S.H. Wang

1. calibrated submanifolds with boundary.

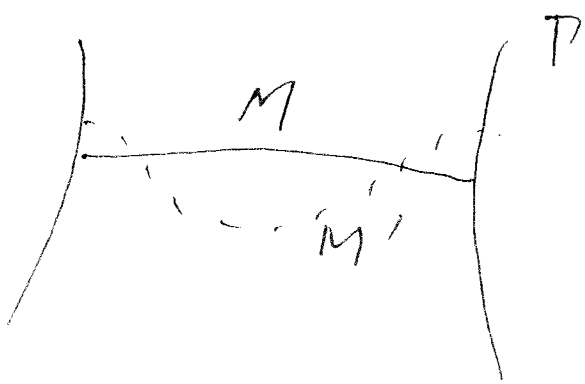
$\phi \in \Omega^p(X)$ a calibration on a Riemannian manifold X

A ϕ-submanifold $M \hookrightarrow X$ is an oriented submanifold s.t.

$$\phi|_M = dvol_M .$$

Another distinguished class of submanifold associated to ϕ is ϕ-null submanifold $P \hookrightarrow X$, that is

$$\phi|_P = 0 .$$



fact. Suppose $M \hookrightarrow X$ is a ϕ-submanifold with $\partial M \subset P$, where $\phi|_P = 0$. Then M is mass minimizing in $[M] \in H_p(X, P)$.

pf) $M - M' = \partial A$ mod P. □

Thus a ϕ-null submanifold serve as a natural constraint for ϕ-manifolds with boundary.

Since such pair $M \perp P$ along ∂M, this leads to.

$\boxed{\text{reflection principle}}$ Let $\tau : X \to X$ be an isometric involution of X s.t. $\tau^* \phi = -\phi$. Suppose the fixed point loci of τ, P, is a smooth submanifold, which is then necessarily ϕ-null. If $M \hookrightarrow X$ is a ϕ-submfd which is C^1 up to $\partial M \subset P$, then M can be continued across ∂M via τ.

ex).

X	ϕ	M	P	moduli
Kähler	ω	holo. curve.	Lag.	
Calabi-Yau	$\mathrm{Re}(\Upsilon)$	Slag	complex hypersurface	$\{\theta \in \Omega^1(M) \mid d\theta=0, \delta\theta=0, \nu \lrcorner \theta = 0\}$
"	"	"	anti-slag	$\{\theta \in \Omega^1(M) \mid d\theta=0, \delta\theta=0, \theta\|_{\partial M} = 0\}$
G_2	ϕ	associative	coassociative	
"	$*\phi$	coassociative	$*\phi$ isotropic 4 fold $\iota_* \phi = 0$	
$Spin(7)$	Φ	Cayley	isotropic 4 fold	

Rmk. One can further generalize the notion of calibration to general variational problem in the presence of differential system.

- $(X, \mathcal{I})$ $\quad \mathcal{I}$ a differential ideal $\subset \Omega^*(X)$.

Suppose $V_p(\mathcal{I}) = \{ E \in Gr(p, TX) \mid \mathcal{I}|_E = 0 \}$

$$\subset Gr^+(p, TX).$$

Let $\Lambda \in \Omega^p(X)$, and consider the variational problem

$$\int_N \Lambda .$$

A p-form $\phi \in \Omega^p(X)$ is a calibration for the variational problem if

$$\begin{cases} \phi \le \Lambda & \text{on } V_p(\mathcal{I}) \\ d\phi & \text{is at least quadratic in } \mathcal{I}. \end{cases}$$

An integral manifold N of $\mathcal{I}$ is calibrated by ϕ if

$$\phi - \Lambda = 0 \quad \text{on } N.$$

In that case, N is $\int \Lambda$ minimizing in its ~~category~~ homotopy classes of integral manifold of $\mathcal{I}$.

2. minimal surfaces of genus ≥ 2 in a unit sphere.

* S^3 Lawson, every genus, reflection principle

S^4 Bryant, every Riemann surface, $\mathbb{C}P^3 \longrightarrow S^4 = \mathbb{H}P^1$

→ S^5 special Legendrian.

S^6 Bryant, every Riemann surface, $Gr^+(2, \mathbb{R}^7) \longrightarrow S^6 \subset \mathrm{Im}\,\mathbb{O}$
(Branched immersion)

They all share the property that the associated differential system is involutive with last Cartan-Character $s_1 = 2$.

** The problem we are interested in is to construct special Legendrian surfaces, link of special Leg cones in $\mathbb{C}^3$, of high genus ≥ 2 in S^5.

$$\begin{array}{lccc} \text{special Leg.} & \Sigma & \subset & S^5 \\ & \pi \downarrow & & \\ \text{min. Lag} & M & \subset & \mathbb{C}P^2 \end{array}$$

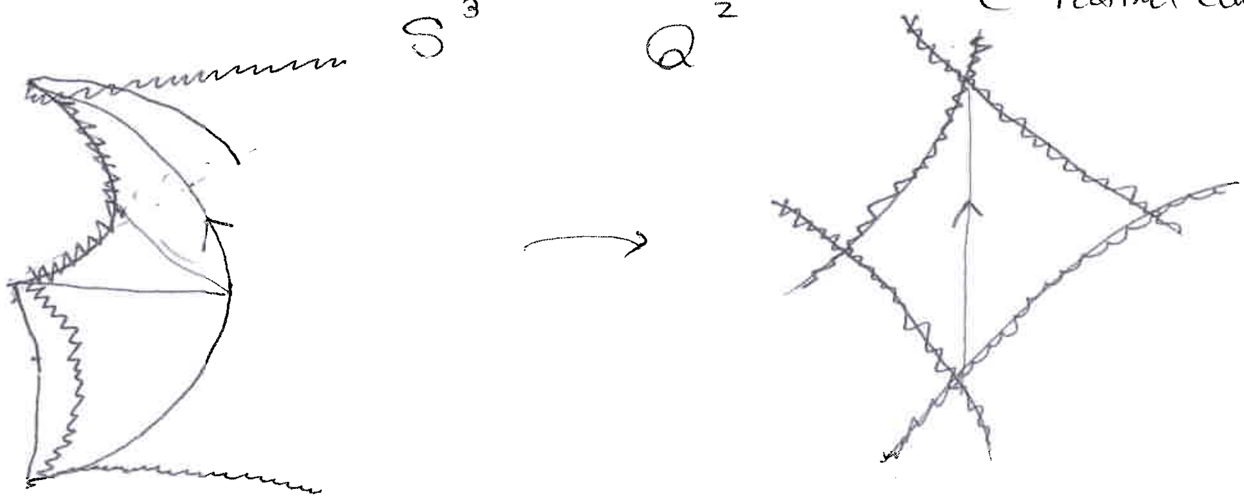
π is $1:1$ or $3:1$
or
$2:1$ or $6:1$ (M is nonorientable)

ex) $(e_0, v) \in S(S^3) = SO(4)/SO(2)$

$$\begin{array}{ccc} & (e_0, v) \in S(S^3) = SO(4)/SO(2) & \\ \swarrow & & \searrow \pi \\ e_0 \in S^3 & & (e_0 \wedge v) \in Gr^+(2, \mathbb{R}^4) = Q^2 \subset \mathbb{C}P^2 \end{array}$$

For ~~an~~ minimal surface $M \subset S^3$, the associated image in Q^2 is minimal Lagrangian.

S^3 $\qquad$ Q^2 $\qquad$ C reflected curve



reflection across 2-plane $\longrightarrow$ $Q^2 = \{[z_0, z_1, z_2, z_3] \mid \sum z_i^2 = 0\}$

$[z_0, z_1, z_2, z_3] \to [z_0, z_1, z_2, -z_3]$

3 minimal Lagrangian surfaces in $\mathbb{C}P^2$.

Here are some relevant observations and ideas.

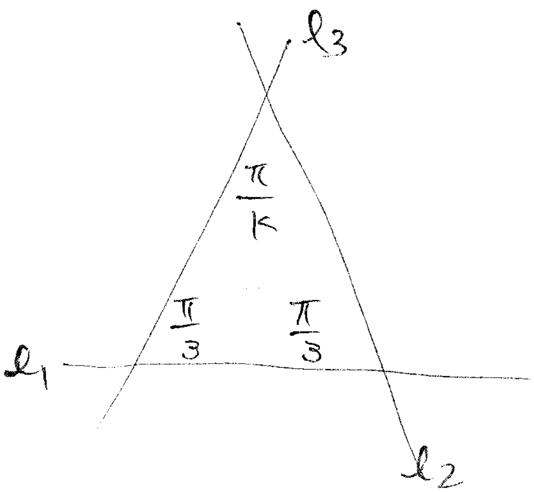
① geodesic polygon.

$$\ell_1 = \{ z_1 = \varepsilon_k z_2 \} \qquad \varepsilon_k = e^{i\frac{2\pi}{k}}.$$

$$\ell_2 = \{ z_1 = z_2 \}$$

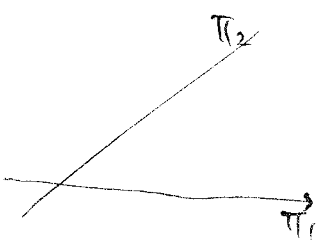
$$\ell_3 = \{ z_2 = z_3 \}$$

G_k generated by reflections across ℓ_1, ℓ_2, ℓ_3 is finite of order $6k^2$ (Coxeter).



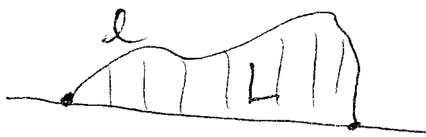
$$g = 1 + \frac{k(k-3)}{2}.$$

② angle. Π_1, Π_2 : complex lines in $\mathbb{C}^2$.



Given a real line $\ell_1 \subset \Pi_1$, $\exists$ unique $\ell_2 \subset \Pi_2$ s.t. $\ell_1 \wedge \ell_2$ is Lagrangian.

(3) relative isoperimetric inequality.

fact. 

π complex line $\subset \mathbb{C}^2$

$\exists$ Lagrangian Half disk L bounded by π and l

s.t. $\quad \text{Area}(L) \leq c \, \text{length}^2(l)$.

$+$

Schoen-Wolfson

$\parallel$

boundary regularity.

(4) the only Hamiltonian-stationary Lagrangian cone in $\mathbb{C}^2$ with reflection symmetry across a complex ~~plane~~ line is flat.

$\#$