MANIFOLDS WITH SPECIAL HOLONOMY
LECTURE 4:
G_2 AND $\mathrm{Spin}(7)$

ROBERT L. BRYANT

DUKE UNIVERSITY

14 AUGUST 2003

We are now ready to consider the exceptional cases, starting with the one in dimension 7.

n	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$-module
n	$\mathfrak{so}(n)$	$\mathbb{R} \oplus \mathsf{S}^2_0(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus \mathsf{S}^{1,1}_0(\mathbb{C}^m)^{\mathbb{R}} \oplus \mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$\mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus \mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$\mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$\mathsf{V}^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$\mathsf{V}^{0,2,0} \simeq \mathbb{R}^{168}$

5. The exceptional holonomy G_2. As usual, we'll start with linear algebra. Consider the 3-form ϕ_0 defined on $\mathbb{R}^7$ by the formula

$$\phi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{356} - \mathrm{d}x^{347}$$

where $\mathrm{d}x^{ijk}$ means $\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge dx^k$. Now define

$$\mathrm{G}_2 = \left\{ \ A \in \mathrm{GL}(7, \mathbb{R}) \,\middle|\, A^*(\phi_0) = \phi_0 \ \right\}.$$

It is not obvious from this what sort of group G_2 is (though it is a closed subgroup, by definition). A few subgroups can be discerned, though:

Writing $\mathbb{R}^7 = \mathbb{R}{\cdot}e_1 \oplus e_1^\perp = \mathbb{R}{\cdot}e_1 \oplus \mathbb{R}^6 = \mathbb{R}{\cdot}e_1 \oplus \mathbb{C}^3$ and setting

$$z^1 = x^2 + \mathrm{i}\, x^3, \qquad z^2 = x^4 + \mathrm{i}\, x^5, \qquad z^3 = x^6 + \mathrm{i}\, x^7$$

allows us to express ϕ_0 in the form

$$\phi_0 = \mathrm{d}x^1 \wedge \frac{\mathrm{i}}{2}\big(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1} + \mathrm{d}z^2 \wedge \mathrm{d}\overline{z^2} + \mathrm{d}z^3 \wedge \mathrm{d}\overline{z^3}\big) + \mathrm{Re}\big(\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3\big).$$

Thus, $\mathrm{G}_2 \supset \{1\} \times \mathrm{SU}(3)$. Similarly $\mathrm{G}_2 \supset \mathrm{SU}(3) \times \{1\}$. (Exercise.) As a result, $\mathrm{G}_2 \cdot e_1 \supset S^6 \subset \mathbb{R}^7$. In fact, $\mathrm{G}_2 \cdot e_1 = S^6$.

Moreover, the subgroup of G_2 that fixes e_1 is $\{1\} \times \mathrm{SU}(3)$. This makes G_2 into a principal $\mathrm{SU}(3)$-bundle over S^6:

$$\begin{array}{ccc} \mathrm{SU}(3) & \to & \mathrm{G}_2 \\ & & \downarrow \\ & & S^6 \end{array}$$

Thus G_2 is compact and of dimension 14. By the long exact seq. in homotopy, it is 2-connected and $\pi_3(G_2) = \pi_3\big(\mathrm{SU}(3)\big) = \mathbb{Z}$, so it is simple. It is the first of the five exceptional compact simple Lie groups, realized here as a subgroup of $\mathrm{SO}(7)$.

Define

$$\Lambda^3_+(\mathbb{R}^7)^* = \left\{ A^*\phi_0 \,\middle|\, A \in \mathrm{GL}(7,\mathbb{R}) \right\} \simeq \mathrm{GL}(7,\mathbb{R})/\,\mathrm{G}_2\,.$$

This orbit has dimension $49 - 14 = 35 = \dim \Lambda^3(\mathbb{R}^7)^*$, so it's an open subset of this vector space. More generally, if V is a vector space of dimension 7, define the *positive* 3*-forms* on V as

$$\Lambda^3_+(V^*) = \left\{ A^*\phi_0 \,\middle|\, A : V \to \mathbb{R}^7 \text{ is an isomorphism } \right\}.$$

Because G_2 lies in $\mathrm{SO}(7)$, to each positive form $\phi = A^*\phi_0$ on V there are a well-defined inner product and orientation on V defined by

$$g_\phi = A^* g_0, \qquad\qquad *_\phi 1 = A^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^7).$$

Given a 7-dimensional vector space V, a G_2-*structure* on V is a positive 3-form ϕ on V. The inner product g_ϕ and orientation $*_\phi 1 \in \Lambda^7(V^*)$ are said to be *associated* to ϕ.

It is not difficult to prove that ϕ_0 and its dual 4-form $*_{\phi_0}\phi_0$ generate the ring of G_2-invariant forms on $\mathbb{R}^7$. Harvey and Lawson proved that they are calibrations (more on this below).

Historical Remark: This was not the orginal definition of G_2, nor the most common. Nowadays, G_2 is most commonly defined as the group of automorphisms of the *octonions* $\mathbb{O}$, the unique normed division algebra of dimension 8 (Cartan, 1908). The definition given here is due to Schouten (ca. 1928), and is, in a certain sense, optimal.

The comass of ϕ_0. We can use the fact that G_2 stabilizes ϕ_0 to prove that it is a calibration and to compute $\mathsf{G}(\phi_0)$:

Use the fact that G_2 acts transitively on S^6 with stabilizer of e_1 equal to $\{1\}\times\mathrm{SU}(3)$ to see that, for any triple of orthonormal vectors (v_1, v_2, v_3) in $\mathbb{R}^7$, there is an element $A\in\mathrm{G}_2$ and a constant θ such that

$$A(v_1) = e_1, \qquad A(v_2) = e_2, \qquad A(v_3) = \cos\theta\, e_3 + \sin\theta\, e_4.$$

Then

$$\phi_0(v_1, v_2, v_3) = \phi_0(e_1, e_2, \cos\theta\, e_3 + \sin\theta\, e_4) = \cos\theta \leq 1.$$

Moreover, $v_1\wedge v_2\wedge v_3 \in \mathsf{Gr}_3^+(\mathbb{R}^7)$ is ϕ_0-calibrated if and only if it is equivalent to $e_1\wedge e_2\wedge e_3$ via G_2. Thus,

$$\mathsf{G}(\phi_0) = \mathrm{G}_2\cdot(e_1\wedge e_2\wedge e_3) = \mathrm{G}_2\,/\,\mathrm{SO}(4).$$

These calibrated 3-planes were dubbed *associative* by Harvey and Lawson because, regarding $\mathbb{R}^7$ as $\mathrm{Im}\,\mathbb{O}$, these planes generate associative subalgebras of $\mathbb{O}$.

Harvey and Lawson also showed that every 2-plane $E \subset \mathbb{R}^7$ lies in a unique associative 3-plane $E^+ \in \mathsf{G}(\phi_0)$.

In fact, they prove that every real analytic surface $S \subset \mathbb{R}^7$ lies in a unique analytically irreducible ϕ_0-calibrated 3-fold $\Sigma \subset \mathbb{R}^7$.

Thus, there are lots of 'associative' 3-manifolds in $\mathbb{R}^7$. These have become the focus of intense interest by $\mathcal{M}$-theorists as of late.

Of course, the Hodge dual 4-form $*_0\phi_0$ is also a calibration, calibrating the oriented orthogonals to the associative 3-planes:

$$\mathsf{G}(*_0\phi_0) = \mathrm{G}_2 \cdot (e_4 \wedge e_5 \wedge e_6 \wedge e_7) = \mathrm{G}_2 \,/\, \mathrm{SO}(4).$$

It follows that ϕ_0 vanishes on any 'coassociative' 4-plane in $\mathbb{R}^7$. In fact, Harvey and Lawson show that a 3-plane $E \in \mathsf{G}r_3(\mathbb{R}^7)$ lies in a coassociative 4-plane if and only iff ϕ_0 vanishes on E.

Correspondingly, they show that any real analytic 3-fold $S \subset \mathbb{R}^7$ to which ϕ_0 pulls back to zero lies in a unique analytically irreducible $*_0\phi_0$-calibrated 4-fold $\Sigma \subset \mathbb{R}^7$.

Manifolds. Let M^7 be a 7-manifold. A G_2-*structure* on M is a 3-form ϕ on M such that ϕ_x is a positive 3-form on T_xM for all $x \in M$. Such a ϕ exists on M iff M is orientable and spinnable.

A G_2-structure ϕ on M defines an associated Riemannian metric g_ϕ and an orientation (i.e., volume form) $*_\phi 1 \in \Omega^7(M)$.

If (M^7, g) is a Riemannian manifold with holonomy conjugate to a subgroup of G_2, then M supports a g-parallel 3-form ϕ that is positive and so defines a G_2-structure on M. Of course, this 3-form is both closed and co-closed.

Conversely, if M supports a g-parallel positive 3-form, then the holonomy of g is conjugate to a subgroup of G_2.

Theorem: (Fernandez–Gray) Let M be a 7-manifold and let ϕ be a G_2-structure on M, with associated metric g_ϕ and orientation $*_\phi 1$. If ϕ is closed and co-closed (with respect to g_ϕ and $*_\phi 1$), then ϕ is g_ϕ-parallel. (In particular, g_ϕ has holonomy conjugate to a subgroup of G_2.)

Calibrations. If $\phi \in \Omega^3_+(M^7)$ is a closed G_2-structure, then it is a calibration for M endowed with its associated metric g_ϕ. This defines a distinguished family of minimizing submanifolds of (M, g_ϕ).

In the case that ϕ is closed and real analytic in local coordinates, Harvey and Lawson's arguments for $(\mathbb{R}^7, \phi_0)$ generalize to (M^7, ϕ), showing that every real analytic surface $S \subset M^7$ lies in a unique analytically irreducible ϕ-calibrated 3-fold $\Sigma \subset M^7$.

Similarly, if $\phi \in \Omega^3_+(M^7)$ satisfies $\mathrm{d}\phi = \mathrm{d}(*_\phi \phi) = 0$, then $*_\phi \phi$ is a calibration on (M, g_ϕ).

Again, Harvey and Lawson's arguments generalize in this case without change to show that any real analytic 3-fold $S \subset M^7$ to which ϕ pulls back to zero lies in a unique analytically irreducible $*_\phi \phi$-calibrated 4-fold $\Sigma \subset M^7$.

Local Existence. Now, we are 'reduced' to studying the equations

$$\mathrm{d}\phi = \mathrm{d}\big(*_\phi\phi\big) = 0$$

for positive 3-forms $\phi \in \Omega^3_+(M)$. *A priori*, this is $35 + 21 = 56$ equations for the 35 unknown coefficients of the 3-form ϕ, but, in fact, these equations overlap by 7, so it's only 49 (quasi-linear, first order) partial differential equations.

These equations aren't elliptic because they are invariant under the diffeomorphism group.

Remember (from the table), that $K(\mathfrak{g}_2) \simeq \mathbb{R}^{77}$ and is irreducible as a G_2-module. It follows that if ϕ solves these equations, then g_ϕ is Ricci-flat (Bonan) and hence real analytic in g_ϕ-harmonic coordinates (DeTurck-Kazdan). In particular, ϕ will also be real analytic in g_ϕ-harmonic coordinates.

This suggests the augmented (coordinate-dependent) system

$$\mathrm{d}\phi = \mathrm{d}\big(*_\phi\phi\big) = \mathrm{d}\big(*_\phi \mathrm{d}x^i\big) = 0$$

for $\phi \in \Omega^3_+(\mathbb{R}^7)$. This system is (overdetermined and) elliptic.

I can now state the 'local generality' analogs for G_2 of the results that we found for the various complex cases:

Theorem: (B—) The system $\mathrm{d}\phi = \mathrm{d}\big({*_\phi}\phi\big) = 0$ for $\phi \in \Omega^3_+(M)$ is involutive. Modulo diffeomorphism, the general solution depends on six arbitrary functions of 6 variables.

Moreover, for the 'generic' solution ϕ, the values of the curvature form at a 'generic' point generate the entire holonomy algebra $\mathfrak{g}_2$. Thus, for the 'generic' solution, the holonomy of g_ϕ is all of G_2.

It is useful to have a test for reduced holonomy in the G_2 case:

Proposition: If M^7 is simply connected g is a metric on M that has holonomy a subgroup of G_2, then the holonomy is a proper subgroup of G_2 iff M supports a nonzero g-parallel 1-form.

Proof: By Berger's classification and the de Rham splitting theorem, the only connected proper subgroups of G_2 that can be holonomy groups in dimension 7 are $\mathbf{1}_1 \times \mathrm{SU}(3)$, $\mathbf{1}_3 \times \mathrm{SU}(2)$, and $\mathbf{1}_7$.

Explicit examples: Once we knew they were there, finding explicit examples wasn't so hard:

A conical example: The first known example was the cone metric on the flag manifold $\mathrm{SU}(3)/T^2$, endowed with its natural metric. In fact, write the left-invariant form on $\mathrm{SU}(3)$ in the form

$$g^{-1}\,\mathrm{d}g = \begin{pmatrix} -\mathrm{i}\,\theta_3 & -\overline{\omega_1} & \mathrm{i}\,\omega_2 \\ \omega_1 & -\mathrm{i}\,\theta_2 & -\overline{\omega_3} \\ \mathrm{i}\,\overline{\omega_2} & \omega_3 & -\mathrm{i}\,\theta_2 \end{pmatrix}$$

Then the quadratic form $\mathrm{d}\sigma^2 = \omega_1{\circ}\overline{\omega_1} + \omega_2{\circ}\overline{\omega_2} + \omega_3{\circ}\overline{\omega_3}$ defines a Riemannian metric on $\mathrm{SU}(3)/T^2$, where T^2 is the maximal torus in $\mathrm{SU}(3)$ consisting of diagonal matrices. Setting

$$\Omega = \frac{\mathrm{i}}{2}\big(\omega_1\wedge\overline{\omega_1} + \omega_2\wedge\overline{\omega_2} + \omega_3\wedge\overline{\omega_3}\big), \qquad \Psi = \omega_1\wedge\omega_2\wedge\omega_3\,,$$

these two forms also drop to $\mathrm{SU}(3)/T^2$ and satisfy

$$\mathrm{d}\Omega = 3\,\mathrm{Re}(\Psi), \qquad\qquad \mathrm{d}\Psi = -2\mathrm{i}\,\Omega^2.$$

Now, on $M = (0,\infty) \times \mathrm{SU}(3)/T^2$, consider the (obviously closed) 3-form

$$\phi = \tfrac{1}{3}\,\mathrm{d}\big(t^3\Omega\big) = t^2\,\mathrm{d}t \wedge \Omega + t^3\,\mathrm{Re}(\Psi).$$

This is a positive 3-form. One computes that its associated metric is the 'cone' metric

$$g_\phi = \mathrm{d}t^2 + t^2\,\mathrm{d}\sigma^2.$$

Moreover, the g_ϕ-dual of ϕ is found to be

$$*_\phi\phi = -\tfrac{1}{4}\,\mathrm{d}\big(t^4\mathrm{Im}(\Psi)\big)$$

and this is obviously closed as well. Thus, the holonomy of g_ϕ is conjugate to a subgroup of G_2.

One can now compute that there are no g_ϕ-parallel 1-forms, so the holonomy must be all of G_2.

Later, Bryant and Salamon constructed complete G_2-holonomy metrics on $\Lambda^2_+(\mathbb{CP}^2)$ and on $S^3 \times \mathbb{R}^4$.

These G_2-metrics have cohomogeneity 1 and Gukov, *et al* and Page, *et al* have now constructed many more examples of such.

Compact examples? On the other hand, a compact example cannot have any continuous symmetries (because it's Ricci-flat), so it's truly a 7-dimensional problem.

Here are a few of the known easy necessary conditions if a compact M is to support a metric g with holonomy G_2:

(1) M must be orientable and spinnable.
(2) $b_1(M) = 0$. (Ricci flat $\Rightarrow$ harmonic 1-forms parallel.)
(3) $\pi_1(M)$ must be finite. (Cheeger-Gromoll)
(4) $b_3(M) = b_4(M) > 0$. (ϕ, $*_\phi\phi$ are parallel forms)
(5) $p_1(M) \neq 0$. ($\int_M \phi\wedge p_1 = c \int_M ||R||^2 * 1$, where $c \neq 0$.)
(6) M cannot be a nontrivial product. (Donaldson)

There is also a relatively easy 'Torelli theorem':

Theorem: Suppose that (M^7, g) is a compact Riemannian manifold with holonomy G_2. Then the marked moduli space of G_2-holonomy metrics near g modulo the diffeomorphisms of M near the identity is a smooth manifold near g whose tangent space is canonically isomorphic to $H^3_{dR}(M, \mathbb{R})$.

In the early 1990's Dominic Joyce produced the first examples of compact Riemannian manifolds with holonomy G_2. There were two aspects to his methods:
Analytic: Joyce proved that, if one has a compact M^7 that supports a *closed* positive 3-form ϕ with the property that its associated metric g_ϕ satisfies a combination of

(1) a upper bound on its curvature and volume,
(2) a lower bound on its injectivity radius, and
(3) a upper bound on the size of $\|\mathrm{d} *_\phi \phi\|$,

then one can perturb ϕ by adding a small exact 3-form so as to get a new $\hat{\phi}$ that is positive, closed, and co-closed with respect to its associated metric.
Geometric: Joyce constructed examples of (M^7, ϕ) that satisfy the above hypotheses by starting with a flat metric on the 7-torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, dividing by a discrete group to get a manifold with singularities, resolving those singularities to get the manifold M^7 and a ϕ on M^7 satisfying the above conditions.

Finally, we are ready to consider the last case, which only occurs in dimension 8.

n	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$-module
n	$\mathfrak{so}(n)$	$\mathbb{R} \oplus \mathsf{S}_0^2(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus \mathsf{S}_0^{1,1}(\mathbb{C}^m)^{\mathbb{R}} \oplus \mathsf{S}_0^{2,2}(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$\mathsf{S}_0^{2,2}(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus \mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$\mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$\mathsf{V}^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$\mathsf{V}^{0,2,0} \simeq \mathbb{R}^{168}$

6. The exceptional holonomy $\mathrm{Spin}(7)$**.** As usual, we begin with linear algebra. Let $\mathbb{R}^8 = \mathbb{R}{\cdot}e_0 \oplus \mathbb{R}^7$ and consider the 4-form

$$\Phi_0 = \mathrm{d}x^0 \wedge \phi_0 + *_0\phi_0 \,.$$

Now 'define'

$$\mathrm{Spin}(7) = \{\ A \in \mathrm{GL}(8,\mathbb{R}) \,\big|\, A^*\Phi_0 = \Phi_0\ \}.$$

The definition and previous discussion $\Rightarrow$ $\mathrm{Spin}(7) \supset \{1\}\times \mathrm{G}_2$. Moreover, if we identify $\mathbb{R}^8$ with $\mathbb{C}^4$ by setting

$$z^0 = x^0 + \mathrm{i}\,x^1, \quad z^1 = x^2 + \mathrm{i}\,x^3, \quad z^2 = x^4 + \mathrm{i}\,x^5, \quad z^3 = x^6 + \mathrm{i}\,x^7,$$

then you can check that

$$\begin{aligned}\Phi_0 = \frac{1}{2}\left(\frac{\mathrm{i}}{2}\big(\mathrm{d}z^0 \wedge \mathrm{d}\overline{z^0} + \mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1} + \mathrm{d}z^2 \wedge \mathrm{d}\overline{z^2} + \mathrm{d}z^3 \wedge \mathrm{d}\overline{z^3}\big)\right)^2 \\ + \mathrm{Re}\big(\mathrm{d}z^0 \wedge \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3\big),\end{aligned}$$

so $\mathrm{Spin}(7) \supset \mathrm{SU}(4)$. Finally, can show that $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ and that the subgroup of $\mathrm{Spin}(7)$ that fixes e_0 is exactly $\{1\}\times \mathrm{G}_2$.

This makes Spin(7) into a principal G_2-bundle over S^7:

$$\begin{array}{ccc} \mathrm{G}_2 & \rightarrow & \mathrm{Spin}(7) \\ & & \downarrow \\ & & S^7 \end{array}$$

Thus Spin(7) is compact and of dimension 21. By the long exact seq. in homotopy, it is 2-connected and $\pi_3\big(\mathrm{Spin}(7)\big) = \pi_3(G_2) = \mathbb{Z}$, so it is simple. By the classification, there are only two possibilities: Spin(7) is Sp(3) or the universal cover of SO(7). The former has no 8-dimensional irreducible representation, so Spin(7) must be the universal cover of SO(7). Define

$$\Lambda^4_+(\mathbb{R}^8)^* = \left\{ A^*\Phi_0 \,\middle|\, A \in \mathrm{GL}(7, \mathbb{R}) \right\} \simeq \mathrm{GL}(8, \mathbb{R}) / \mathrm{Spin}(7).$$

This orbit has dimension $64 - 21 = 43 < \dim \Lambda^4(\mathbb{R}^8)^*$, so it's *not* an open subset of this vector space. More generally, if V is a vector space of dimension 8, define the *Cayley forms* on V to be

$$\Lambda^4_+(V^*) = \left\{ A^*\Phi_0 \,\middle|\, A : V \to \mathbb{R}^8 \text{ is an isomorphism} \right\}.$$

Because Spin(7) lies in SO(8), to each Cayley form $\Phi = A^*\Phi_0$ on V there are a well-defined inner product and orientation on V defined by

$$g_\Phi = A^* g_0, \qquad\qquad *_\Phi 1 = A^*(\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^7).$$

Given an 8-dimensional vector space V, a Spin(7)-*structure* on V is a Cayley form $\Phi \in \Lambda^4_+(V^*)$. The inner product g_Φ and orientation $*_\Phi 1 \in \Lambda^8(V^*)$ are said to be *associated* to Φ.

It is not difficult to prove that $\Phi_0 = *_{\Phi_0}\Phi_0$ generates the ring of Spin(7)-invariant forms on $\mathbb{R}^8$. Harvey and Lawson proved that it is a calibration (more on this below).

Historical Remark: This is a very idiosyncratic definition of Spin(7). Usually, Spin(7) is defined abstractly as the universal cover of SO(7). Its embedding into SO(8) is derived via Clifford constructions or by using the octonions, e.g., Spin(7) is the subgroup of the rotations in $\mathbb{O}$ that is generated by right multiplications by unit imaginary octonions.

Manifolds. Let M^8 be an 8-manifold. A $\mathrm{Spin}(7)$-*structure* on M is a 4-form Φ on M such that Φ_x is a Cayley form on T_xM for all $x \in M$.

A $\mathrm{Spin}(7)$-structure on M defines an associated Riemannian metric g_Φ and orientation (i.e., volume form) $*_\Phi 1 \in \Omega^8(M)$ with respect to which Φ is self-dual, i.e., $*_\Phi \Phi = \Phi$.

If (M^8, g) is a Riemannian manifold with holonomy conjugate to a subgroup of $\mathrm{Spin}(7)$, then M supports a g-parallel Cayley form Φ and this defines a $\mathrm{Spin}(7)$-structure on M. This 4-form is closed (and co-closed, of course).

Conversely, if M supports a g-parallel Cayley form, then the holonomy of g is conjugate to a subgroup of $\mathrm{Spin}(7)$.

Theorem: (Fernandez–Gray) Let M be a 8-manifold and let Φ be a $\mathrm{Spin}(7)$-structure on M, with associated metric g_Φ and orientation $*_\Phi 1$. If Φ is closed, then Φ is g_Φ-parallel. (In particular, g_Φ has holonomy conjugate to a subgroup of $\mathrm{Spin}(7)$.)

Thus, we are 'reduced' to studying the equations

$$\mathrm{d}\Phi = 0$$

for Cayley forms $\Phi \in \Omega^4_+(M)$. This is 56 (quasi-linear, first order) partial differential equations for the 43 unknown coefficients of the form Φ.

These equations aren't elliptic because they are invariant under the diffeomorphism group.

Remember (from the table), that $K(\mathfrak{spin}(7)) \simeq \mathbb{R}^{168}$ and is irreducible as a Spin(7)-module. It follows that if Φ solves these equations, then g_Φ is Ricci-flat (Bonan) and hence real analytic in g_Φ-harmonic coordinates (DeTurck-Kazdan). In particular, Φ will also be real analytic in g_Φ-harmonic coordinates.

This suggests the augmented (coordinate-dependent) system

$$\mathrm{d}\Phi = \mathrm{d}\big(*_\Phi \mathrm{d}x^i\big) = 0$$

for $\Phi \in \Omega^4_+(\mathbb{R}^8)$. This system of $56 + 8 = 64$ equations is (overdetermined and) elliptic.

I can now state the 'local generality' analogs for Spin(7):

Theorem: (B—) The system $\mathrm{d}\Phi = 0$ for $\Phi \in \Omega^4_+(M)$ is involutive and, modulo diffeomorphism, the general solution depends on twelve arbitrary functions of 7 variables.

Moreover, for the 'generic' solution Φ, the values of the curvature form at a 'generic' point generate the entire holonomy algebra $\mathfrak{spin}(7)$. Thus, for the 'generic' solution, the holonomy of g_Φ is all of Spin(7).

Proposition: If M^8 is simply connected and g is a metric on M that has holonomy a subgroup of Spin(7), then the holonomy is a proper subgroup of Spin(7) iff M supports a nonzero g-parallel 1-form or 2-form.

Proof: By Berger's classification and the de Rham splitting theorem, the only connected proper subgroups of Spin(7) that can be holonomy groups in dimension 7 are subgroups of either $\mathbf{1}_1 \times \mathrm{G}_2$ or SU(4).

Explicit examples: Again, constructing cohomogeneity 1 examples isn't hard:

A conical example: The first known example was the cone metric on the homogeneous space $Y^7 = \mathrm{SO}(5)/H_5$, where $H_5 \subset \mathrm{SO}(5)$ is a subgroup isomorphic to SO(3) and acting irreducibly on $\mathbb{R}^5$. Up to constant multiples, there is a unique SO(5)-invariant metric on Y^7.

I won't give the full details, but the basic point is that Y^7 is a rational homology 7-sphere that has a unique (up to constant multiples) SO(5) invariant 3-form ϕ. (This follows from the representation theory of SO(3).)

Now, this form ϕ is actually a positive 3-form, so it defines a G_2-structure on Y, with associated metric g_ϕ and orientation $*_\phi 1$.

Now, it is not difficult to show that $\mathrm{d}\phi = \lambda *_\phi \phi$ for some constant $\lambda \neq 0$. By scaling ϕ, we can assume that

$$\mathrm{d}\phi = 4 *_\phi \phi.$$

Now, on $M^8 = (0, \infty) \times Y^7$, consider the (obviously closed) 4-form

$$\Phi = \tfrac{1}{4}\,\mathrm{d}\big(t^4\phi\big) = t^3\,\mathrm{d}t \wedge \phi + t^4 \ *_\phi \phi.$$

This is clearly a Cayley form. One computes that its associated metric is the 'cone' metric

$$g_\Phi = \mathrm{d}t^2 + t^2\, g_\phi.$$

Thus, the holonomy of g_Φ is conjugate to a subgroup of Spin(7).

One can now compute that there are no g_Φ-parallel 1-forms or 2-forms so the holonomy must be all of Spin(7).

Later, Bryant and Salamon constructed a complete Spin(7)-holonomy metric on $\mathbb{S}_+(S^4)$, the semi-spinor bundle over S^4.

These Spin(7)-metrics have cohomogeneity 1 and Gukov, *et al* and Page, *et al* have now constructed many more examples of such.

Compact examples? On the other hand, a compact example cannot have any continuous symmetries (because it's Ricci-flat), so it's truly an 8-dimensional problem.

Here are a few of the known easy necessary conditions if a compact M^8 is to support a metric g with holonomy Spin(7):

(1) M must be orientable and spinnable.
(2) The Euler class of a spinor bundle of M must be zero.
(3) The $\hat{A}$-genus of M must be 1.
(4) $b_1(M) = 0$. (Ricci flat $\Rightarrow$ harmonic 1-forms parallel.)
(5) $\pi_1(M)$ must be finite. (Cheeger-Gromoll)
(6) $b_4(M) > 0$. (Φ is g-parallel)
(7) $p_1(M) \neq 0$. ($\int_M \Phi \wedge p_1 = c \int_M ||R||^2 * 1$, where $c \neq 0$.)

Theorem: Suppose that (M^8, g) is a compact Riemannian manifold with holonomy Spin(7). Then the marked moduli space of Spin(7)-holonomy metrics near g modulo the diffeomorphism of M near the identity is a smooth manifold near g whose tangent space is canonically isomorphic to $H^4_+(M, \mathbb{R})$.

It may seem that Spin(7) metrics are hopelessly hard to come by. However, here's a sample of the kind of minimal conditions that can produce such a metric:

Proposition: Let M^8 be a compact oriented spin manifold with $\hat{A}(M) = 1$. Suppose that g is a metric on M with $\mathrm{Scal}(g) \geq 0$. Then $\mathrm{Scal}(g) \equiv 0$ and the holonomy of g is Spin(7).

In the early 1990's Dominic Joyce produced the first examples of compact Riemannian manifolds with holonomy Spin(7). Again, there were two aspects to his methods, an analytic aspect and a geometric aspect.

It would take me too long to describe these here, so I'll just refer you to his beautiful book, "Compact manifolds with special holonomy" for further details.