

MANIFOLDS WITH SPECIAL HOLONOMY

LECTURE 4:

$G_2$  AND Spin(7)

ROBERT L. BRYANT

DUKE UNIVERSITY

14 AUGUST 2003

We are now ready to consider the exceptional cases, starting with the one in dimension 7.

$n$	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$ -module
$n$	$\mathfrak{so}(n)$	$\mathbb{R} \oplus S_0^2(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus S_0^{1,1}(\mathbb{C}^m)^{\mathbb{R}} \oplus S_0^{2,2}(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$S_0^{2,2}(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus S^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$S^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$V^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$V^{0,2,0} \simeq \mathbb{R}^{168}$

**5. The exceptional holonomy  $G_2$ .** As usual, we'll start with linear algebra. Consider the 3-form  $\phi_0$  defined on  $\mathbb{R}^7$  by the formula

$$\phi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{356} - dx^{347}$$

where  $dx^{ijk}$  means  $dx^i \wedge dx^j \wedge dx^k$ . Now define

$$G_2 = \{ A \in GL(7, \mathbb{R}) \mid A^*(\phi_0) = \phi_0 \}.$$

It is not obvious from this what sort of group  $G_2$  is (though it is a closed subgroup, by definition). A few subgroups can be discerned, though:

Writing  $\mathbb{R}^7 = \mathbb{R} \cdot e_1 \oplus e_1^\perp = \mathbb{R} \cdot e_1 \oplus \mathbb{R}^6 = \mathbb{R} \cdot e_1 \oplus \mathbb{C}^3$  and setting

$$z^1 = x^2 + ix^3, \quad z^2 = x^4 + ix^5, \quad z^3 = x^6 + ix^7$$

allows us to express  $\phi_0$  in the form

$$\phi_0 = dx^1 \wedge \frac{i}{2} (dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) + \text{Re}(dz^1 \wedge dz^2 \wedge dz^3).$$

Thus,  $G_2 \supset \{1\} \times SU(3)$ . Similarly  $G_2 \supset SU(3) \times \{1\}$ . (Exercise.) As a result,  $G_2 \cdot e_1 \supset S^6 \subset \mathbb{R}^7$ . In fact,  $G_2 \cdot e_1 = S^6$ .

Moreover, the subgroup of  $G_2$  that fixes  $e_1$  is  $\{1\} \times \mathrm{SU}(3)$ . This makes  $G_2$  into a principal  $\mathrm{SU}(3)$ -bundle over  $S^6$ :

$$\begin{array}{ccc} \mathrm{SU}(3) & \rightarrow & G_2 \\ & & \downarrow \\ & & S^6 \end{array}$$

Thus  $G_2$  is compact and of dimension 14. By the long exact seq. in homotopy, it is 2-connected and  $\pi_3(G_2) = \pi_3(\mathrm{SU}(3)) = \mathbb{Z}$ , so it is simple. It is the first of the five exceptional compact simple Lie groups, realized here as a subgroup of  $\mathrm{SO}(7)$ .

Define

$$\Lambda_+^3(\mathbb{R}^7)^* = \{A^* \phi_0 \mid A \in \mathrm{GL}(7, \mathbb{R})\} \simeq \mathrm{GL}(7, \mathbb{R}) / G_2.$$

This orbit has dimension  $49 - 14 = 35 = \dim \Lambda^3(\mathbb{R}^7)^*$ , so it's an open subset of this vector space. More generally, if  $V$  is a vector space of dimension 7, define the *positive 3-forms* on  $V$  as

$$\Lambda_+^3(V^*) = \{A^* \phi_0 \mid A : V \rightarrow \mathbb{R}^7 \text{ is an isomorphism}\}.$$

Because  $G_2$  lies in  $SO(7)$ , to each positive form  $\phi = A^*\phi_0$  on  $V$  there are a well-defined inner product and orientation on  $V$  defined by

$$g_\phi = A^*g_0, \quad *_\phi 1 = A^*(dx^1 \wedge \cdots \wedge dx^7).$$

Given a 7-dimensional vector space  $V$ , a  $G_2$ -*structure* on  $V$  is a positive 3-form  $\phi$  on  $V$ . The inner product  $g_\phi$  and orientation  $*_\phi 1 \in \Lambda^7(V^*)$  are said to be *associated* to  $\phi$ .

It is not difficult to prove that  $\phi_0$  and its dual 4-form  $*_{\phi_0}\phi_0$  generate the ring of  $G_2$ -invariant forms on  $\mathbb{R}^7$ . Harvey and Lawson proved that they are calibrations (more on this below).

*Historical Remark:* This was not the original definition of  $G_2$ , nor the most common. Nowadays,  $G_2$  is most commonly defined as the group of automorphisms of the *octonions*  $\mathbb{O}$ , the unique normed division algebra of dimension 8 (Cartan, 1908). The definition given here is due to Schouten (ca. 1928), and is, in a certain sense, optimal.

The comass of  $\phi_0$ . We can use the fact that  $G_2$  stabilizes  $\phi_0$  to prove that it is a calibration and to compute  $G(\phi_0)$ :

Use the fact that  $G_2$  acts transitively on  $S^6$  with stabilizer of  $e_1$  equal to  $\{1\} \times \mathrm{SU}(3)$  to see that, for any triple of orthonormal vectors  $(v_1, v_2, v_3)$  in  $\mathbb{R}^7$ , there is an element  $A \in G_2$  and a constant  $\theta$  such that

$$A(v_1) = e_1, \quad A(v_2) = e_2, \quad A(v_3) = \cos \theta e_3 + \sin \theta e_4.$$

Then

$$\phi_0(v_1, v_2, v_3) = \phi_0(e_1, e_2, \cos \theta e_3 + \sin \theta e_4) = \cos \theta \leq 1.$$

Moreover,  $v_1 \wedge v_2 \wedge v_3 \in \mathrm{Gr}_3^+(\mathbb{R}^7)$  is  $\phi_0$ -calibrated if and only if it is equivalent to  $e_1 \wedge e_2 \wedge e_3$  via  $G_2$ . Thus,

$$G(\phi_0) = G_2 \cdot (e_1 \wedge e_2 \wedge e_3) = G_2 / \mathrm{SO}(4).$$

These calibrated 3-planes were dubbed *associative* by Harvey and Lawson because, regarding  $\mathbb{R}^7$  as  $\mathrm{Im} \mathbb{O}$ , these planes generate associative subalgebras of  $\mathbb{O}$ .

Harvey and Lawson also showed that every 2-plane  $E \subset \mathbb{R}^7$  lies in a unique associative 3-plane  $E^+ \in \mathbf{G}(\phi_0)$ .

In fact, they prove that every real analytic surface  $S \subset \mathbb{R}^7$  lies in a unique analytically irreducible  $\phi_0$ -calibrated 3-fold  $\Sigma \subset \mathbb{R}^7$ .

Thus, there are lots of ‘associative’ 3-manifolds in  $\mathbb{R}^7$ . These have become the focus of intense interest by  $\mathcal{M}$ -theorists as of late.

Of course, the Hodge dual 4-form  $*_0\phi_0$  is also a calibration, calibrating the oriented orthogonals to the associative 3-planes:

$$\mathbf{G}(*_0\phi_0) = \mathbf{G}_2 \cdot (e_4 \wedge e_5 \wedge e_6 \wedge e_7) = \mathbf{G}_2 / \mathrm{SO}(4).$$

It follows that  $\phi_0$  vanishes on any ‘coassociative’ 4-plane in  $\mathbb{R}^7$ . In fact, Harvey and Lawson show that a 3-plane  $E \in \mathbf{Gr}_3(\mathbb{R}^7)$  lies in a coassociative 4-plane if and only iff  $\phi_0$  vanishes on  $E$ .

Correspondingly, they show that any real analytic 3-fold  $S \subset \mathbb{R}^7$  to which  $\phi_0$  pulls back to zero lies in a unique analytically irreducible  $*_0\phi_0$ -calibrated 4-fold  $\Sigma \subset \mathbb{R}^7$ .

**Manifolds.** Let  $M^7$  be a 7-manifold. A  $G_2$ -structure on  $M$  is a 3-form  $\phi$  on  $M$  such that  $\phi_x$  is a positive 3-form on  $T_xM$  for all  $x \in M$ . Such a  $\phi$  exists on  $M$  iff  $M$  is orientable and spinnable.

A  $G_2$ -structure  $\phi$  on  $M$  defines an associated Riemannian metric  $g_\phi$  and an orientation (i.e., volume form)  $*_\phi 1 \in \Omega^7(M)$ .

If  $(M^7, g)$  is a Riemannian manifold with holonomy conjugate to a subgroup of  $G_2$ , then  $M$  supports a  $g$ -parallel 3-form  $\phi$  that is positive and so defines a  $G_2$ -structure on  $M$ . Of course, this 3-form is both closed and co-closed.

Conversely, if  $M$  supports a  $g$ -parallel positive 3-form, then the holonomy of  $g$  is conjugate to a subgroup of  $G_2$ .

**Theorem:** (Fernandez–Gray) Let  $M$  be a 7-manifold and let  $\phi$  be a  $G_2$ -structure on  $M$ , with associated metric  $g_\phi$  and orientation  $*_\phi 1$ . If  $\phi$  is closed and co-closed (with respect to  $g_\phi$  and  $*_\phi 1$ ), then  $\phi$  is  $g_\phi$ -parallel. (In particular,  $g_\phi$  has holonomy conjugate to a subgroup of  $G_2$ .)



*Calibrations.* If  $\phi \in \Omega_+^3(M^7)$  is a closed  $G_2$ -structure, then it is a calibration for  $M$  endowed with its associated metric  $g_\phi$ . This defines a distinguished family of minimizing submanifolds of  $(M, g_\phi)$ .

In the case that  $\phi$  is closed and real analytic in local coordinates, Harvey and Lawson's arguments for  $(\mathbb{R}^7, \phi_0)$  generalize to  $(M^7, \phi)$ , showing that every real analytic surface  $S \subset M^7$  lies in a unique analytically irreducible  $\phi$ -calibrated 3-fold  $\Sigma \subset M^7$ .

Similarly, if  $\phi \in \Omega_+^3(M^7)$  satisfies  $d\phi = d(*_\phi\phi) = 0$ , then  $*_\phi\phi$  is a calibration on  $(M, g_\phi)$ .

Again, Harvey and Lawson's arguments generalize in this case without change to show that any real analytic 3-fold  $S \subset M^7$  to which  $\phi$  pulls back to zero lies in a unique analytically irreducible  $*_\phi\phi$ -calibrated 4-fold  $\Sigma \subset M^7$ .

*Local Existence.* Now, we are ‘reduced’ to studying the equations

$$d\phi = d(*_{\phi}\phi) = 0$$

for positive 3-forms  $\phi \in \Omega_+^3(M)$ . *A priori*, this is  $35 + 21 = 56$  equations for the 35 unknown coefficients of the 3-form  $\phi$ , but, in fact, these equations overlap by 7, so it’s only 49 (quasi-linear, first order) partial differential equations.

These equations aren’t elliptic because they are invariant under the diffeomorphism group.

Remember (from the table), that  $K(\mathfrak{g}_2) \simeq \mathbb{R}^{77}$  and is irreducible as a  $G_2$ -module. It follows that if  $\phi$  solves these equations, then  $g_{\phi}$  is Ricci-flat (Bonan) and hence real analytic in  $g_{\phi}$ -harmonic coordinates (DeTurck-Kazdan). In particular,  $\phi$  will also be real analytic in  $g_{\phi}$ -harmonic coordinates.

This suggests the augmented (coordinate-dependent) system

$$d\phi = d(*_{\phi}\phi) = d(*_{\phi}dx^i) = 0$$

for  $\phi \in \Omega_+^3(\mathbb{R}^7)$ . This system is (overdetermined and) elliptic.

I can now state the ‘local generality’ analogs for  $G_2$  of the results that we found for the various complex cases:

**Theorem:** (B—) The system  $d\phi = d(*_\phi\phi) = 0$  for  $\phi \in \Omega_+^3(M)$  is involutive. Modulo diffeomorphism, the general solution depends on six arbitrary functions of 6 variables.

Moreover, for the ‘generic’ solution  $\phi$ , the values of the curvature form at a ‘generic’ point generate the entire holonomy algebra  $\mathfrak{g}_2$ . Thus, for the ‘generic’ solution, the holonomy of  $g_\phi$  is all of  $G_2$ .

---

It is useful to have a test for reduced holonomy in the  $G_2$  case:

**Proposition:** If  $M^7$  is simply connected  $g$  is a metric on  $M$  that has holonomy a subgroup of  $G_2$ , then the holonomy is a proper subgroup of  $G_2$  iff  $M$  supports a nonzero  $g$ -parallel 1-form.

*Proof:* By Berger’s classification and the de Rham splitting theorem, the only connected proper subgroups of  $G_2$  that can be holonomy groups in dimension 7 are  $\mathbf{1}_1 \times \mathrm{SU}(3)$ ,  $\mathbf{1}_3 \times \mathrm{SU}(2)$ , and  $\mathbf{1}_7$ .

**Explicit examples:** Once we knew they were there, finding explicit examples wasn't so hard:

*A conical example:* The first known example was the cone metric on the flag manifold  $SU(3)/T^2$ , endowed with its natural metric. In fact, write the left-invariant form on  $SU(3)$  in the form

$$g^{-1} dg = \begin{pmatrix} -i\theta_3 & -\bar{\omega}_1 & i\omega_2 \\ \omega_1 & -i\theta_2 & -\bar{\omega}_3 \\ i\bar{\omega}_2 & \omega_3 & -i\theta_1 \end{pmatrix}$$

Then the quadratic form  $d\sigma^2 = \omega_1 \circ \bar{\omega}_1 + \omega_2 \circ \bar{\omega}_2 + \omega_3 \circ \bar{\omega}_3$  defines a Riemannian metric on  $SU(3)/T^2$ , where  $T^2$  is the maximal torus in  $SU(3)$  consisting of diagonal matrices. Setting

$$\Omega = \frac{i}{2} (\omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2 + \omega_3 \wedge \bar{\omega}_3), \quad \Psi = \omega_1 \wedge \omega_2 \wedge \omega_3,$$

these two forms also drop to  $SU(3)/T^2$  and satisfy

$$d\Omega = 3 \operatorname{Re}(\Psi), \quad d\Psi = -2i\Omega^2.$$

Now, on  $M = (0, \infty) \times \mathrm{SU}(3)/T^2$ , consider the (obviously closed) 3-form

$$\phi = \frac{1}{3} d(t^3 \Omega) = t^2 dt \wedge \Omega + t^3 \mathrm{Re}(\Psi).$$

This is a positive 3-form. One computes that its associated metric is the ‘cone’ metric

$$g_\phi = dt^2 + t^2 d\sigma^2.$$

Moreover, the  $g_\phi$ -dual of  $\phi$  is found to be

$$*_\phi \phi = -\frac{1}{4} d(t^4 \mathrm{Im}(\Psi))$$

and this is obviously closed as well. Thus, the holonomy of  $g_\phi$  is conjugate to a subgroup of  $G_2$ .

One can now compute that there are no  $g_\phi$ -parallel 1-forms, so the holonomy must be all of  $G_2$ .

Later, Bryant and Salamon constructed complete  $G_2$ -holonomy metrics on  $\Lambda_+^2(\mathbb{C}\mathbb{P}^2)$  and on  $S^3 \times \mathbb{R}^4$ .

These  $G_2$ -metrics have cohomogeneity 1 and Gukov, *et al* and Page, *et al* have now constructed many more examples of such.

*Compact examples?* On the other hand, a compact example cannot have any continuous symmetries (because it's Ricci-flat), so it's truly a 7-dimensional problem.

Here are a few of the known easy necessary conditions if a compact  $M$  is to support a metric  $g$  with holonomy  $G_2$ :

- (1)  $M$  must be orientable and spinnable.
- (2)  $b_1(M) = 0$ . (Ricci flat  $\Rightarrow$  harmonic 1-forms parallel.)
- (3)  $\pi_1(M)$  must be finite. (Cheeger-Gromoll)
- (4)  $b_3(M) = b_4(M) > 0$ . ( $\phi, *_{\phi}\phi$  are parallel forms)
- (5)  $p_1(M) \neq 0$ . ( $\int_M \phi \wedge p_1 = c \int_M \|R\|^2 * 1$ , where  $c \neq 0$ .)
- (6)  $M$  cannot be a nontrivial product. (Donaldson)

There is also a relatively easy 'Torelli theorem':

**Theorem:** Suppose that  $(M^7, g)$  is a compact Riemannian manifold with holonomy  $G_2$ . Then the marked moduli space of  $G_2$ -holonomy metrics near  $g$  modulo the diffeomorphisms of  $M$  near the identity is a smooth manifold near  $g$  whose tangent space is canonically isomorphic to  $H_{dR}^3(M, \mathbb{R})$ .

In the early 1990's Dominic Joyce produced the first examples of compact Riemannian manifolds with holonomy  $G_2$ . There were two aspects to his methods:

*Analytic:* Joyce proved that, if one has a compact  $M^7$  that supports a *closed* positive 3-form  $\phi$  with the property that its associated metric  $g_\phi$  satisfies a combination of

- (1) a upper bound on its curvature and volume,
- (2) a lower bound on its injectivity radius, and
- (3) a upper bound on the size of  $\|d * \phi\|$ ,

then one can perturb  $\phi$  by adding a small exact 3-form so as to get a new  $\hat{\phi}$  that is positive, closed, and co-closed with respect to its associated metric.

*Geometric:* Joyce constructed examples of  $(M^7, \phi)$  that satisfy the above hypotheses by starting with a flat metric on the 7-torus  $T^7 = \mathbb{R}^7 / \mathbb{Z}^7$ , dividing by a discrete group to get a manifold with singularities, resolving those singularities to get the manifold  $M^7$  and a  $\phi$  on  $M^7$  satisfying the above conditions.

Finally, we are ready to consider the last case, which only occurs in dimension 8.

$n$	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$ -module
$n$	$\mathfrak{so}(n)$	$\mathbb{R} \oplus S_0^2(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus S_0^{1,1}(\mathbb{C}^m)^{\mathbb{R}} \oplus S_0^{2,2}(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$S_0^{2,2}(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus S^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$S^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$V^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$V^{0,2,0} \simeq \mathbb{R}^{168}$



**6. The exceptional holonomy  $\text{Spin}(7)$ .** As usual, we begin with linear algebra. Let  $\mathbb{R}^8 = \mathbb{R} \cdot e_0 \oplus \mathbb{R}^7$  and consider the 4-form

$$\Phi_0 = dx^0 \wedge \phi_0 + * \phi_0.$$

Now ‘define’

$$\text{Spin}(7) = \{ A \in \text{GL}(8, \mathbb{R}) \mid A^* \Phi_0 = \Phi_0 \}.$$

The definition and previous discussion  $\Rightarrow \text{Spin}(7) \supset \{1\} \times G_2$ .  
Moreover, if we identify  $\mathbb{R}^8$  with  $\mathbb{C}^4$  by setting

$$z^0 = x^0 + i x^1, \quad z^1 = x^2 + i x^3, \quad z^2 = x^4 + i x^5, \quad z^3 = x^6 + i x^7,$$

then you can check that

$$\begin{aligned} \Phi_0 = & \frac{1}{2} \left( \frac{i}{2} (dz^0 \wedge d\bar{z}^0 + dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \right)^2 \\ & + \text{Re}(dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3), \end{aligned}$$

so  $\text{Spin}(7) \supset \text{SU}(4)$ . Finally, can show that  $\text{Spin}(7) \subset \text{SO}(8)$  and that the subgroup of  $\text{Spin}(7)$  that fixes  $e_0$  is exactly  $\{1\} \times G_2$ .

This makes  $\text{Spin}(7)$  into a principal  $G_2$ -bundle over  $S^7$ :

$$\begin{array}{ccc} G_2 & \rightarrow & \text{Spin}(7) \\ & & \downarrow \\ & & S^7 \end{array}$$

Thus  $\text{Spin}(7)$  is compact and of dimension 21. By the long exact seq. in homotopy, it is 2-connected and  $\pi_3(\text{Spin}(7)) = \pi_3(G_2) = \mathbb{Z}$ , so it is simple. By the classification, there are only two possibilities:  $\text{Spin}(7)$  is  $\text{Sp}(3)$  or the universal cover of  $\text{SO}(7)$ . The former has no 8-dimensional irreducible representation, so  $\text{Spin}(7)$  must be the universal cover of  $\text{SO}(7)$ . Define

$$\Lambda_+^4(\mathbb{R}^8)^* = \{ A^* \Phi_0 \mid A \in \text{GL}(7, \mathbb{R}) \} \simeq \text{GL}(8, \mathbb{R}) / \text{Spin}(7).$$

This orbit has dimension  $64 - 21 = 43 < \dim \Lambda_+^4(\mathbb{R}^8)^*$ , so it's *not* an open subset of this vector space. More generally, if  $V$  is a vector space of dimension 8, define the *Cayley forms* on  $V$  to be

$$\Lambda_+^4(V^*) = \{ A^* \Phi_0 \mid A : V \rightarrow \mathbb{R}^8 \text{ is an isomorphism} \}.$$

Because  $\text{Spin}(7)$  lies in  $\text{SO}(8)$ , to each Cayley form  $\Phi = A^*\Phi_0$  on  $V$  there are a well-defined inner product and orientation on  $V$  defined by

$$g_\Phi = A^*g_0, \quad *_\Phi 1 = A^*(dx^0 \wedge \cdots \wedge dx^7).$$

Given an 8-dimensional vector space  $V$ , a  $\text{Spin}(7)$ -*structure* on  $V$  is a Cayley form  $\Phi \in \Lambda_+^4(V^*)$ . The inner product  $g_\Phi$  and orientation  $*_\Phi 1 \in \Lambda^8(V^*)$  are said to be *associated* to  $\Phi$ .

It is not difficult to prove that  $\Phi_0 = *_{\Phi_0}\Phi_0$  generates the ring of  $\text{Spin}(7)$ -invariant forms on  $\mathbb{R}^8$ . Harvey and Lawson proved that it is a calibration (more on this below).

*Historical Remark:* This is a very idiosyncratic definition of  $\text{Spin}(7)$ . Usually,  $\text{Spin}(7)$  is defined abstractly as the universal cover of  $\text{SO}(7)$ . Its embedding into  $\text{SO}(8)$  is derived via Clifford constructions or by using the octonions, e.g.,  $\text{Spin}(7)$  is the subgroup of the rotations in  $\mathbb{O}$  that is generated by right multiplications by unit imaginary octonions.

**Manifolds.** Let  $M^8$  be an 8-manifold. A  $\text{Spin}(7)$ -structure on  $M$  is a 4-form  $\Phi$  on  $M$  such that  $\Phi_x$  is a Cayley form on  $T_x M$  for all  $x \in M$ .

A  $\text{Spin}(7)$ -structure on  $M$  defines an associated Riemannian metric  $g_\Phi$  and orientation (i.e., volume form)  $*_\Phi 1 \in \Omega^8(M)$  with respect to which  $\Phi$  is self-dual, i.e.,  $*_\Phi \Phi = \Phi$ .

If  $(M^8, g)$  is a Riemannian manifold with holonomy conjugate to a subgroup of  $\text{Spin}(7)$ , then  $M$  supports a  $g$ -parallel Cayley form  $\Phi$  and this defines a  $\text{Spin}(7)$ -structure on  $M$ . This 4-form is closed (and co-closed, of course).

Conversely, if  $M$  supports a  $g$ -parallel Cayley form, then the holonomy of  $g$  is conjugate to a subgroup of  $\text{Spin}(7)$ .

**Theorem:** (Fernandez–Gray) Let  $M$  be a 8-manifold and let  $\Phi$  be a  $\text{Spin}(7)$ -structure on  $M$ , with associated metric  $g_\Phi$  and orientation  $*_\Phi 1$ . If  $\Phi$  is closed, then  $\Phi$  is  $g_\Phi$ -parallel. (In particular,  $g_\Phi$  has holonomy conjugate to a subgroup of  $\text{Spin}(7)$ .)

Thus, we are ‘reduced’ to studying the equations

$$d\Phi = 0$$

for Cayley forms  $\Phi \in \Omega_+^4(M)$ . This is 56 (quasi-linear, first order) partial differential equations for the 43 unknown coefficients of the form  $\Phi$ .

These equations aren’t elliptic because they are invariant under the diffeomorphism group.

Remember (from the table), that  $K(\mathfrak{spin}(7)) \simeq \mathbb{R}^{168}$  and is irreducible as a  $\text{Spin}(7)$ -module. It follows that if  $\Phi$  solves these equations, then  $g_\Phi$  is Ricci-flat (Bonan) and hence real analytic in  $g_\Phi$ -harmonic coordinates (DeTurck-Kazdan). In particular,  $\Phi$  will also be real analytic in  $g_\Phi$ -harmonic coordinates.

This suggests the augmented (coordinate-dependent) system

$$d\Phi = d(*_\Phi dx^i) = 0$$

for  $\Phi \in \Omega_+^4(\mathbb{R}^8)$ . This system of  $56 + 8 = 64$  equations is (over-determined and) elliptic.

I can now state the ‘local generality’ analogs for Spin(7):

**Theorem:** (B—) The system  $d\Phi = 0$  for  $\Phi \in \Omega_+^4(M)$  is involutive and, modulo diffeomorphism, the general solution depends on twelve arbitrary functions of 7 variables.

Moreover, for the ‘generic’ solution  $\Phi$ , the values of the curvature form at a ‘generic’ point generate the entire holonomy algebra  $\mathfrak{spin}(7)$ . Thus, for the ‘generic’ solution, the holonomy of  $g_\Phi$  is all of Spin(7).

---

**Proposition:** If  $M^8$  is simply connected and  $g$  is a metric on  $M$  that has holonomy a subgroup of Spin(7), then the holonomy is a proper subgroup of Spin(7) iff  $M$  supports a nonzero  $g$ -parallel 1-form or 2-form.

*Proof:* By Berger’s classification and the de Rham splitting theorem, the only connected proper subgroups of Spin(7) that can be holonomy groups in dimension 7 are subgroups of either  $\mathbf{1}_1 \times \mathbf{G}_2$  or SU(4).

**Explicit examples:** Again, constructing cohomogeneity 1 examples isn't hard:

*A conical example:* The first known example was the cone metric on the homogeneous space  $Y^7 = \mathrm{SO}(5)/H_5$ , where  $H_5 \subset \mathrm{SO}(5)$  is a subgroup isomorphic to  $\mathrm{SO}(3)$  and acting irreducibly on  $\mathbb{R}^5$ . Up to constant multiples, there is a unique  $\mathrm{SO}(5)$ -invariant metric on  $Y^7$ .

I won't give the full details, but the basic point is that  $Y^7$  is a rational homology 7-sphere that has a unique (up to constant multiples)  $\mathrm{SO}(5)$  invariant 3-form  $\phi$ . (This follows from the representation theory of  $\mathrm{SO}(3)$ .)

Now, this form  $\phi$  is actually a positive 3-form, so it defines a  $G_2$ -structure on  $Y$ , with associated metric  $g_\phi$  and orientation  $*_\phi 1$ .

Now, it is not difficult to show that  $d\phi = \lambda *_\phi \phi$  for some constant  $\lambda \neq 0$ . By scaling  $\phi$ , we can assume that

$$d\phi = 4 *_\phi \phi.$$

Now, on  $M^8 = (0, \infty) \times Y^7$ , consider the (obviously closed) 4-form

$$\Phi = \frac{1}{4} d(t^4 \phi) = t^3 dt \wedge \phi + t^4 *_{\phi} \phi.$$

This is clearly a Cayley form. One computes that its associated metric is the ‘cone’ metric

$$g_{\Phi} = dt^2 + t^2 g_{\phi}.$$

Thus, the holonomy of  $g_{\Phi}$  is conjugate to a subgroup of  $\text{Spin}(7)$ .

One can now compute that there are no  $g_{\Phi}$ -parallel 1-forms or 2-forms so the holonomy must be all of  $\text{Spin}(7)$ .

Later, Bryant and Salamon constructed a complete  $\text{Spin}(7)$ -holonomy metric on  $\mathbb{S}_+(S^4)$ , the semi-spinor bundle over  $S^4$ .

These  $\text{Spin}(7)$ -metrics have cohomogeneity 1 and Gukov, *et al* and Page, *et al* have now constructed many more examples of such.



*Compact examples?* On the other hand, a compact example cannot have any continuous symmetries (because it's Ricci-flat), so it's truly an 8-dimensional problem.

Here are a few of the known easy necessary conditions if a compact  $M^8$  is to support a metric  $g$  with holonomy  $\text{Spin}(7)$ :

- (1)  $M$  must be orientable and spinnable.
- (2) The Euler class of a spinor bundle of  $M$  must be zero.
- (3) The  $\hat{A}$ -genus of  $M$  must be 1.
- (4)  $b_1(M) = 0$ . (Ricci flat  $\Rightarrow$  harmonic 1-forms parallel.)
- (5)  $\pi_1(M)$  must be finite. (Cheeger-Gromoll)
- (6)  $b_4(M) > 0$ . ( $\Phi$  is  $g$ -parallel)
- (7)  $p_1(M) \neq 0$ . ( $\int_M \Phi \wedge p_1 = c \int_M \|R\|^2 * 1$ , where  $c \neq 0$ .)

**Theorem:** Suppose that  $(M^8, g)$  is a compact Riemannian manifold with holonomy  $\text{Spin}(7)$ . Then the marked moduli space of  $\text{Spin}(7)$ -holonomy metrics near  $g$  modulo the diffeomorphism of  $M$  near the identity is a smooth manifold near  $g$  whose tangent space is canonically isomorphic to  $H_+^4(M, \mathbb{R})$ .

It may seem that  $\text{Spin}(7)$  metrics are hopelessly hard to come by. However, here's a sample of the kind of minimal conditions that can produce such a metric:

**Proposition:** Let  $M^8$  be a compact oriented spin manifold with  $\hat{A}(M) = 1$ . Suppose that  $g$  is a metric on  $M$  with  $\text{Scal}(g) \geq 0$ . Then  $\text{Scal}(g) \equiv 0$  and the holonomy of  $g$  is  $\text{Spin}(7)$ .

In the early 1990's Dominic Joyce produced the first examples of compact Riemannian manifolds with holonomy  $\text{Spin}(7)$ . Again, there were two aspects to his methods, an analytic aspect and a geometric aspect.

It would take me too long to describe these here, so I'll just refer you to his beautiful book, "Compact manifolds with special holonomy" for further details.