \partial/\partial z_j + \partial/\partial z_j$, $Je_j = i(\partial/\partial z_j - \partial/\partial z_j)$, and

$$e_j \wedge Je_j = \frac{2}{i}\, \partial/\partial z_j \wedge \partial/\partial z_j. \quad \text{Consequently,}$$

$$\xi = \left(\frac{2}{i}\right)^h \partial/\partial z_1\ \partial/\partial z_1 \wedge \ldots \wedge \partial/\partial z_h \wedge \partial/\partial z_h \wedge (\partial/\partial z_{h+1} + \partial/\partial z_{h+1}) \wedge \ldots \wedge (\partial/\partial z_\varepsilon + \partial/\partial \bar{z}_\varepsilon)$$

is the sum of non-zero terms $\xi_{r,s}$ of bidimension r,s with

$r = h+i$, $s = h+\rho-i$ and $0 \le i \le \rho$. Hence

$s-r = 2\rho - i$ varies from ρ to $-\rho$ as desired.

Now we can describe the integral elements of I. Suppose W_k is a k dimensional subspace of T_{z_0} and let $\xi(W)$ denote a corresponding k vector. Then W is an integral element of I if and only if

$$\xi = \sum_{r,s\le p} \xi_{r,s}$$

(That is $\xi_{r,s} = 0$ if either $r > p$ or $s > p$.)

Comparing this with the above decomposition

$\xi = \sum_{\substack{r,s\le\varepsilon \\ r+s=k}} \xi_{r,s}$, we have that:

W is an integral element if and only if the complex envelope dimension $\varepsilon(W)$ is less than or equal to p.

Next we discuss regular integral elements. Since each subspace of dimension $\le p$ is integral we have:

<u>Case 1 ($k < p$)</u>: Each subspace W_k is a regular integral element, and $E_k(W_k)$, the space of enlargment possibilities, is all of $\mathbb{C}^n$. In particular $r_k = 2h-k-1$.

<u>Case 2 ($k \ge p$)</u>: A subspace W_k is a regular integral element if and only if the complex envelope dimension $\tilde{E}(W_k)$ is equal

to p, and $E_k(W_k)$, the space of enlargement possibilities for a regular integral element W_k, is the complex envelope $\tilde{E}(W_k)$ of W_k. In particular, the minimum value of $r_k(W)$ is

$$r_k = 2\varepsilon - k-1 = 2p - k-1 = \rho - 1.$$

Case 1 is immediate since each W_k is integral for $k \leq p$. In order to prove the statement in Case 2 we compute $E_k(W_k)$ the space of enlargement possibilities. If $\varepsilon(W_k) < p$ then $\varepsilon(W + [v]) \leq p$ for all v and hence $E_k(W) = \mathbb{C}^n$. However, if $\varepsilon(W_k) = p$ then $\varepsilon(W + [v]) = p$ if and only if $v \in \tilde{E}(W)$ the complex envelope of W.

In summary, for an integral element W,

$$E_k(W) = \mathbb{C}^n \quad \text{if} \quad \varepsilon(W) < p, \quad \text{and}$$

$$E_k(W) = \tilde{E}(W) \quad \text{if} \quad \varepsilon(W) = p.$$

The conclusions listed under Case 2 now follow.

Note $r_1 = 2n-2$, $r_2 = 2n-3$, $\ldots$, $r_{p-1} = 2n-p$; $r_p = p-1$,

$r_{p+1} = p-2$, $\ldots$, $r_{2p-1} = 0$, $r_{2p} = -1$, so that the

genus is $g \equiv 2p$.

A submanifold N is said to be a <u>C.-R. submanifold</u> if the dimension of the holomorphic part of the tangent space to N is locally constant, or equivalently, if the dimension of the complex envelope of the tangent space to N is locally constant.

The Cartan-Kähler Theorem applied to the differential ideal I defined above, now reduces to the following.

Suppose N is a k-dimensional C.-R. submanifold of a complex manifold with complex envelope dimension equal to p (note $p \leq k$ so we are in Case 2 above). Prescribe a constraining submanifold P of codimension $r_k = 2p-k-1 = \rho - 1$ containing N and transverse to the complex envelope of the tangent space to N. Then there exists a unique C.-R. submanifold M of dimension $k+1$ with $N \subset M \subset P$ and the complex envelope of T_zM is the same as the complex envelope of T_zN.

By repeating the application of the Cartan-Kähler Theorem to M etc. we obtain a complex p-dimensional submanifold $\tilde{N}$ containing N. Since, at each stage, the space of enlargement possibilities is the same, namely the complex envelope of T_zN, arguing exactly as in the case of Foliations to get uniqueness we have the following consequence of the Cartan-Kähler Theorem.

<u>Theorem</u>. <u>Suppose</u> N <u>is a k-dimensional</u> C.-R. <u>submanifold with the dimension of the complex envelope</u> $\tilde{E}(T_zN)$ <u>of the tangent space equal to</u> p. <u>There exists a unique complex submanifold</u> $\tilde{N}$ <u>of complex dimension</u> p <u>with</u> $N \subset \tilde{N}$.

Conversely, this result implies the previous result. Simply take $M \equiv P \cap \tilde{N}$.

Remark 1: The analogous results are false will real-analytic replaced by C^∞. However, if one weakens the conclusion to read that "$\tilde{N}$ is complex to infinite order along N" the result is still valid.

Remark 2: There is a brief elementary proof of this Theorem which avoids the Cartan-Kähler Theorem. (This will not be true for our later examples.) Consider $0 \in U \subset R^k$ where U is open and let $f : U \to N \subset C^n$ be a local real analytic coordinate chart. There exists $r > 0$ such that for $|x| < r$, f is represented by a power series $f(x_1, \ldots, x_k) = \Sigma\, a_I x^I$ with complex vector coefficients. Set $F(z_1, \ldots, z_{2k}) = \Sigma\, a_I z^I$ where $z_j = x_j + iy_j$ and $|z| < r$. The hypothesis on N implies that the rank of the complex Jacobian of F is exactly p for $|z| < r$ and z real (i.e. at points of N). Hence, this is true for all z near zero. It follows that the image under F is an p-dimensional complex submanifold $\tilde{N} \supset N$. The uniqueness is obvious.

A C.R. submanifold S is said to be generic or minimally complex if the complex envelope of the tangent space to N is the ambient tangent space. The above Theorem can be reformulated by considering N and $\tilde{N}$ to be graphed over their tangent space.

Theorem. Suppose S is a generic C.-R. submanifold of $\mathbb{C}^p$ and f is a C.-R. function on S. Then there exists a unique

<u>holomorphic function</u> F <u>on</u> $\mathbb{C}^p$ <u>with</u> $F|_S = f$.

<u>Proof</u>: Let N = graph f over S. It is easy to see that the following are equivalent:

1) N has complex envelope dimension p.
2) N has the same holomorphic dimension as S.
3) The differential of f restricted to the holomorphic part of the tangent space is complex linear (i.e., f is a C.-R. function).

Similarly, let $\tilde{N}$ = graph F and note that $\tilde{N}$ is a complex submanifold if and only if F is holomorphic. Now it is easy to see the above two Theorems are equivalent.

<u>Example III. (Symplectic Geometry)</u>

Suppose we are given a symplectic manifold with symplectic form ω. Let I denote the ideal generated by ω. The integral elements are the subspaces W_k of the tangent space which satisfy:

1) $$i_W^* \omega = 0.$$

where $i_W : W \hookrightarrow T_{z_0}$ denotes the natural inclusion of W in the tangent space. Given tangent vectors u, v if $\omega(u \wedge v) = 0$ then we say u, v are <u>skew-orthogonal</u>, denoted $u \angle v$. Also let $W^{\angle}$ denote the subspace of tangent vectors skew orthogonal to W, the condition 1) that W be an integral element can be reformulated as

1)' $$u \angle v \quad \text{for all} \quad u, v \in W \quad (\text{i.e. } W \subset W^{\angle}).$$

The space of enlargment possibilities for an integral W is just

$$E_k(W) = W^{\angle}$$

since, for a given vector u,

$$u \angle (W + [u]) \quad \text{if and only if} \quad u \angle W$$

It is easy to see that, given an integral element W_k, there exists a symplectic basis $e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_n$ with $e_1, \ldots, e_k$ a basis for W and

$$e_1, \ldots, e_n, \tilde{e}_{k+1}, \ldots, \tilde{e}_n \quad \text{a basis for} \quad W^{\angle} \equiv E_k(W).$$

In particular, if W_k is an integral element then $k \le n$. The integral elements W_k are called <u>isotropic subspaces</u> and if $k = n$ <u>Lagrangian subspaces</u>. The space $W_k^{\angle} = E_k(W_k)$ has dimension $2n-k$. In particular, the $r_k(W_k) = 2n-2k-1$ does not depend on the integral element W_k, or the point on the manifold, so that all integral elements are regular.

Note that $r_1 = 2n-3, \ldots, r_{n-1} = 1, r_n = -1,$

with the genus $g = n$.

Even in the case of an integral (isotropic) submanifold N of dimension $n-1$ a constraining hypersurface P must be prescribed since the dimension of possibilities for enlarging the tangent space to N is $1+r_{n-1} = 2$.

Theorem. Suppose N is an isotropic submanifold of dimension $n-1$. Given a hypersurface P containing N and transverse to the $n+1$ plane field $T_z(N)^{\angle}$, there exists a uniquely determined Lagrangian submanifold M with $N \subset M \subset P$.

This follows immediately from the Cartan-Kähler Theorem.

Remark: Of course there is the standard proof of this result. Suppose P is defined implicitly by the "Hamiltonian function" p. Let X_p denote the associated Hamiltonian vector field defined by $dp = X_p \lrcorner\, \omega$. Then M consists of the union of all the integral curves of X_p passing through N. The vector field X_p is transverse to N if and only if P is transverse to $T_z(N)^{\angle}$, since $X_p \in T_z(P)^{\angle} \subset T_z(N)^{\angle}$.

The above Theorem can be reformulated by considering N and M to be graphed over their tangent spaces. More generally, suppose that $N \subset \mathbb{R}^{2n} = T^*\mathbb{R}^n$ projects non-degenerately onto a hypersurface S in $\mathbb{R}^n$ and let $f: S \to \mathbb{R}^n$ denote the graphing function for N. The following are equivalent:

1) N is isotropic.

2) f is a compatible 1-jet on S.

3) There exist "Cauchy data" ϕ, $\frac{\partial \phi}{\partial n}$ on S (for a second order equation) with $f = \nabla\phi$ on S.

Moreover, the condition $M \subset P$ and the transversality determine $\frac{\partial \phi}{\partial n}$ in 3) along S.

Consequently, the above Theorem is equivalent to the following special case of the Cauchy-Kowalewski Theorem.

__Theorem__. Given a hypersurface S in $\mathbb{R}^n$ and a function ϕ on S, if $\frac{\partial p}{\partial n}(x,\xi) \neq 0$ along S then there exists a unique extension Φ of ϕ satisfying $p(x,\nabla\Phi(x)) = 0$.

Example IV. (Special Lagrangian Geometry)

Let I denote the ideal generated by $\omega \equiv \sum_i^n \frac{i}{2} dz_j \wedge d\bar{z}_j$ and $\psi \equiv \mathrm{Im}dz = \mathrm{Im}dz_1 \wedge \ldots \wedge dz_n$ on $\mathbb{C}^n$. The integral elements W_k of dimension $k < n$ are exactly as in the previous example, i.e. there are the isotropic subspaces. The n-dimensional integral elements are the special Lagrangian subspaces, since $|dz(\xi)| \le |\xi|$ (with equality if and only if the n-vector ξ is Lagrangian) and $|dz(\xi)|^2 = \phi(\xi)^2 + \psi(\xi)^2$ where $\phi \equiv \mathrm{Re}dz$. Consequently, as noted in the previous example, if W_k is an integral element of dimension $k \le n-2$ then $E(W_k) = W_k^-$ and $r_k(W_k) = 2n-2k-1$ is independent of W_k and the point in $\mathbb{C}^n$. If W_{n-1} is an integral element (i.e. isotropic) then there exists a unique special Lagrangian n-plane W_n containing W_{n-1}. To see this we may assume W_{n-1} is spanned by $e_1, \ldots, e_{n-1}$ where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{C}^n$. Then W_{n-1}^- is spanned by $e_1, \ldots, e_{n-1}, e_n, Je_n$. Let $u_\theta \equiv \cos\theta\, e_n + \sin\theta\, Je_n$. Then the Lagrangian n-plane $W(\theta)$ spanned by $e_1, \ldots, e_{n-1}, u_\theta$ is special Lagrangian if and only if $\mathrm{Im}dz(e_1 \wedge \ldots \wedge e_{n-1}\, u_\theta) = \mathrm{Im}dz(u_\theta) = \sin\theta = 0$. Thus $SL(W_{n-1}) \equiv \mathrm{span}\, e_1, \ldots, e_n$ is the unique special Lagrangian n-plane containing W_{n-1}. Therefore

$$E(W_{n-1}) = SL(W_{n-1}) \quad \text{and} \quad r(W_{n-1}) = 0.$$

Thus $r_{n-1} = 0$, $r_n = -1$ and the genus is n. In particular, this proves that all integral elements are regular and since $r_{n-1} = 0$ no constraining submanifold P is necessary. The Cartan-Kähler Theorem, in this case, says the following.

<u>Theorem</u>. <u>Given an isotropic</u> $n-1$ <u>dimensional submanifold</u> N <u>of</u> $\mathbb{C}^n$ <u>there exists a unique special Lagrangian submanifold</u> M <u>containing</u> N.

Of course, each $n-1$ dimensional submanifold N, of a special Lagrangian submanifold, is isotropic.

<u>Remark</u>: Note that the above discussion goes through if we replace ψ by $\psi_\theta = \mathrm{Im}(e^{i\theta} dz_1 \wedge \ldots \wedge dz_n)$ for any $\theta \in R$. For a given isotropic $(n-1)$-manifold $N \subset C^n$ we thereby produce a 1-parameter family M_θ of minimal submanifolds all intersecting transversely in the manifold N.

This Theorem may be reformulated by considering N and M to be graphed over $\mathbb{R}^n \subset \mathbb{C}^n$ (say over the tangent space to M at some point). Recall the discussion and notation of Example III above. In particular, prescribing N^{n-1} to be isotropic in $\mathbb{C}^n$ is equivalent to prescribing the Cauchy data ϕ, $\frac{\partial \phi}{\partial n}$ on the hypersurface S in $\mathbb{R}^n$; while M is Lagrangian if and only if the graphing function is $\nabla\Phi$ for some scalar function Φ. M is special Lagrangian if and only if the special Lagrangian differential equation

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1} (\text{Hess } \Phi) = 0$$

is satisfied by this scalar potential Φ.

Theorem. Suppose S is a hypersurface in $\mathbb{R}^n$ and ϕ, $\frac{\partial\phi}{\partial n}$ is given Cauchy data on S. Assume that the unique special Lagrangian n-plane containing the isotropic n-1 plane obtaining by graphing $\nabla\Phi(x_0)$ over $T_{x_0}S$ projects non-degenerately onto $\mathbb{R}^n$. Then there exists a unique solution Φ to the special Lagrangian differential equation with the prescribed Cauchy data on S.

Remark: The assumption that the unique special Lagrangian n-plane containing T_{z_0} = graph $\nabla\phi(x_0)$ can be graphed over $\mathbb{R}^n$ can be seen to be equivalent to S being non-characteristic for the special Lagrangian differential equation at the point x_0, $\nabla\Phi(x_0)$. In particular, the above (equivalent) Theorems are a special case of the Cauchy-Kowalewski Theorem for second order equations.

Example V (Associative Submanifolds)

Recall the associator inequality:

$$\varphi(x\wedge y\wedge z)^2 + \tfrac{1}{4}|[x,y,z]|^2 = |x\wedge y\wedge z|^2 \quad \text{for all } x,y,z \in \text{Im } \mathbb{O},$$

where $\varphi(x\wedge y\wedge z) \equiv \langle x,yz\rangle$ defines the 3-form φ on Im $\mathbb{O}$. Let I denote the differential system generated by the forms

$\psi_1, \ldots, \psi_7 \in \Lambda^3(\mathrm{Im}\,\mathbb{O})^*$, obtained by taking components of the $\mathrm{Im}\,\mathbb{O}$ valued alternating 3-form $[x,y,z]$. Since I is generated by 3-forms, each element W_1, W_2 of dimension 1 or 2 is an integral element. By definition, W_3 is an integral element if W_3 is associative (i.e. $W_3 = \mathrm{span}\,\{x,y,z\}$ with $[x,y,z] = 0$).

$E(W_1) = \mathrm{Im}\,\mathbb{O}$ and $r(W_1) = 5$ since each W_2 is an integral element. Hence each W_1 is regular. Given any two plane W_2 in $\mathrm{Im}\,\mathbb{O}$ there exists a unique associative 3-plane W_3 containing W_2. To prove this fact suppose y,z is an orthonormal basis for W_2. Then $W_3 \equiv \mathrm{span}\,\{y\times z,\ y,z\}$ defines an associative 3-plane containing W_2. Note that $\phi(y\times z,\ y,z) = \pm 1$ and hence by the associator inequality $[y\times z,\ y,z] = 0$, so W_3 is associative. If $[x,y,z] = 0$ for any other orthonormal triple with $W_2 = \mathrm{span}\,\{y,z\}$ then $\phi(x\wedge y\wedge z) = \pm 1$ and hence $x = \pm y \times z$ proving that W_3 is unique. Thus $E(\mathrm{span}\,\{y,z\}) = \mathrm{span}\,\{y\times z,\ y,z\}$ and hence $r_2 = r_2(W_2) = 0$. Therefore $r_3 = -1$ and the genus is 3. In particular, each integral element is regular, and the Cartan-Kähler Theorem applies. Since $r_2 = 0$ no constraining submanifold P is required.

<u>Theorem</u>. <u>Each two dimensional submanifold</u> N <u>of</u> $\mathrm{Im}\,\mathbb{O}$ <u>is contained in a unique associative submanifold</u> M^3 <u>of</u> $\mathrm{Im}\,\mathbb{O}$.

Example VI. (Coassociative Submanifolds)

Here we give a local characterization of the boundaries of coassociative submanifolds of $\mathrm{Im}\,\mathbb{O}$. If W_4 is coassociative with orthonormal basis x,y,z,w then $\psi(x\wedge y\wedge z\wedge w) \equiv \langle x, y\times z\times w\rangle$ must equal ± 1. In particular, $x = \pm\, y\times z\times w \in \mathrm{Im}\,\mathbb{O}$ so that $\phi(y\wedge z\wedge w) \equiv \mathrm{Re}\; y\times z\times w = 0$. A 3-plane W_3 in which $\phi \in \Lambda^3(\mathrm{Im}\,\mathbb{O})^*$ vanishes will be called an $\mathbb{O}$ generating 3-plane, since under an automorphism of $\mathbb{O}$ such a 3-plane is spanned by $i,j,e \in \mathrm{Im}\,\mathbb{O}$. Thus we have found a necessary condition for a 3-plane W_3 to be contained in a coassociative 4-plane W_4, namely W_3 must be a $\mathbb{O}$-generating 3-plane. We define an $\mathbb{O}$-generating 3-manifold to be a 3-dimensional submanifold of $\mathrm{Im}\,\mathbb{O}$ whose tangent space at each point is $\mathbb{O}$ generating.

Theorem. Suppose N is an $\mathbb{O}$ generating 3-manifold. Then there exist a unique coassociative 4-manifold M containing N.

Proof: Let I denote the differential system generated by the 3-form $\phi \in \Lambda^3(\mathrm{Im}\,\mathbb{O})^*$ and the 4-forms $\psi_1, \ldots, \psi_7 \in \Lambda^4(\mathrm{Im}\,\mathbb{O})^*$ obtained by taking the seven components of the vector valued alternating form $\mathrm{Im}\; x\times y\times z\times w$.

Each W_1, W_2 is an integral element. W_3 is integral if and only if W_3 is $\mathbb{O}$-generating. W_4 is integral if and only if W_4 is coassociative. Now $E(W_1) = \mathrm{Im}\,\mathbb{O}$ and $r_1 = r_1(W_1) = 5$. $E(\mathrm{span}\,\{x,y\}) = [x\times y]^\perp$, since $W_3 = \mathrm{span}\,\{x,y,u\}$ is $\mathbb{O}$ generating if and only if $\phi(x\wedge y\wedge u) = \langle u, x\times y\rangle = 0$ (i.e. $u \perp x\times y$).

Therefore $r_2 = r_2(W_2) = 3$.

Finally, if $W_3 = \text{span}\,\{y,z,w\}$ is integral (i.e. W_3 is $\mathbb{O}$ generating) then $x \equiv y \times z \times w \in \text{Im}\,\mathbb{O}$ and hence $W_4 = \text{span}\,\{x,y,z,w\}$ is coassociative. By the coassociative inequality there can be no other coassociative 4-plane containing W_3. Thus, for $W_3 \equiv \text{span}\,\{y,z,w\}$ $\mathbb{O}$-generating, $E(\text{span}\,\{y,z,w\}) = \text{span}\,\{y \times z \times w, y, z, w$ and hence $r_3 = r_3(W_3) = 0$. Therefore $r_4 = -1$ and the genus is 4. This proves that every integral element is regular and hence the Cartan-Kähler Theorem is applicable. Since $r_3 = 0$ no constraining submanifold P is required and the theorem follows.

Remark: Suppose $f : \mathbb{H} \to \text{Im}\,\mathbb{H}$. The graph is spanned by $f(1) + e$, $f(i) + ie$, $f(j) + je$, $f(k) + ke$. If $S^3 \subset \mathbb{H}$, $f : S^3 \to \text{Im}\,\mathbb{H}$ and u,v,w is a basis for $T_{pt.}S^3$. Then $f(u) + ue$, $f(v) + ve$, $f(w) + we$ is a basis for the tangent space to the graph of f over S. Therefore, by Lemma B10

$$\begin{aligned}
&\phi((f(u) + ue) \wedge (f(v) + ve) \wedge (f(w) + we)) \\
&= \text{Re}[(f(u) + ue) \times (f(v) + ve) \times (f(w) + we)] \\
&= f(u) \times f(v) \times f(w) + f(u)(v \times w) + f(v)(w \times u) + f(w)(u \times v).
\end{aligned}$$

Thus the differential operator on S induced by the coassociative differential operator can be easily calculated.

Example VII. (Cayley Submanifolds of $\mathbb{O}$)

Let I denote the ideal generated by the 4-forms $\psi_1, \ldots, \psi_7 \in \Lambda^4\mathbb{O}^*$ obtained by taking the components of

Im $x \times y \times z \times w$. Each W_1, W_2, W_3 is an integral element. A 4-plane W_4 is an integral element if and only if W_4 is a Cayley subspace of $\mathbb{O}$. Note $E(W_1) = E(W_2) = \mathbb{O}$. Recall the equality (for all $x,y,z,w \in \mathbb{O}$)

$$\psi(x \wedge y \wedge z \wedge w)^2 + \mathrm{Im}|x \times y \times z \times w|^2 = |x \wedge y \wedge z \wedge w|^2,$$

where $\psi(x \wedge y \wedge z \wedge w) = \langle x, y \times z \times w \rangle$.

Thus each $W_3 = \mathrm{span}\,\{y,z,w\}$ is contained in a unique integral W_4 given by the span of $x \equiv y \times z \times w$, y,z,w. In particular, $E(W_3) = \mathrm{span}\,\{y \times z \times w,\ y,z,w\}$ for $W_3 = \mathrm{span}\,\{y,z,w\}$. In summary, $r_1 = r_1(W_1) = 6$, $r_2 = r_2(W_2) = 5$, $r_3 = r_3(W_3) = 0$, $r_4 = -1$ with genus 4. Therefore each integral element is regular, and hence the Cartan-Kähler Theorem is applicable. Since $r_3 = 0$, no constraining submanifold is required.

Theorem. Each 3-dimensional submanifold N of $\mathbb{O}$ is contained in a unique Cayley submanifold M^4 of $\mathbb{O}$.

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