Representations of surface groups and Hermitian symmetric spaces

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Based on joint work:

- Bradlow, Gothen, GP
  - arXiv: math.AG/9904144
  - arXiv: math.AG/0211428
  - arXiv: math.AG/0211431
- Mundet, GP
  - arXiv: math.GT/0304027
- Other work in progress

X closed oriented surface of genus $g \geq 2$

G reductive Lie group (real / complex)

Moduli space of representations

$$R(G) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(X), G)/G$$

reductive

real / complex analytic variety

Basic questions about $R(G)$

- non-emptiness
- number of connected components
- cohomology, homotopy
- geometry

Address first two questions for

G = non-compact real form of a complex reductive group

(answer is well-known for G = compact / complex reductive)

Explain general approach

- complex geometry
- gauge theory
- Morse theory
- Birational geometry of moduli spaces

Solution for particular cases

- $G = U(p,q)$ (Bradlow – G – Gothen)
- $G = Sp(2n, \mathbb{R})$
  - * $n=2$ (Gothen, G – Mundet)
  - in progress – * arbitrary n (Gothen – Mundet – G)

($n=1$, Goldman)

Previous work on these questions for non-compact real groups (non-complex):

- Goldman : $PSL(2,\mathbb{R})$, $SL(2,\mathbb{R})$
- Hitchin : $PSL(n,\mathbb{R})$, $SL(n,\mathbb{R})$
- Xia : $PGL(2,\mathbb{R})$, $U(p,1)$, $PU(p,1)$
- Gothen: $SU(2,2)$, $Sp(4,\mathbb{R})$ – partial answer
- Markman–Xia : $PU(p,p)$ partial answer

Review of cpx geometry & gauge methods for G = compact, complex reductive

- G compact

e.g. $G = U(n)$

relation to stable vector bundles

E - hol. vector bundle

$\downarrow$

X · hol. structure : Riemann surface

E stable:

$$\mu(E') < \mu(E) \quad \forall E' \subset E$$

$$\mu(E') = \frac{\deg E'}{\operatorname{rk} E'}$$

E polystable if $E = \oplus E_i$

with E_i stable and $\mu(E_i) = \mu(E)$ $\forall i$

$M(n,d)$ – Moduli space of polystable bundles of $rk = n$, $\deg = d$.

<u>Theorem</u> (Narasimhan & Seshadri)

$$R(U(n)) \cong M(n,0)$$

NS: (using a result of Atiyah): $M(n,0)$ is <u>connected</u>

<u>Atiyah – Bott – Donaldson approach</u>:

$E \downarrow X$ $U(n)$ - C^∞ vector bundle, $\deg E = 0$

A – unitary connection

$\mathcal{G}$ – unitary gauge group

$$R(U(n)) \longleftrightarrow \{A \mid F_A = 0\} / \mathcal{G} \longleftrightarrow \{\bar\partial_A \text{ stable}\} / \mathcal{G}^c$$

$\bar\partial_A = d_A^{0,1}$

Donaldson

- $M(n,d)$ also related to representations of:

$$PU(n) = U(n) / U(1)\,\mathbb{1}.$$ – adjoint form of $U(n)$.

- G compact arbitrary

$G^{\mathbb{C}}$ — complexification

$M(G^{\mathbb{C}})$ — moduli space of stable principal $G^{\mathbb{C}}$-bundles over X (topologically trivial).

Ramanathan

Theorem (Ramanathan)

$$R(G) \cong M(G^{\mathbb{C}})$$

• G^c complex reductive group

e.g. $G^c = GL(n, \mathbb{C})$

Higgs bundles

(E, Φ) $\qquad$ $\Phi : E \to E \otimes K$

canonical bundle

hol. vector bundle $\qquad$ Higgs field

(E, Φ) stable $\Leftrightarrow$ $\mu(E') < \mu(E)$

$\forall\ E' \subset E$ Φ-invariant

(E, Φ) polystable : $(E, \Phi) = \oplus (E_i, \Phi_i)$

stable, $\mu(E_i) = \mu(E)$

$\mathcal{M}$. moduli space of polystable Higgs bundles with $\deg E = 0$

Theorem (Hitchin, Simpson, Corlette, Donaldson)

There is a homeomorphism:

$$R(GL(n, \mathbb{C})) \cong \mathcal{M}$$

$$\mathcal{R}(GL(n,\mathbb{C})) \cong \left\{ \begin{array}{ll} D : GL(n,\mathbb{C})\text{-connection on } E & \underline{\text{reductive}} \\ F_D = D^2 = 0 & \text{flat} \end{array} \right\} \Big/ \mathcal{G}^c$$

Donaldson–Corlette $\updownarrow\ \cong$ $\quad \Big[D = d_A + \theta, \quad d_A$: $U(n)$-connection $\Big] \leftarrow$

$$\left\{ \begin{array}{ll} (d_A, \theta) \text{ s.t.} & \\ F_A + [\theta, \theta] = 0 & \\ d_A \theta = 0 & \Big\} \text{ flatness of } D \\ d_A^* \theta = 0 & \text{— harmonicity} \end{array} \right\} \Big/ \mathcal{G}$$

$\updownarrow\ \cong$ $\quad \Big[\theta = \Phi + \Phi^*, \quad \bar{\partial}_A = d_A^{0,1} \Big] \leftarrow$

$$\mathcal{M} \underset{\text{Simpson}}{\overset{\text{Hitchin}}{\cong}} \left\{ \begin{array}{l} (d_A, \Phi) \text{ s.t.} \\ F_A + [\Phi, \Phi^*] = 0 \\ \bar{\partial}_A \Phi = 0 \end{array} \right\} \Big/ \mathcal{G}$$

$\mathcal{R}(GL(n,\mathbb{C})) \updownarrow \cong \mathcal{M}$

More generally:

$G^{\mathbb{C}}$ Complex reductive group

Higgs bundle with struct. group $G^{\mathbb{C}}$

$$\begin{array}{ll} P & \text{Principal } G^{\mathbb{C}}\text{-holomorphic bundle} \\ \downarrow & \\ X & \end{array}$$

$$\mathrm{ad}P = P \times_{\mathrm{ad}} \mathfrak{g}^{\mathbb{C}}$$

$$\Phi \in H^0(X, \mathrm{ad}P \otimes K).$$

$\mathcal{M}$ – moduli of polystable pairs (P, Φ) (P: C^∞ trivial bundle)

$$\boxed{\mathcal{R}(G^{\mathbb{C}}) \cong \mathcal{M}}$$

G : non-compact real form of G^c (complex reductive)

$H \subset G$ maximal compact subgroup

G/H : symmetric space of non-compact type.

At the Lie algebra level:

$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ involution

$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ Cartan decomposition

+1 eigen space   −1 - eigen space

- $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$
- $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$
- $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$
- $ad(G)|_H : \mathfrak{m} \longrightarrow \mathfrak{m}$ isotropy rep. of H

G - Higgs bundle:

P — H^c - bundle

complexification of H

$$\begin{array}{c} P \\ \downarrow \\ X \end{array}$$

$$\varphi \in H^0(X, P \times_{ad} \mathfrak{m}^c \otimes K)$$

Higgs field in the isotropy rep.

(P, φ) — G - Higgs bundle

K - twisted principal pair in the isotropy rep.

Notion of stability à la <u>Ramanathan</u> studied by Banfield & Mundet, and in this generality by Bradlow - Mundet - G.

<u>Fixing the characteristic class</u> $\underline{c}$ of the H^c-bdle

$\mathcal{M}(c)$ — moduli space of polystable G - Higgs bdles

Back to $\mathcal{R}(G)$

$$\mathrm{Hom}(\pi_1(X), G) \xrightarrow{c} H^2(X, \pi_1(G))$$

$$\cong$$

$$\pi_1(G)$$

$$\rho \longrightarrow c = c(\rho)$$

Geometrically:

$$\rho \longleftrightarrow \text{Flat } G\text{-bundle}$$

reduce structure group to $H \subset G$

H-bundle

$c(\rho) =$ characteristic class of this H-bdle. $\in \pi_1(H)$.

Define:

$$\mathcal{R}(c) = \{\rho \in \mathcal{R}(G) \mid c(\rho) = c\}$$

Theorem There is a homeomorphism

$$\mathcal{R}(c) \cong \mathcal{M}(c)$$

Pf:

Corlette

$$\mathcal{R}(c) \cong \left\{ \begin{array}{l} \text{solutions to} \\ \text{harmonic eqns} \\ \text{for flat } G\text{-bundle} \end{array} \right\}$$

$$\wr\|$$

$$\mathcal{M}(c) \cong \left\{ \begin{array}{l} \text{solutions to} \\ \text{Hitchin eqns} \\ \text{for } (P, \varphi) \end{array} \right\}$$

Bradlow - Mundet - GP

In some cases it follows from

Hitchin, Simpson

$$\{G\text{-Higgs bundles}\} \to \{\text{Higgs } G^c\text{-bundles}\}$$

$$(P, \varphi) \longmapsto (\tilde{P}, \Phi)$$

- Extending structure group $H^c \subset G^c$
- Inclusion $m^c \subset g^c \Rightarrow P \times_{ad} m^c \subset \tilde{P} \times_{ad} g^c$

$$\varphi \longmapsto \Phi$$

Warning! • different equivalence
• different moduli) generally.

Morse theory techniques

Proper function (Hitchin)

$$f : \mathcal{M}(c) \longrightarrow \mathbb{R}$$

$$(A, \varphi) \longmapsto \int_X |\varphi|^2$$

Local minima:

$$\mathcal{N}(c) \subset \mathcal{M}(c)$$

f proper $\Rightarrow$ f has a minimum on each connected comp. of $\mathcal{M}(c)$

number of connected components of $\mathcal{N}(c)$ $\geqslant$ number of connected components of $\mathcal{M}(c)$

Have to:

- Identify $\mathcal{N}(c)$
- Study connectedness of $\mathcal{N}(c)$
- In particular

$\mathcal{N}(c)$ connected $\Rightarrow$ $\mathcal{M}(c)$ connected

Examples : $U(p,q)$, $Sp(2n,\mathbb{R})$

- $G = U(p,q)$ (Bradlow – Gothen – G)

Maximal compact subgroup

$$H = U(p) \times U(q)$$

$$\mathfrak{g} = \mathfrak{u}(p) \oplus \mathfrak{u}(q) + \mathfrak{m}$$

$$H^{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$$

$$\mathfrak{m}^{\mathbb{C}} = Hom(\mathbb{C}^q, \mathbb{C}^p) \oplus Hom(\mathbb{C}^p, \mathbb{C}^q)$$

$U(p,q)$ – Higgs bundle : (P, φ)

$P \rightarrow GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ - bundle

$$\varphi \in H^0(X, P \times_{ad} \mathfrak{m}^{\mathbb{C}} \otimes K)$$

Using standard representations of $GL(p,\mathbb{C}), GL(q,\mathbb{C})$ have holomorphic vector bundles:

$$\underbrace{V}_{\mathrm{rk}\ p}, \quad \underbrace{W}_{\mathrm{rk}\ q}$$

$$\varphi \in H^0(X, \underbrace{\mathrm{Hom}(W,V)\otimes K}_{\beta} \oplus \underbrace{\mathrm{Hom}(V,W)\otimes K}_{\gamma})$$

A $U(p,q)$-Higgs bundle is given by the data

$$(V, W, \beta, \gamma)$$

V - rank p holomorphic bundle

W - rank q " "

$\beta : W \to V\otimes K$

$\gamma : V \to W\otimes K$

$$(P, \varphi) \rightsquigarrow (\tilde{P}, \Phi)$$

G-Higgs bdl — Higgs bundle with structure gp $G^{\mathbb{C}}$.

$$(V, W, \beta, \gamma) \rightsquigarrow (E, \Phi)$$

$$E = V \oplus W, \qquad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

- Topological invariant

$$c = (a, b) \qquad \begin{aligned} a &= \deg V \\ b &= \deg W \end{aligned}$$

Toledo invariant: $\tau = 2 \dfrac{qa - pb}{p+q}$

$\mathcal{M}(a,b)$ is empty unless:

$$\boxed{|\tau| \leq \min\{p,q\}(2g-2)}$$

(Domic–Toledo)

(Milnor–Wood type inequality

Local minima: $\mathcal{N}(a,b) \subset \mathcal{M}(a,b)$

$$\mathcal{N}(a,b) = \left\{ (E,\Phi) \mid \beta = 0 \text{ or } \gamma = 0 \right\}$$

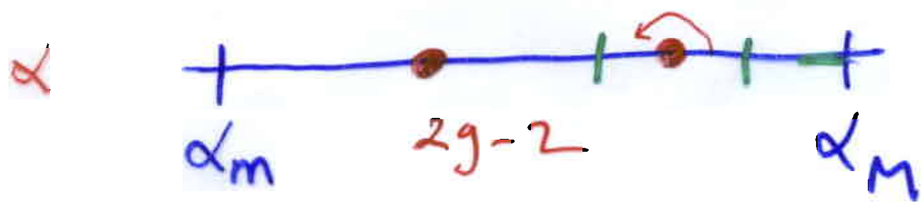
- $\overset{1-1}{\longleftrightarrow} \left\{ W \xrightarrow{\beta} V \otimes K \text{ if } \underline{\tau \leq 0} \right\}$ triples
- $\overset{1-1}{\longleftrightarrow} \left\{ V \xrightarrow{\gamma} W \otimes K \text{ if } \underline{\tau \geq 0} \right\}$

Have to study moduli space of triples

$\mathcal{N}_\alpha(a,b)$ (Bradlow – G)

$\alpha \in \mathbb{R}$

- $\mathcal{N}(a,b) = \mathcal{N}_{2g-2}(a,b)$



- $\mathcal{N}_\alpha(a,b)$ <u>irreducible</u> if $\alpha \geq 2g-2$!

→ in fact birational to a projective bundle over a product of moduli spaces of v. bdls

- $G = Sp(2n, \mathbb{R})$

Real symplectic group: Linear Transformations preserving the standard symplectic vector space $(\mathbb{R}^{2n}, \omega)$.

Maximal compact subgroup

$$H = U(n) \subset Sp(2n, \mathbb{R})$$

$$H^c = GL(n, \mathbb{C})$$

$$\mathfrak{h} = \mathfrak{u}(n)$$

$$\mathfrak{h}^c = \mathfrak{gl}(n, \mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$$

$$\mathfrak{m}^c = S^2(\mathbb{C}^n) \oplus S^2(\mathbb{C}^{n*})$$

$$\mathfrak{sp}(2n, \mathbb{C}) = \mathrm{End}(\mathbb{C}^n) \oplus S^2(\mathbb{C}^n) \oplus S^2(\mathbb{C}^{n*})$$

$Sp(2n,\mathbb{R})$ - Higgs bundle: (P,φ)

P - $GL(n,\mathbb{C})$ - bundle

$$\varphi \in H^0(X, P\times_{ad} \mathfrak{m}^{\mathbb{C}} \otimes K)$$

If V is the vector bdle associated to the standard repn of $GL(n,\mathbb{C})$:

$$\varphi \in H^0(X, \underbrace{S^2V\otimes K}_{\beta} \oplus \underbrace{S^2V^*\otimes K}_{\gamma})$$

- $Sp(2n,\mathbb{R})$ - Higgs bundle

(V,β,γ)

$$\beta : V^* \to V\otimes K$$
$$\gamma : V \to V^*\otimes K$$

Symmetric : quadratic bundles.

$$\left(E = V\oplus V^*,\ \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$$

corresponding $Sp(2n,\mathbb{C})$ Higgs bdle.

$$\mathcal{M}(d) \cong R(d) \subset R(Sp(2n,\mathbb{R}))$$

$\rho : \pi \longrightarrow Sp(2n,\mathbb{R})$

of degree d = Chern class

of a reduction to $U(n)$

Milnor-Wood inequality

(Turaev, Domic-Toledo)

$\mathcal{M}(d) \cong R(d)$ empty unless

$$|d| \leq n(g-1)$$

Minima

(Gothen - Mundet - GP
$n=2 \to$ Gothen $-Sp(4,\mathbb{R})$)

- $|d| < n(g-1)$

Minima: $\beta = 0$ or $\gamma = 0$ (depends on sign of d)

if $d=0 \Rightarrow \beta=\gamma=0$

- $|d| = n(g-1)$

Minima $\begin{cases} \beta = 0 \quad \text{or} \quad \gamma = 0 \\ + \\ \text{other} \end{cases}$ $\longleftarrow$ many components

- general n and $d = \pm n(g-1)$

$E = V \oplus V^*$ $\quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

$\beta : V^* \to V \otimes K$
$\gamma : V \to V^* \otimes K$ $\Big\}$ symmetric

Fact: (E, Φ) polystable $\Rightarrow$

- $\gamma : V \to V^* \otimes K$ isomorphism $\quad d = n(g-1)$
- $\beta : V^* \to V \otimes K$ isomorphism $\quad d = -n(g-1)$

Say $d = n(g-1)$

$L_0 = K^{1/2}$ $\quad$ Theta characteristic / spin structure

$W := V \otimes L_0^{-1}$

$Q := \gamma \otimes 1_{L_0^{-1}} \in H^0(X, S^2 W^*)$ $\quad$ quadratic form

$\Psi = \beta \otimes 1_{L_0} \circ \gamma \otimes 1_{L_0^{-1}} \in H^0(X, \mathrm{End}(W) \otimes K^2)$.

symmetric with respect to Q

$V \otimes L_0^{-1} \xrightarrow{\gamma \otimes 1_{L_0^{-1}}} V^* \otimes K \otimes L_0^{-1} \xrightarrow{\beta \otimes 1_{L_0}} V \otimes K \otimes L_0$

Fact:

$$\mathcal{M}(n(g-1)) \simeq \left\{ \begin{array}{c} \text{moduli of polystable} \\ (W, Q, \varphi) \end{array} \right\}$$

$Q \rightsquigarrow O(n, \mathbb{C})$ structure group

$O(n, \mathbb{R}) \subset O(n, \mathbb{C}) \rightsquigarrow$ Stiefel-Whitney classes w_1, w_2

$\Rightarrow$ $n = 2$ (Gothen)

$Q \rightsquigarrow (\Lambda^2 W)^2 \cong \mathcal{O}$

$$\Rightarrow \Lambda^2 W \in H^1(X, \mathbb{Z}/2)$$

$$= w_1(W)$$

$\Rightarrow \Lambda^2 W \simeq \mathcal{O} \iff w_1(W) = 0$

$\Rightarrow$ reduction $SO(2, \mathbb{C}) \subset O(2, \mathbb{C})$

- $\mathbb{C}^* \simeq SO(2, \mathbb{C}) \qquad \lambda \longleftrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

$W = L \oplus L^{-1} \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Now $w_2(W) = c_1(L) \bmod 2$

Can assume : $\deg L \geq 0$

Must have $\deg L \leq 2g-2$ (stability)

$$\mathcal{M}(2g-2) = \Big(\bigcup_{u,v} \mathcal{M}_{u,v}\Big) \cup \Big(\bigcup_{l=0}^{2g-2} \mathcal{M}_{0,l}\Big)$$

where :

$$\mathcal{M}_{u,v} = \left\{ (W, Q, \varphi) \text{ polystable} ;\ \begin{array}{l} u \in H^1(X; \mathbb{Z}/_2), \\ \ \ \ne 0 \\ v \in H^2(X; \mathbb{Z}/_2) \end{array} \right\}$$

$$\mathcal{M}_{0,l} = \{ (W, Q, \varphi) \text{ polystable} ;\ w_1(W) = 0,\ \deg L = l \}$$

Gothen:

$\mathcal{M}_{u,v}$ — connected

$\mathcal{M}_{0,l}$ — connected $0 \leq l < 2g-2$

$\mathcal{M}_{0,2g-2}$ — 2^{2g} connected components.

↳ choice of square roots of K^2
$(L^2 \simeq K^2)$.

Hitchin–Teichmüller components !

- $|d| < n(2g-2)$ $(d \leq 0)$

$$\mathcal{N}(d) = \mathcal{N}_0(d)$$

$$\mathcal{N}_\alpha(d) = \{\alpha\text{-polystable } (V,\beta),\ \beta \in H^0(S^2 V \otimes K)\}$$

Theorem (Muñoz-GP) $n=2$

$\mathcal{N}_\alpha(d)$ is connected for $\alpha \geq 0$, and non-empty.

Theorem $\mathcal{M}(Sp(4,\mathbb{R}))$

- $\mathcal{M}(d) \neq \phi$ connected if $|d| < 2g-2$
- $\mathcal{M}(\pm(2g-2))$ has $3 \cdot 2^{2g} + 2g-4$ non-empty connected components

- Gothen: $d=0$, $d = \pm(2g-2)$
- Muñoz-GP: $|d| < 2g-2$

G	G^c	Higgs bundles (V, Φ)
$SL(n, \mathbb{R})$	$SL(n, \mathbb{C})$	V – $SO(n, \mathbb{C})$ bdle $\Phi \in H^0(S^2 V \otimes K)$
$SO(n, \mathbb{H})$	$SO(2n, \mathbb{C})$	$V \oplus V^*$, $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ $\beta \in H^0(X, \Lambda^2 V^* \otimes K)$ $\gamma \in H^0(X, \Lambda^2 V^* \otimes K)$
$SO_0(p, q)$	$SO(p+q, \mathbb{C})$	$V \oplus W$, $\Phi = \begin{pmatrix} 0 & \beta \\ -\beta^t & 0 \end{pmatrix}$ V – $SO(p, \mathbb{C})$ bdle W – $SO(q, \mathbb{C})$ bdle $\beta \in \mathrm{Hom}(W, V \otimes K)$

• $G = G_2$ — real (split) form

$\cup$

$H = SU(2) \times SU(2)$

$H^c = sl(V) \oplus sl(W)$

$m^c = V \otimes S^3 W$

G_2 - Higgs bdle:

(V, W, φ),

$\varphi \in H^0(X, V^* \otimes S^3 W \otimes K)$

$\varphi : V \otimes W \to S^2(W) \otimes K$

$\rightsquigarrow$

$\left(E = V \otimes W \oplus S^2 W, \ \Phi = \begin{pmatrix} & \varphi \\ -\varphi^t & \end{pmatrix} \right)$

G_2^c - Higgs bundle.

Table 3. Irreducible Symmetric Spaces of Type III

Helgason's type	G	H	$\dim G/H$	rank	Isotropy representation[1])	Kähler or not	Geometric interpretation
A I	$SL(n, \mathbb{R})$	$SO(n)$	$\frac{(n-1)(n+2)}{2}$	$n-1$	$\wedge^p SO(n) \underline{\otimes} \wedge^p SO(n)$ $p = 2$ if n odd $p = 3$ if n even	No	Set of Euclidean structures on $\mathbb{R}^n$ or set of the $\mathbb{R}P^n_{hyp}$'s in $\mathbb{C}P^n_{hyp}$
A II	$SU^*(2n) = SL(n, \mathbb{H})$	$Sp(n)$	$(n-1)(2n+1)$	$n-1$	$\underline{\wedge}^2 Sp(n)$	No	Set of the $\mathbb{H}P^{n-1}_{hyp}$'s in $\mathbb{C}P^{2n-1}_{hyp}$
A III	$SU(p, q)$	$S(U(p) \times U(q))$ $p \leqslant q$	$2pq$	$\min(p, q)$	$S(U(p) \otimes U(q))$	Yes	Grassman manifold of positive definite $\mathbb{C}^p$'s in $\mathbb{C}^{p,q}$, or set of the $\mathbb{C}P^{p-1}_{hyp}$'s in $\mathbb{C}P^{p+q+1}_{hyp}$ (in particular, complex hyperbolic space $\mathbb{C}P^q_{hyp}$ if $p = 1$)
BD I	$SO_0(p, q)$	$SO(p) \times SO(q)$ $p \leqslant q$	pq	$\min(p, q)$	$SO(p) \otimes SO(q)$	Yes if and only if $p = 2$	Grassman manifold of positive definite $\mathbb{R}^p$'s in $\mathbb{R}^{p,q}$, or set of the $\mathbb{R}P^{p-1}_{hyp}$'s in $\mathbb{R}P^{p+q+1}_{hyp}$ (in particular, real hyperbolic space $\mathbb{R}P^q_{hyp}$—denoted by H^q in 1.37—if $p = 1$)
D III	$SO^*(2n) = SO(n, \mathbb{H})$	$U(n)$	$n(n-1)$	$[n/2]$	$\wedge^2 U(n)$	Yes	Set of quaternionic quadratic forms on $\mathbb{R}^{2n}$, or set of the $\mathbb{C}P^{n-1}_{hyp}$'s in $\mathbb{R}P^{2n-1}_{hyp}$
C I	$Sp(n, \mathbb{R})$	$U(n)$	$n(n+1)$	n	$U(n) \underline{\otimes} U(n)$	Yes	Set of Lagrangian subspaces of $\mathbb{R}^{2n}$ or set of the $\mathbb{C}P^n_{hyp}$'s in $\mathbb{H}P^n_{hyp}$
C II	$Sp(p, q)$	$Sp(p) \times Sp(q)$ $p \leqslant q$	$4pq$	$\min(p, q)$	$Sp(p) \otimes Sp(q)$	No	Grassman manifold of positive definite $\mathbb{H}^p$'s in $\mathbb{H}^{p,q}$, or set of the $\mathbb{H}P^{p-1}_{hyp}$'s in $\mathbb{H}P^{p+q+1}_{hyp}$ (in particular, quaternionic hyperbolic space $\mathbb{H}P^q_{hyp}$ if $p = 1$)

Table 3 (continued)

Helgason's type	G	H	dim G/H	rank	Isotropy representation[1]	Kähler or not	Geometric interpretation
E I	E_6^6	$Sp(4)$	42	6	$\underline{\Lambda}^4 Sp(4)$	No	**Anti-chains of $(\mathbb{C} \otimes \mathbb{C}a)P^2_{hyp}$**
E II	E_6^2	$SU(6) \times SU(2)$	40	4	$\Lambda^3 SU(6) \otimes SU(2)$	No	**Set of the $(\mathbb{C} \otimes \mathbb{H})P^2_{hyp}$'s in $(\mathbb{C} \otimes \mathbb{C}a)P^2_{hyp}$**
E III	E_6^{-14}	$SO(10) \times SO(2)$	32	2	$\mathrm{Spin}(10) \cdot SO(2)$	Yes	**Rosenfeld's hyperbolic projective plane $(\mathbb{C} \otimes \mathbb{C}a)P^2_{hyp}$**
E IV	E_6^{-26}	F_4	26	2	F_4	No	**Set of the $\mathbb{C}aP^2_{hyp}$'s in $(\mathbb{C} \otimes \mathbb{C}a)P^2_{hyp}$**
E V	E_7^7	$SU(8)$	70	7	$\Lambda^4 SU(8)$	No	**Anti-chains of $(\mathbb{H} \otimes \mathbb{C}a)P^2_{hyp}$**
E VI	E_7^{-5}	$SO(12) \times SU(2)$	64	4	$\mathrm{Spin}(12) \otimes SU(2)$	No	**Rosenfeld's hyperbolic projective plane $(\mathbb{H} \otimes \mathbb{C}a)P^2_{hyp}$**
E VII	E_7^{-25}	$E_6 \times SO(2)$	54	3	$E_6 \otimes SO(2)$	Yes	**Set of the $(\mathbb{C} \otimes \mathbb{C}a)P^2_{hyp}$'s in $(\mathbb{H} \otimes \mathbb{C}a)P^2_{hyp}$**
E VIII	E_8^8	$SO(16)$	128	8	$\mathrm{Spin}(16)$	No	**Rosenfeld's hyperbolic projective plane "$(\mathbb{C}a \otimes \mathbb{C}a)P^2_{hyp}$" [2]**
E IX	E_8^{-24}	$E_7 \times SU(2)$	112	4	$\Lambda^2 E_7 \otimes SU(2)$	No	**Set of the $(\mathbb{H} \otimes \mathbb{C}a)P^2_{hyp}$'s in "$(\mathbb{C}a \otimes \mathbb{C}a)P^2_{hyp}$" [2]**
F I	F_4^4	$Sp(3) \times SU(2)$	28	4	$\underline{\Lambda}^3 Sp(3) \otimes SU(2)$	No	**Set of the $\mathbb{H}P^2_{hyp}$'s in $\mathbb{C}aP^2_{hyp}$**
F II	F_4^{-20}	$SO(9)$	16	1	$\mathrm{Spin}(9)$	No	**Hyperbolic Cayley projective plane $\mathbb{C}aP^2_{hyp}$**
G I	G_2^2	$SU(2) \times SU(2)$	8	2	$\underline{\otimes}^3 SU(2) \otimes SU(2)$	No	Set of the non-division quaternionic sub-algebras of the non-division Cayley algebra

[1]) here Λ (resp. $\otimes$) denotes the exterior (resp. tensor) product representation and $\underline{\Lambda}$ (resp. $\underline{\otimes}$) denotes the natural irreducible representation deduced from it

[2]) up to this day an algebraic definition of this hyperbolic plane over $\mathbb{C}a \otimes \mathbb{C}a$ seems pending, see [Fre] and [Ros]