Compactness

$\mathcal{M}_{\text{asd}}$ is noncompact due to conformal invariance of $*$ (and so the asd equations) in dimension 4.

On $\mathbb{R}^4$ can rescale to peak instantons about one point; in the limit get Dirac delta concentration of $\frac{-\text{tr}}{4\pi^2} F_A \wedge F_A$. E.g. for $c_2 = 1$ on S^4 moduli space is open ball B^5; radial direction parameterising "concentratedness".

Natural compactification $\bar{B}^5$ allows "ideal" connections with Dirac delta singularities – "bubbles" – mopping up integer units of charge $c_2 \geq 0$.

Theorem 11 [Uhlenbeck]
Passing to a subsequence of asd connections A_i on $E \to M$, we have the following

$\frac{-1}{4\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i})$ *converges as a current to a smooth 4-form plus k Dirac deltas $\sum_{j=1}^{k} \delta_{p_i}$, $k \leq c_2(E)$. (Allow points with multiplicity.)*

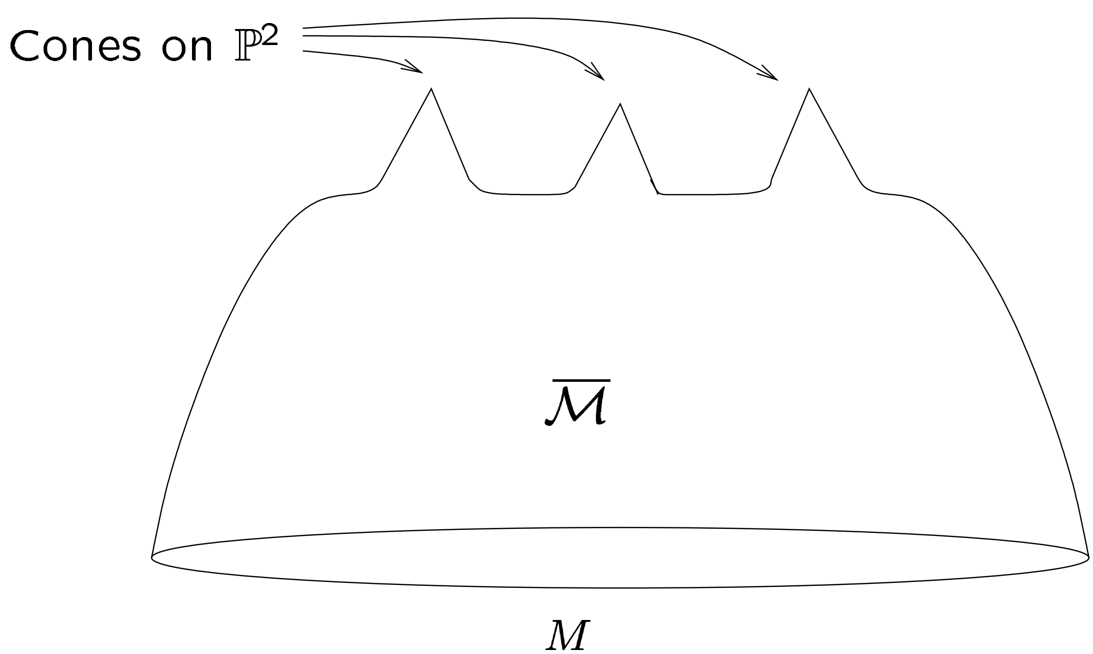
On $M\backslash\{p_i\}$ we can choose gauge transformations g_i such that $g_i^*(A_i)$ converge to an asd connection which extends uniquely to an asd connection on a new bundle $E' \to M$ with $c_2(E') = c_2(E) - k$.

So we can compactify $\mathcal{M}_k$ inside $\mathcal{M}_k \cup (\mathcal{M}_{k-1} \times M) \cup (\mathcal{M}_{k-2} \times S^2M) \cup \ldots$. The topological L^2 bound on curvature is crucial, forcing $c_2(E') \geq 0$, so that there can be at most k bubbles.

Example – negative definite 4-manifolds

If $b^+ = 0$ we can't avoid reducibles, but otherwise can get a smooth compact moduli space. Donaldson's first result came from considering $\overline{\mathcal{M}}_1$, of dimension 8-3=5.

Reducibles $\Leftrightarrow$ $L \oplus L^{-1}$ with $c_1(L)^2 = -1$. Harmonic representative for $2\pi i c_1$ is asd so gives solution. So we get a cone on $\mathbb{P}^2$ for every class $e \in H^2(X, \mathbb{Z})$ with $e^2 = -1$:



$\mathcal{M}$ provides a cobordism between X and $\coprod \mathbb{P}^2$. Invariance of signature under cobordism proves that $H^2(X,\mathbb{Z})$ is generated *over* $\mathbb{Z}$ by classes of square -1. Thus its intersection form is *standard*. (C.f. topological C^0 4-manifolds; by Freedman these can have nonstandard negative definite forms.)

Donaldson's polynomial invariants

There need not be a universal bundle $\mathbb{E}$ (with universal connection $\mathbb{A}$) over $\mathcal{M} \times M$ due to the stabilisers $C(SU(2)) = \{\pm 1\}$. But there is a universal projective bundle $\mathbb{P}(\mathbb{E})$ and adjoint bundle $\mathrm{End}_0\, \mathbb{E}$, both with universal connections $\mathbb{A}$. $\mathbb{A}$ is a $\mathbb{P}SU(2) = SU(2)/\pm 1 = SO(3)$ connection, and so we can still make sense of c_2 and $\mathrm{tr}(F_{\mathbb{A}} \wedge F_{\mathbb{A}})$.

Definition 12 Donaldson's μ-map
$\mu\colon H_2(M,\mathbb{Z}) \to H^2(\mathcal{M},\mathbb{Z}), \quad \mu(a) = a\backslash c_2(\mathbb{E})$.
I.e. $\mu(a)(b) = \langle c_2(\mathbb{E}), a \times b\rangle$ *for* $a \in H_2(M)$, $b \in H_2(\mathcal{M})$.

Together with the corresponding 4-dimensional class $\nu \in H^4(\mathcal{M},\mathbb{Z})$ these generate the cohomology of the space $\mathcal{B}^*$ of isomorphism classes of irreducible connections. They also extend over the Uhlenbeck compactification.

We can now define the *Donaldson polynomials* of M as integrals of these μ-classes over $\mathcal{M}$. For $b^+ > 1$ we can choose a generic metric to make the $\mathcal{M}_{k-i} \times S^iM$ smooth of the right dimension *except* for the trivial connection $i = k$. So we work in the *stable range*

$$k = c_2(E) \geq \frac{1}{4}(3b^+ + 5) \Leftrightarrow d(k) \geq 4c_2(E) + 2,$$

so that the dimension of the lowest stratum S^kM, $k = c_2(E)$ is codimension ≥ 2 in $\overline{\mathcal{M}}$.

Then given an element of $S^{d(k)/2}H_2(M,\mathbb{Z})$ we can apply μ and wedge to get a class in $H^{d(k)}(\overline{\mathcal{M}}_k)$ which we integrate over its fundamental class. For $b^+ > 1$ the result is invariant of the (generic) metric: different moduli spaces are cobordant and so homologous in $\mathcal{B}$. So we get invariants of the differential topology of M,

$$S^{d(k)/2}H_2(M,\mathbb{Z}) \to \mathbb{Z},$$

i.e. polynomials on $H_2(M,\mathbb{Z})$.

Connected sums

Theorem 13 [Donaldson]
The invariants vanish for a connect sum $M = M_1 \# M_2$ with $b^+(M_i) > 0$.

The idea of the proof is that, roughly (i.e. over a big open set), the moduli space is a union of pieces

$$\mathcal{M}_{M_1 \# M_2, k} = \bigcup_{k_1+k_2=k} \mathcal{M}_{M_1,k_1} \times \mathcal{M}_{M_2,k_2} \times SU(2).$$

$H^2(M) = H^2(M_1) \oplus H^2(M_2)$ comes from the M_i only, so the μ-classes do not see the homology in the $SU(2)$ so the integral is zero.

I.e. we cut down by Poincaré duals of μ classes to get to zero dimensions in $\mathcal{M}$, but this procedure splits into the two parts of the above splitting making one negative dimensional:

$$\dim(\mu(\ldots) \cap \mathcal{M}_{M_1}) + \dim(\mu(\ldots) \cap \mathcal{M}_{M_2}) + 3 = 0,$$

and so empty.

Compare Wall's theorem: two simply connected 4-manifolds with the same signature become diffeomorphic after connect summing with some $(S^2 \times S^2)^{\# k}$. So new invariants have to be unstable with respect to # to distinguish 4-manifolds.

But the invariants are not always trivial.

Complex surfaces

Simply connected $\Rightarrow$ after deformation may assume projective algebraic (Kodaira).

HYM $\Leftrightarrow$ asd, so can describe $\mathcal{M}$ algebro geometrically, and compactify it using moduli of semistable coherent sheaves. Kähler metric not generic, but theorem of Donaldson ensures moduli space generically smooth of correct dimension for $c_2 \gg 0$.

Also, $\mu(\omega_M) = \omega_{\overline{\mathcal{M}}}$ is a Kähler form on $\overline{\mathcal{M}}$.

Theorem 14 [Donaldson]

$$\int_{\overline{\mathcal{M}}} \mu(\omega_M)^d = \int_{\overline{\mathcal{M}}} (\omega_{\overline{\mathcal{M}}})^d > 0.$$

So complex surfaces cannot be connect sums $M_1 \# M_2$ with $b^+(M_i) > 0$. Compare blowing up, which connect sums a $\overline{\mathbb{P}^2}$, with $b^+(\overline{\mathbb{P}^2}) = 0$.

E.g. a $K3$ surface (diffeomorphic to a quartic surface in $\mathbb{P}^3$) has intersection form $2(-E_8) \oplus 3H$, where H is the hyperbolic intersection form of $S^2 \times S^2$. But Donaldson's result shows that this cannot be represented diffeomorphically: $K3$ is not $Y \#(S^2 \times S^2)$ (Freedman $\Rightarrow$ it is homeomorphically). Notice we cannot conclude anything about $K3 \# K3$, because all invariants now vanish.

The same theorem is true of symplectic manifolds, but to prove this the analysis was too difficult until the introduction of the Seiberg-Witten equations in 1994. See Lecture 4.

Calibrated geometry and higher dimensional gauge theory

Tian has given a link between Ω-asd connections

$$*(F_A \wedge \Omega) = -F_A,$$

and calibrated geometry. Using the resulting $||F_A||^2_{L^2}$ estimates he shows that any sequence of Ω-asd connections converges (after passing to a subsequence) away from a "blow-up locus" Z of finite Hausdorff codimension-4 measure (c.f. finite numbers of points in 4-manifolds).

He shows Z is *rectifiable* (tangent cones exist and are unique at H^{n-4}-almost-all points). Blowing up perpendicular to the tangent cones we get a limit Ω-asd connection on the tangent space T_xX ($x \in Z$ such that $T_xZ \subset T_xX$ exists) which is the pullback of an instanton B on $T_xZ^{\perp}$.

From the linear algebra of the Ω-asd equation it follows that Ω calibrates $T_x Z \subset T_x X$:

Write $\Omega = \alpha\,\mathrm{vol}_{T_x Z} + \sigma$, where $\sigma|_{T_x Z} = 0$, then the equation becomes

$$\begin{aligned} F_B &= -\alpha(*F_B) \\ *(\sigma \wedge F_A) &= 0 \end{aligned}$$

and so $\alpha = \pm 1$ and Ω calibrates $T_x Z$.

It follows that the blow-up locus defines a calibrated current, and so a calibrated cycle – (a multiple of) something smooth away from a codimension-2 subset. Its mass is less than $c_2(E).\Omega$.

(Similar results for general YM eqns, giving stationary currents – their generalised mean curvature vanishes.)

Tian also proves a removal of singularities result; that over the top stratum of Z, after gauge transformations, the limit connection extends smoothly. I.e. after passing to a subsequence and applying gauge transformations away from Z, the limit connection is well defined and extends to a connection (on a different bundle) smooth away from a set of H^{n-4}-measure zero. $c_2.\Omega$ can be defined for this new connection, and it differs from the old one by the mass of Z.

So we get the beginnings of a compactness result for higher dimensional gauge theories by introducing "ideal" connections that are Ω-asd connections plus calibrated cycles.

There are many more analogies between calibrated cycles and higher dimensional gauge theories, e.g. (putative) Floer theories for calibrated cycles. Both theories need much more solid analytical foundations. We turn finally to one such theory where algebraic geometry can be used to handle the technicalities and so make a rigorous theory.

The Casson invariant and its holomorphic analogue

(Taubes' version of) the Casson invariant of M^3 counts $SU(2)$ flat connections on a fixed vector bundle $E \to M$. Formally the curvature F_A defines a closed one-form

$$a \mapsto \frac{1}{4\pi^2}\int_M \mathrm{tr}(a \wedge F_A), \qquad a \in \Omega^1(\mathfrak{su}E),$$

on $\mathcal{A}$. This is gauge invariant, and so descends to $\mathcal{B} = \mathcal{A}/\mathcal{G}$.

Fixing a basepoint $A_0 \in \mathcal{A}$ this one-form is dCS for a locally defined function, the Chern-Simons functional:

$$CS(A_0+a) = \frac{1}{4\pi^2}\int_M \operatorname{tr}\left(\frac{1}{2}d_{A_0}a \wedge a + \frac{1}{3}a \wedge a \wedge a\right)$$

which is independent of gauge transformations connected to the identity, and well defined modulo $\mathbb{Z}$ on $\mathcal{B}$.

(Bound M by a 4-manifold N and extend (E, A) to $(\mathbb{E}, \mathbb{A})$ over N. Then $CS(A) = \frac{1}{4\pi^2}\int_N \operatorname{tr} F_{\mathbb{A}} \wedge F_{\mathbb{A}}$ which is well defined up to integrals of $-c_2$ over closed 4-mfds, i.e. up to an integer.)

In particular, at a zero of the one form, i.e. a flat connection, the deformation complex of a flat connection is self-dual – i.e. the Hessian of CS is symmetric – as then are its cohomology groups

$$H^i_A(\mathfrak{su}(E)) \cong H^{3-i}_A(\mathfrak{su}(E))^*$$

(Poincaré duality).

Therefore the virtual dimension of the moduli space of flat connections

$$d = \sum_{i=0}^{3} (-1)^{i+1} \dim H^i_A(\mathfrak{su}(E)) = 0$$

vanishes, and we could hope to count them. Formally then the Casson invariant counts flat connections, i.e. critical points of CS, so can be thought of as $\chi(\mathcal{B})$. More generally one could try to use CS as a Morse function on $\mathcal{B}$ and do Morse homology; this yields Floer homology of M.

(Original definition of Casson invariant is via a Heegard splitting

$$M = M_1 \cup_S M_2$$

along a (symplectic) surface S. Restriction defines Lagrangian subspaces $\mathcal{M}_{M_i} \hookrightarrow \mathcal{M}_S$ of the symplectic space $\mathcal{M}_S$. Intersect them.)

Now we formally complexify onto a Calabi-Yau 3-fold X (smooth, compact, Kähler with a trivialisation $\theta \in H^{3,0}$ of the canonical bundle $K_X = \Lambda_X^{3,0} \cong \mathcal{O}_X$).

Naively, replace x by z, d by $\bar{\partial}$, Poincaré duality by Serre duality, and $\int_{M^{\mathrm{or}}}$ becomes $\int_X \cdot \wedge \theta$.

So consider the space $\mathcal{A} = \mathcal{A}^{0,1}$ of $\bar{\partial}$-operators on a fixed C^∞-bundle $E \to X$, and the closed one-form given by $F_A^{0,2}$:

$$a \mapsto \frac{1}{4\pi^2} \int_X \mathrm{tr}(a \wedge F_A^{0,2}) \wedge \theta,$$

for $a \in \Omega^{0,1}(\mathrm{End}_0(E))$. Again this is gauge invariant and descends to $\mathcal{B}$.

Fixing $A_0 \in \mathcal{A}$ this one-form is d of the locally defined *holomorphic* Chern-Simons functional:

$$CS(A_0 + a) = \frac{1}{4\pi^2} \int_X \operatorname{tr} \left(\frac{1}{2} \bar{\partial}_{A_0} a \wedge a + \frac{1}{3} a \wedge a \wedge a \right) \wedge \theta,$$

independent of gauge transformations connected to the identity.

(Suppose X is a smooth effective anticanonical divisor in a 4-fold Y defined by $s \in H^0(K_Y^{-1})$. If $(E, \bar{\partial}_A)$ extends to $(\mathbb{E}, \mathbb{A})$ on Y, then, modulo periods,

$$CS(A) = \frac{1}{4\pi^2} \int_Y \operatorname{tr} F_{\mathbb{A}}^{0,2} \wedge F_{\mathbb{A}}^{0,2} \wedge s^{-1}.)$$

Periods dense, but the zeroes of the one-form, i.e. the critical points of CS, are well defined: they are integrable holomorphic structures on the bundle E,

$$\bar{\partial}_A^2 = F_A^{0,2} = 0.$$

For the same reasons the deformation complex of a holomorphic connection is self-dual as are its cohomology groups

$$H_A^{0,i}(\mathrm{End}_0(E)) \cong H_A^{0,3-i}(\mathrm{End}_0(E))^*,$$

(Serre duality and the trivialisation θ of the canonical bundle K_X). So the virtual dimension of the moduli space of holomorphic bundles

$$d = \sum_{i=0}^{3} (-1)^{i+1} \dim H^{0,i}(\mathfrak{su}(E); A) = 0$$

vanishes, and we could hope to count the bundles to formally compute $e(T^*\mathcal{B})$.

(There is an analogue, due to Donaldson and Tyurin, of the original definition of Casson invariant via a Heegard splitting. Suppose

$$X = X_1 \cup_S X_2$$

is a normal crossing CY, two Fanos X_i glued along a common anticanonical divisor – a holomorphic-symplectic surface S ($K3$ or T^4). Restriction generically defines complex Lagrangians

$$\mathcal{M}_{M_i} \hookrightarrow \mathcal{M}_S$$

of the symplectic space $\mathcal{M}_S$. Now intersect them.)

To make the moduli space compact (to get a finite number) restrict to *stable* bundles or count HYM connections. Could try to use Tian's compactification (with bubbles along curves) but no transversality – moduli space usually high dimensional and singular.

So use algebraic geometry, and compactify to a projective $\overline{\mathcal{M}}$ using semistable coherent sheaves (Gieseker, Maruyama, Simpson). We then want a *virtual moduli cycle* in $\overline{\mathcal{M}}$ of dimension 0 – the "right" moduli space representing the zero locus of a transverse perturbation of the equations.

Virtual moduli cycles

Suppose a manifold M (later $\overline{\mathcal{M}}$) sits inside a smooth ambient n-fold Z, cut out by a section s of a rank r vector bundle $E \to Z$. Virtual dimension $(M) = (n - r)$.

Suppose s transverse only to a subbundle $E' \subset E$, then the "correct" $(n-r)$-cycle is

$$e(E/E') \in H_{n-r}(M)$$

as this is homologous to the zero set of any transverse perturbation of s.

(E.g. perturb to $s \oplus e \in \Gamma(E' \oplus \ E/E')$.)

So in this smooth case we take the Euler class of the obstruction bundle (fibre H^2 of the deformation complex; in our case this is $(H^1)^* = T^*\overline{\mathcal{M}}$.)

In general Fulton-MacPherson intersection theory gives a class in $H_{n-r}(M)$ (whose pushfoward to $H_{n-r}(Z)$ is $e(E)$).

(s induces a cone $\lim_{\lambda\to\infty} \lambda s$ in $E|_M$. Intersect this with the zero set $M \subset E|_M$.)

Worked entirely on M (not in Z) so method extends to moduli problems where the ambient space Z does not exist. Instead need the deformation theory of the moduli problem to give the infinitesimal version of (Z, E, s) on M, namely

$$0 \to TM \to TZ|_M \xrightarrow{ds} E|_M \to \mathrm{ob} \to 0,$$

for some cokernel ob which in the moduli problem becomes the obstruction sheaf H^2.

Actually require a global version: a two term locally free resolution

$$0 \to \mathcal{T}_1 \to E_1 \to E_2 \to \mathcal{T}_2 \to 0,$$

of the *tangent-obstruction functors* [Li-Tian].

Here E_1 and E_2 play the roles of $TZ|_M$ and $E|_M$ in the above (Z smooth, E a bundle $\Rightarrow$ ranks constant $\Rightarrow$ locally free) and are required to have difference in ranks equal to the virtual dimension of the moduli problem. [Li-Tian], [Behrend-Fantechi] show that such data on $\mathcal{M}$ gives a virtual moduli cycle with the correct properties.

Theorem 15 *Stable sheaves on Calabi-Yau 3-folds admit such a 2-term locally free resolution of virtual dimension 0.*

So if we consider only moduli of sheaves where semistable $\Rightarrow$ stable (e.g. sheaves E with $c_\bullet(E)$ satisfying various numerical conditions) then can define a projective 0-dimensional virtual moduli cycle in $\overline{\mathcal{M}}$ whose length we define to be the holomorphic Casson invariant. This has good properties such as deformation invariance under deformations of *polarised* manifolds.

Example

We describe a pretty example of Donaldson.

Fix a smooth quadric Q_0 in $\mathbb{P}^5$, in a fixed $\mathbb{P}^2$-family of quadrics spanned by Q_0, Q_1 and Q_2, say. The singular quadrics in the family lie on the sextic curve

$$C = \Big\{[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2 : \\ \det(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2) = 0\Big\} \subset \mathbb{P}^2$$

where the quadratic form defining the quadric becomes singular.

Each smooth quadric $\cong \mathrm{Gr}(2, \mathbb{C}^4)$ so has two tautological rank 2 bundles A and B over it.
(Dual of the subspace bundle and universal quotient bundle, i.e. $0 \to A^* \to \mathbb{C}^4 \to B \to 0$.)
So each point of $\mathbb{P}^2 \backslash C \Rightarrow$ 2 bundles over the $K3$ surface

$$S = Q_0 \cap Q_1 \cap Q_2.$$

In fact (Mukai) this makes $\mathcal{M}_S$ the double cover of $\mathbb{P}^2$ branched along C – also a (complex symplectic) $K3$ surface.

Similarly the Fano $X_1 = Q_0 \cap Q_1$ lies in the $\mathbb{P}^1$ pencil $\langle Q_0, Q_1 \rangle$. The restriction of the double cover $\mathcal{M} \to \mathbb{P}^2$ to this $\mathbb{P}^1$ is $\mathcal{M}_{X_1}$. Similarly for $X_2 = Q_0 \cap Q_2$ and the cover of the line $\langle Q_0, Q_2 \rangle \subset \mathbb{P}^2$.

Their intersection, namely the double cover of the intersection point $\{Q_0\}$ of the lines in $\mathbb{P}^2$, corresponds to the two stable bundles A_{Q_0} and B_{Q_0} on the singular Calabi-Yau $X_1 \cup_S X_2$.

Deforming this singular quartic in Q_0 to a smooth Calabi-Yau motivates the following.

Theorem 16 *Let Q_0 be a smooth quadric in $\mathbb{P}^5$, and let X be a generic smooth quartic hypersurface in Q_0. Then the bundles A and B on Q_0 restrict to stable, isolated bundles of the same topological type on X, and they are the only semistable sheaves in the moduli space. Thus the corresponding holomorphic Casson invariant is 2.*

By deformation invariance of the invariant, then, it is also 2 for all smooth such X, even though $\overline{\mathcal{M}}$ may not be so simple.