Seiberg-Witten theory
Spin^c structures

Given a $2m$-dimensional real inner product space V we can form its *Clifford algebra* $C(V)$ as the free tensor algebra $T^\bullet V$ modulo the relation $v \otimes v = -||v||^2$, or, as a vector space,

$$C(V) = \Lambda^\bullet V,$$

of dimension 2^{2m}, with product

$$v.w := v \wedge w - v \lrcorner w.$$

$C_{\mathbb{C}}(V) := C(V) \otimes_{\mathbb{R}} \mathbb{C}$ has itself as a Clifford module, but it is highly reducible. We produce an irreducible module in the following (non-canonical) way.

$\dim V = 2m \Rightarrow$ pick compatible complex structure on it $\Rightarrow$ complex m-dimensional hermitian vector space. Then

$$\Lambda^\bullet_{\mathbb{C}} V$$

($\dim_{\mathbb{C}} = 2^m$, $\dim_{\mathbb{R}} = 2^{m+1}$, so much smaller)
is a $C_{\mathbb{C}}(V)$-module, with $C_{\mathbb{C}}(V)$-action ρ given by

$$\rho(v)w := v \wedge w - v \lrcorner\, w.$$

Picking an orthonormal basis $\{\frac{v}{\|v\|^2}, \ldots\}$ of V can see this satisfies the Clifford relations $\rho(v)^2 = -||v||^2$ so extends to $C_{\mathbb{C}}(V)$ (by the same formula).

(Using injectivity and a dimension count) this exhibits $C_{\mathbb{C}}(V)$ as a matrix algebra

$$C_{\mathbb{C}}(V) \cong \mathrm{End}(\Lambda^\bullet_{\mathbb{C}} V) \cong M_{2^m \times 2^m}(\mathbb{C}).$$

(Or can prove this inductively using $C_{\mathbb{C}}(\mathbb{R}^{n+2}) \cong C_{\mathbb{C}}(\mathbb{R}^n) \otimes_{\mathbb{C}} C_{\mathbb{C}}(\mathbb{R}^2)$ without direct construction.)

So given a rank $2m$ bundle V on a manifold, can form its Clifford bundle $C_{\mathbb{C}}(V)$, and this is a bundle of endomorphisms, i.e. *locally* $\operatorname{End} W$ for some complex vector bundle W. If V is *globally* complex then W can be chosen globally as $\Lambda^{\bullet}_{\mathbb{C}} V$, or

$$W = (\Lambda^{\bullet}_{\mathbb{C}} V) \otimes \mu,$$

but in general W might not exist globally. Local choices of W (i.e. defined locally only up to tensoring with a line bundle μ) $\Rightarrow$ global obstructions.

(i.e. $\mathbb{P}(W)$ well defined; can this $\mathbb{P}$-bundle be lifted to a vector bundle W ?)

Definition 17 *A spin^c structure on V is a choice of complex vector bundle W such that $C_{\mathbb{C}}(V) \cong$* End W.

A spin^c structure on M is a choice of W such that $C_{\mathbb{C}}(TM) \cong$ End W.

So if M^{2m} is almost complex (and hermitian) have a canonical spin^c structure; $TM \cong \Lambda^{0,1}_M$ over $\mathbb{C}$, and

$$W = \bigoplus_i \Lambda^{0,i}_M.$$

All others are of the form $W \otimes \mu$.

(Spin^c=Unitary autos of W that commute with autos $SO(V)$ of V. Have an additional $U(1)$ – the centraliser of End(W): isomorphism between irreps is a scalar – i.e.

$$1 \to U(1) \to \text{Spin}^c(V) \to SO(V) \to 1.$$

A Spin^c structure is a lift of structure group from $SO(2m)$ to Spin$^c(2m)$. $U(m) \hookrightarrow SO(2m)$ lifts naturally to Spin$^c(2m)$.)

Theorem 18 *An oriented Riemannian 4-manifold M has a Spin^c structure; the set of them is an affine space modelled on $H^2(M;\mathbb{Z})$.*

Proof. We can pick an almost complex structure J on M away from finitely many points p_i: J is equivalent to a self-dual 2-form $\omega = g(\cdot\,, J\,\cdot)$, and $e(\Lambda^+) = 0$.

So we get a Spin^c structure on $M\backslash\{p_i\}$. Pick any spin^c structure on $B_{p_i}(\epsilon)$. These differ over $\partial B_{p_i} \cong S^3$ by a $U(1)$ line bundle, but line bundles on S^3 are trivial: $[S^2, U(1)] = \pi_2(S^1) = 0$.

Any other spin^c structure differs by twisting by a line bundle μ. □

$i^{m(m+1)}$vol $\in \Lambda^\bullet V \otimes \mathbb{C} \simeq C_{\mathbb{C}}(V)$ anticommutes with $V \subset C_{\mathbb{C}}(V)$ and has square -1, so it splits

$$W = W^+ \oplus W^-$$

such that $\rho|_V$ maps $W^\pm \to W^\mp$.

In the almost complex case, $W = \Lambda^{0,\bullet} \otimes \mu$, $W^+ = \Lambda^{0,\text{even}} \otimes \mu$, $W^- = \Lambda^{0,\text{odd}} \otimes \mu$, and $v \wedge \cdot - v \lrcorner \cdot$ swaps these.

The line bundle L associated associated to a Spin^c structure W on M^4 is the determinant of W^+. Under $W \mapsto W \otimes \mu$, $L \mapsto L \otimes \mu^{\otimes 2}$.

Connections and Dirac operators

The Levi-Civita connection on M determines a projective connection on W, i.e. a connection on $\mathbb{P}(W)$.

To induce a connection B on W then, it is enough to give a connection A on L.

Definition 19 *The Dirac operator D_A associated to a connection A on L is given by the composition*

$$\Gamma(W) \xrightarrow{d_B} \Gamma(T^*X \otimes W) \xrightarrow{g} \Gamma(TX \otimes W) \xrightarrow{\rho} \Gamma(W).$$

Since $\rho\big|_{TX}$ switches $W^{\pm}$, so does $D_A : \Gamma(W^{\pm}) \to \Gamma(W^{\mp})$.

D_A is self adjoint.

Theorem 20 *For M Kähler, $W = \Lambda_M^{0,*} \otimes \mu$, and connection A in $L = K_M^* \otimes \mu^2$,*

$$D_A = \sqrt{2}\,(\bar\partial_B + \bar\partial_B^*).$$

Proof. ($\mu = \mathcal{O}$ for simplicity.) Chasing the identifications of $\Lambda_M^{1,0} \cong T^*M \cong TM \ni v$, the Clifford action of v on $\Lambda_M^{0,*}$ is

$$w \mapsto \sqrt{2}\,\pi^{0,*}(v \wedge w - v \lrcorner\, w).$$

So the Dirac operator is given by

$$\sqrt{2}\,\pi^{0,*}(\,\wedge\, - \,\lrcorner\,) \circ \nabla_{LC}.$$

Since ∇_{LC} is torsion-free, $\pi^{0,*}(\,\wedge\,) \circ \nabla_{LC} = \pi^{0,*} \circ d = \bar\partial$.

Similarly, $\pi^{0,*}(\,\lrcorner\,) \circ \nabla_{LC} = \pi^{0,*}(\bar{*} \wedge \bar{*})\nabla_{LC} = \bar{*}\pi^{n,*}(\,\wedge\,)\nabla_{LC}\bar{*} = \bar{*}\pi^{n,*}\,d\bar{*} = \bar{*}\bar\partial\bar{*} = -\bar\partial^*$. □

Since F_A is a section of $\Lambda^2(i\mathbb{R}) \subset C_{\mathbb{C}}(T^*_M)$, we have $\rho(F_A) \in \mathrm{End}(W)$. Under this identification, $\Lambda^+ \otimes \mathbb{C} \cong \mathrm{End}_0 W^+$.

The Seiberg-Witten equations

The equations, for a section $\Phi \in \Gamma(W^+)$ and a unitary connection A on L, are

$$\begin{aligned} D_A\Phi &= 0, \\ \rho(F_A^+) &= (\Phi\Phi^*)_0. \end{aligned}$$

Being only mildly nonlinear, with abelian gauge group, these equations behave well analytically.

The moduli space $\mathcal{M}$

The equations minimise a functional, giving L^2-estimates as before. By the Lichnerowicz formula

$$D_A^* D_A \Phi = 0 = \nabla_A^* \nabla_A \Phi + \frac{1}{2} F_A^+(\Phi) + \frac{1}{4} s \Phi.$$

Using the 2nd SW equation,

$$\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle \leq -\frac{1}{2} |\Phi|^4 - \frac{1}{4} \min(s) |\Phi|^2.$$

Now $\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle = < \nabla_A \Phi, \nabla_A \Phi \rangle + \frac{1}{2} \Delta |\Phi|^2$, so

$$\Delta |\Phi|^2 \leq -|\Phi|^4 - \frac{1}{2} \min(s) |\Phi|^2.$$

At the maximum of $|\Phi|$ the LHS ≥ 0, so

$$\max |\Phi|^2 \leq -\frac{1}{2} \min(s).$$

This gives a similar pointwise bound on F_A^+, and compactness follows easily. (And for $s \geq 0$ the only solutions are $\Phi = 0$, $F_A^+ = 0$, with consequences for M's topology.)

Similarly, perturbing the equations by $\rho(F_A^+ + i\eta) = (\Phi\Phi^*)_0$ for any $\eta \in \Omega^+$, we can make them transverse and $\mathcal{M}$ compact, of dimension

$$d = \frac{1}{4}(c_1(L)^2 - 2\chi(M) - 3\tau(M)).$$

Recall we always take $b_1 = 0$.

The gauge group $\mathcal{G} = \Gamma(U(1) \times M)$ acts on W by multiplication and so on L with weight two:

$$\Phi \mapsto g.\Phi,\ B \mapsto B - g^{-1}dg,\ A \mapsto A - 2g^{-1}dg,$$

leaving the equations unaltered. The only fixed points ("reducibles") occur when $\Phi = 0$ and so $F_A^+ = 0 \ \Rightarrow$ asd elements of $H^2(M, \mathbb{Z})$ again. So as before, for $b^+ > 0$ we can avoid reducibles for the generic metric, and for $b^+ > 1$ any two such $\mathcal{M}$s are cobordant.

Definition 21 *For $b^+(M) > 1$, we define the SW invariants of a Spin^c structure on M with associated line bundle L such that*

$$c_1(L)^2 = 2\chi(M) + 3\tau(M)$$

to be the (signed) count of points in $\mathcal{M}$.

Conjecture 22 (Seiberg-Witten) *The Donaldson polynomials of (simple type) M are determined by an explicit formula in the SW invariants.*

The complex case

On a Kähler surface write $\Phi = (\alpha, \beta) \in \Omega^0(\mu) \oplus \Omega^{0,2}(\mu) = \Gamma(W^+)$. Then the equations become

$$\begin{aligned} \bar{\partial}_B \alpha + \bar{\partial}_B^* \beta &= 0, \\ F_A^{0,2} &= \bar{\alpha}\beta, \\ i\omega.F_A^{1,1} &= -\frac{1}{2}(|\alpha|^2 - |\beta|^2). \end{aligned}$$

Theorem 23

If $\deg L \leq 0$ *then* $\beta \equiv 0$ *and*

$$\begin{aligned} \bar{\partial}_B \alpha &= 0, \\ F_A^{0,2} &= 0, \\ i\omega.F_A^{1,1} &= -\frac{1}{2}|\alpha|^2. \end{aligned}$$

If $\deg\ L \geq 0$ *then* $\alpha \equiv 0$ *and*

$$\begin{aligned} \bar{\partial}_B^* \beta &= 0, \\ F_A^{0,2} &= 0, \\ i\omega.F_A^{1,1} &= \frac{1}{2}|\beta|^2. \end{aligned}$$

(In each case $F_B^{0,2} = \frac{1}{2}F_B^{0,2} = 0$ since $L = K_M^* \otimes \mu^2$ and K_M is holomorphic. So both are holomorphic line bundles and again we have a holomorphic interpretation of a gauge theory equation.)

Proof. Applying $\bar\partial_B$ to $\bar\partial_B\alpha + \bar\partial_B^*\beta = 0$ and using $F_B^{0,2} = \frac{1}{2}\bar\alpha\beta$ gives

$$\frac{1}{2}|\alpha|^2\beta + \bar\partial_B\bar\partial_B^*\beta = 0.$$

$\int_M\langle\,\cdot\,,\beta\rangle \;\Rightarrow\; \bar\alpha\beta \equiv 0 \equiv \bar\partial_B^*\beta$. Thus one of α, β is $\equiv 0$, and $\bar\partial_B\alpha = 0$.

$$\deg L = \frac{i}{2\pi}\int_M \omega.F_A^{1,1} = -\frac{1}{4\pi}\int(|\alpha|^2 - |\beta|^2)$$

determines which of α, β vanishes from the sign of $\deg L$. □

The $\dim \mathcal{M} = \frac{1}{4}(c_1(L)^2 - 2\chi(M) + 3\tau(M)) = 0$ simple type condition says that $L = K_M^{\pm 1}$ – the standard Spin^c structures. We take the case where μ is trivial and $L = K_M^*$; the other is related by Serre duality. For $b^+ > 1$ we require $h^{2,0} = h^0(K_M) > 0$ and so $(-\deg L) = \deg K_M > 0$.

So $\beta \equiv 0$ and α is a holomorphic function, i.e. a constant determined by $\deg K_M^*$ by the $i\omega.F_A^{1,1} = -\frac{1}{2}|\alpha|^2$ equation. This is easily solved uniquely and

Theorem 24 *The Seiberg-Witten invariant of a Kähler surface is* ± 1*.*

In fact Taubes has proved

Theorem 25 [Taubes]
The Seiberg-Witten invariant of the canonical Spin^c *structure on a symplectic surface is* ± 1*.*

Taubes also relates SW invariants to Gromov-Witten invariants (via the curve cut out by α). Other applications include the Thom conjecture and very many generalisations of it (including to curves in symplectic manifolds).

Theorem 26 [Kronheimer-Mrowka]
Smooth complex curves have the minimal genus of curves representing a class in $H_2(\mathbb{P}^2)$.

Bauer-Furuta invariants
The Pontryagin-Thom construction

If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is *proper* then its fibres are all *compact* and we can count the signed number of points in a generic fibre.

But f also extends to the one-point compactifications $S^n \to S^n$, and we can instead consider the homotopy class of this map $\pi_n(S^n) \cong \mathbb{Z}$ and get the same answer – the *degree* of f.

Similarly, if $f\colon \mathbb{R}^N \to \mathbb{R}^n,\ N > n$ is proper, we can consider its compact fibres up to cobordism or the homotopy class $\pi_N(S^n)$ of its one-point compactification.

In fact the compact fibres have trivialised normal bundle (f^* of the trivial normal bundle T_pS^n of a point $p \in S^n$) – we say the normal bundle is *framed* – and the cobordism is also framed (as the normal bundle to a curve in S^n is also trivial). So to an element of $\pi_N(S^n)$ we can associate a framed cobordism class.

Conversely, given $M^{N-n} \subset S^N$ with framed normal bundle

$$\nu_{M \subset S^N} \cong \mathbb{R}^n \times M,$$

we get an isomorphism between a tubular neighbourhood of $M \subset S^N$ and $\mathbb{R}^n \times M$; projecting this to $\mathbb{R}^n \subset S^n$ and mapping the rest of S^N to $\infty \in S^n$ gives an element of $\pi_N(S^n)$.

Stabilising this (replacing $f\colon \mathbb{R}^N \to \mathbb{R}^n$ by $f \times \mathrm{id}_{\mathbb{R}^p}\colon \mathbb{R}^{N+p} \to \mathbb{R}^{n+p}$ and trivial normal bundles by stably trivial normal bundles) we get an isomorphism between the stably framed cobordism group of $(N-n)$-manifolds and

$$\pi^{st}_{N-n} := \lim_{r\to\infty} \pi_{N+r}(S^{n+r}).$$

($\pi^{st}_0 \cong \mathbb{Z}$ still counts the number of points in a framed cobordism class of dimension 0.)

Bauer and Furuta extend this to infinite dimensions (so that $N-n$ becomes a Fredholm index), lifting our construction of a cobordism class of moduli space to one with stably framed normal bundle. This would be hard in the asd case because the equations are only locally (via gauge fixing) the zero set of a Fredholm map; globally they are a quotient by $\mathcal{G}$ of such.

The SW gauge group is abelian and we can gauge fix globally. That is, picking a base connection A_0 on L and setting $A = A_0 + a$, the Seiberg-Witten moduli space is the zero set of

$$\begin{aligned} \Gamma(W^+) \oplus \Omega^1(i\mathbb{R}) &\to \Gamma(W^-) \oplus \Omega^+(i\mathbb{R}) \oplus \Omega^0(i\mathbb{R}) \\ (\Phi, a) \quad &\mapsto \quad (D_A\Phi, F_A^+ - (\Phi\Phi^*)_0, d^*a), \end{aligned}$$

modulo the $U(1)$ of covariantly constant gauge transformations.

Since this map (completed to Sobolev spaces with one more derivative on the LHS) is of the form

$$F + c$$

with F Fredholm and linear and c compact, then we can take finite dimensional approximations to it

$$F + c|_{F^{-1}(V)} : F^{-1}(V) \to V$$

with V finite dimensional and containing $(\operatorname{im} F)^{\perp}$ (mapping onto $\operatorname{coker} F$). Bauer-Furuta show that everything can be done S^1-equivariantly, and the equations are proper so extend to one-point compactifications. The upshot is an invariant

$$\mu(M) \in \pi^{st}_{d+1}, \quad d = \frac{1}{4}(c_1(L)^2 - 2\chi(M) - 3\tau(M)).$$

Bauer-Furuta also keep track of the S^1-action to get a refined equivariant invariant; because of problems with equivariant transversality this is technical.

($b^+ > 1$, $c_1(L)^2 = 2\chi(M) + 3\tau(M)$ $\Rightarrow$ 1-dimensional stable cobordism class; counting S^1 orbits gives the usual SW invariant.)

Connected sums

Recall that the Donaldson (and SW) invariants of connected sums vanish. The gluing procedure described for the asd equations (stretch the neck of $M_1 \# M_2$, solutions concentrate to a product of solutions on both pieces, $\mathcal{M} \simeq \mathcal{M}_{M_1} \times \mathcal{M}_{M_2} \times U(1)$) can be generalised to the following.

Theorem 27 [Bauer] $\mu(M_1 \# M_2) \in \pi^{st}_{d_1+d_2+1}$ *is the smash product*

$$\mu = \mu(M_1) \wedge \mu(M_2)$$

of the invariants $\mu(M_i) \in \pi^{st}_{d_i+1}$.

(At the cobordism level, smash product is product of manifolds; for stabilised maps of spheres it is composition.)

The invariant of $K3$ is the stable homotopy class $[\eta]$ of the Hopf map $S^3 \to S^2$ (divide by S^1 to recover $SW = 1$). So

$$\mu(K3 \# K3) = [\eta]^2 \neq 0 \in \pi^{st}_2.$$

But $\mu(S^2 \times S^2) = 0$, so $K3 \# K3$ cannot be written as a smooth connect sum $Y \#(S^2 \times S^2)$.

In fact all of the above is true for a ($b_1 = 0$) symplectic manifold with $b^+ = 3 \mod 4$.