Second Order Families of Special Lagrangian 4-folds in $\mathbb{C}^4$

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SL = Special Lagrangian

Problem: Classify the families of SL submflds in $\mathbb{C}^4$ whose fundamental cubic has nontrivial stabilizer in $SO(4)$ at a generic point.

Extends the results of R. Bryant who completely solve the problem in dimension 3.

Motivation: SL not well understood in $\mathbb{C}^n$, $n \geq 3$

Examples of SL in $\mathbb{C}^n$ invariant under certain group action in $\mathbb{C}^n$ were written down by Harvey & Lawson, M. Haskins, Joyce, S. Marshall, etc.

Idea: $\rightarrow$ classify families of SL submanifolds characterized by invariant geometric conditions.

In $\mathbb{C}^n$, the second fundam. form is the lowest order invariant of a SL submfld.

The points on SL n-folds where the $SO(n)$-stabilizer is nontrivial are the analogs of of the umbilical points in the classical theory of surfaces.

Let $(M, \omega_0, g_0, \Omega_0)$ Calabi Yau, $M = \mathbb{C}^n$ here.

$g_0 = |dz_1|^2 + \dots + |dz_n|^2$ $\qquad$ $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \dots + dz_n \wedge d\bar{z}_n)$

$\Omega_0 = dz_1 \wedge \dots \wedge dz_n$ $\qquad$ $\rightarrow$ standard Calabi-Yau str. on $\mathbb{C}^n$

Defn: A real oriented submfld $L \subseteq \mathbb{C}^n$ of dim n is called special Lagrangian if it is calibrated by the real n-form $Re(\Omega_0)$. ($\Longleftrightarrow \omega_0|_L = 0$ and $Im(\Omega_0)|_L = 0$).

<u>Results of R. Bryant</u> ($n = 3$):

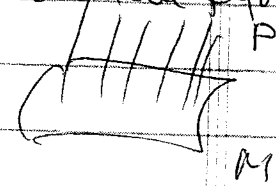
- classified SL 3-folds in $\mathbb{C}^3$ whose fd. cubic C has nontrivial $SO(3)$-stab. at a generic pt.

1) when stab. is $SO(3)$: 3-planes ($C \equiv 0$)
2). when stab. is $SO(2)$: Harvey-Lawson examples invar. under the action of $SO(3)$
3) when stab. is S_3: L is locally a product $\mathbb{R} \times \Sigma$, where $\Sigma \subseteq \mathbb{C}^2$ is a complex curve or a twisted cone.
4). when stab. is $\mathbb{Z}_3$: asymptotically conical SL 3-folds
5). when stab. is $\mathbb{Z}_2$: Lawlor-Harvey-Joyce examples (SL extensions of 2-dim quadratic surfaces $E \subset P$, where P = Lagr. 3-plane in $\mathbb{C}^3$)
6). when stab is A_4: no Special Lagr.

<u>Structure equations of a SL submfld and the fundam. cubic</u>:

Let $M = \mathbb{C}^n$ with the standard Calabi-Yau str. g_0, ω_0, Ω_0.

Let $\pi: P \to M$ be the bundle of $\mathbb{C}^n$-valued Calabi-Yau coframes.
↳ pp right $SU(n)$-bdl over M.



ξ = canonical form on P. ($\mathbb{C}^n$-valued)

$\xi = (\xi_i)_{i=\overline{1,n}}$

$\boxed{d\xi_i = -\psi_{ij} \wedge \xi_j}$ (1) Cartan's first str. eq.

(ψ_{ij}) takes values in $su(n)$
↑ connection form on P.

$\boxed{d\psi = -\psi \wedge \psi}$ (2) Cartan's second str. eq.

Adapt frames: $\xi_i = \omega_i + \sqrt{-1}\,\eta_i$

$\psi_{ij} = \alpha_{ij} + \sqrt{-1}\,\beta_{ij}$

(1) $\Leftrightarrow \begin{cases} d\omega_i = -\alpha_{ij} \wedge \omega_j + \beta_{ij} \wedge \eta_j \\ d\eta_i = -\beta_{ij} \wedge \omega_j - \alpha_{ij} \wedge \eta_j \end{cases}$ where $\alpha_{ij} = -\alpha_{ji}$, $\beta_{ij} = \beta_{ji}$ and $\sum_{i=1}^{n} \beta_{ii} = 0$

(2) $\Leftrightarrow \begin{cases} d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} \text{ (Gauss eq)} \\ d\beta_{ij} = -\beta_{ik} \wedge \beta_{kj} - \alpha_{ik} \wedge \beta_{kj} \text{ (Codazzi eq).} \end{cases}$

P_L = sbdl of L-adapted coframes, pp. right $SO(n)$-bdl over L.
$x \in L$, $u \in P_x$ is L-adapted if $u(T_xL) = \mathbb{R}^n \subseteq \mathbb{C}^n$ and u preserves orientation

On P_L: $\eta_i = 0 \Rightarrow d\omega_i = -\alpha_{ij} \wedge \omega_j$

$\beta_{ij} \wedge \omega_j = 0 \Rightarrow \beta_{ij} = h_{ijk}\, \omega_k$, (h_{ijk}) fully symmetric

$C = h_{ijk}\, \omega_i \omega_j \omega_k$ is called the fundam. cubic of the SL subfld. $L \subseteq \mathbb{C}^4$. C is traceless

$\Rightarrow C \in \mathcal{H}^3(\mathbb{R}^n) \leftarrow$ space of traceless cubics in n vars.

~~Hse~~ Bonnet type theorem.

$\mathcal{H}^3(\mathbb{R}^4)$ is an irred $SO(4)$-module of dim 16.

Find all traceless cubics in 4 variables (x_1, x_2, x_3, x_4) that have a nontrivial stabilizer G under the action of $SO(4)$. G can be a continuous sbgr of $SO(4)$ (positive dimension) or G = discrete sbgr.

The max. torus in $SO(4)$ is conjugate to group:

$$H = \left\{ \begin{bmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{bmatrix} \right\}$$

Let $z_1 = x_1 + i x_2$, $z_2 = x_3 + i x_4$ and consider:

$\mathcal{H}^3_{\mathbb{C}} = \mathcal{H}^3(z_1, z_2, \bar{z}_1, \bar{z}_2)$ space of complexified traceless cubics in vars. $(z_1, z_2, \bar{z}_1, \bar{z}_2)$.

$\mathcal{H}^3_{\mathbb{C}}$ decomposes under the action of the max. torus into 8 pairs of opposite weight spaces, each of multipl. 1

$C \in \mathcal{H}^3(\mathbb{R}^4)$ is the sum of elem. drawn from these weight spaces, with the coeffs in opposite weight spaces being complex conjugate.

$\exists$ $C \in \mathcal{H}^3(\mathbb{R}^4)$ fixed $\Leftrightarrow$ $\exists$ nontrivial $g = \begin{bmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{bmatrix} \in H$ that acts trivially on at least one pair of these $V_{(r,s)}$

$\Leftrightarrow$ $3r, r, 2r+s, 2r-s, 2s+r, 2s-r, 3s$ or $s \in \mathbb{Z}$. $(*)$

$$\mathcal{H}^3_{\mathbb{C}} = \bigoplus_{\substack{m+n=3 \\ m,n \in \{0, \pm 1, \pm 2, \pm 3\}}} \left(V_{(mr, ns)} \oplus V_{(-mr, -ns)}\right)$$

Obs: If only (exactly) one of conds $(*)$ is satisfied $\to$ continuous stabilizer.

To get discrete symmetry, look at elements that are at the intersection of at least 2 pairs of non opposite weight spaces.

Up to conjugacy in $O(4)$, there are 6 nontrivial elements in H that acts trivially on more than 2 pairs of $V_{(r,s)}$.

$\to$ elements of order $6, 5, 4, 3, 3, 2$.

Get incomplete classification. To ~~prove~~ prove existence of SL use Cartan-Kähler theory, to describe the families of

SL, integrate the structure equations.

Let $L \subseteq \mathbb{C}^4$ be a connected SL 4-fold in $\mathbb{C}^4$ with fd. cubic C and suppose C has nontrivial stab G in $SO(4)$.

I. If $G \subseteq SO(4)$ is continuous sbgr (i.e. positive dim):

1). $G = SO(4)$: $C \equiv 0 \rightarrow L$ is a real 4-plane

2). $G = SO(3)$: $C = r x_1 (x_1^2 - x_2^2 - x_3^2 - x_4^2)$, $r > 0$

$\rightarrow$ L is an open set of the Harvey-Lawson examples L_c, invariant under SO(4).

$L_c = \{(s+it)\, u \mid u \in S^3 \subseteq \mathbb{R}^4,\ \mathrm{Im}(s+it)^4 = c\}$, $c = \text{cst}$.

3). $G = O(2)$, where $O(2) = S^1 \cup g S^1$, $S^1 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$,

$g = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$, $C = r\{(x_1^2 - x_2^2) x_3 + 2 x_1 x_2 x_4\}$ $\rightarrow$ no SL

4). $G = O(2)$, where $O(2) = S^1 \cup g S^1$, $S^1 = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}$,

g as above $\rightarrow C = r(x_1^3 - 3x_1 x_2^2) + 3\nu x_1 (x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2)$, $\nu > 0$,

$r \neq 3\nu \rightarrow$ 2-parameter family of SL (not able to integrate completely)

5). $G = SO(2) \ltimes S_3$: $C = r(x_1^3 - 3x_1 x_2^2)$, $r > 0$

$\rightarrow$ products $\Sigma \times \mathbb{R}^2$, $\Sigma \subseteq \mathbb{C}^2$ is holo curve w.r.t. an alternative complex str. on $\mathbb{C}^2$.

6). $G = SO(2)$: $C = r(x_1^3 - 3x_1 x_2^2) + s(3x_1^2 x_2 - x_2^3) + 3\nu x_1 (x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2)$, $s, \nu > 0$

L is an open set of a SO(3)-invar SL 4-fold in $\mathbb{C}^4$.

$\rightarrow$ orbits of SO(3)-action are S^2 $L|_{S^2}$ — pseudo holo curve

II. If $G \subseteq SO(4)$ is discrete sbgr:

Prop: A discrete sbgr. of $SO(4)$ that stabilizes a traceless cubic in 4 vars can not have elem. of order > 6.

Classify the SL 4-folds with polyhedral symm on its fd. cubic:

1). $G = T$ → $C = rx_1(x_1^2 - x_2^2 - x_3^2 - x_4^2) + sx_2x_3x_4$, $r, s \geq 0$, $s \neq 2\sqrt{5}r$
tetrah. gr.
→ ~~Harvey-Lawson~~ ex. invar. under T^3 of the form:
$$\begin{cases} |z_1| = |z_2| = |z_3| = |z_4| \\ \mathrm{Re}(z_1 z_2 z_3 z_4) = \sqrt[5]{2} \end{cases}$$

2). $G = O^+$ ← irred. acting octahedral sbgr. $C = \dots$
→ Harvey-Lawson cones on flat 3-dim tori in S^7.

3). $G = I$ ← irred. ly acting icosahedral sbgr. $C = \dots$
→ no SL

Cyclic and dihedral symm:

Prop: If the stab. contains and elem of order 6, 5 or 4 and G discrete and nonpolyhedral, then no SL whose fd. cubic has stab. G.

If G has an elem of order 3 → 2 inequivalent orbits:

1). $g = \begin{pmatrix} e^{4\pi/3} & 0 \\ 0 & I_2 \end{pmatrix}$ fixes C.

a) ↦ $G = D_3$ ⇝ infinite parameter of SL 4-folds depending on 2 fcns of one variables. $C = \dots$

) ↦ $G = D_3$, different C: L is an open sbset of the asymptotically conical SL 4-fold:
$$L_\Sigma = \{(a+ib)u \mid u \in \Sigma,\ \mathrm{Re}(a+ib)^4 c\},$$ where $\Sigma \subset S^7$ is sp. Legendrian with phase i.

c) G = order 18 normal sbgr of $D_3 \times D_3$

$L = \Sigma_1 \times \Sigma_2$, Σ_i - holo curves in $\mathbb{C}^2$ w.r.t. an altern. complex str. on $\mathbb{C}^2$.
$\quad\;\; \cap \quad\;\; \cap$
$\quad\;\; \mathbb{C}^2 \quad \mathbb{C}^2$

d) $G = \mathbb{Z}_3$ → infinite param. family of SL 4-folds in $\mathbb{C}^4$, solution depends on 4 fcns of 1 var. → foliated by congr. holo curves in one direction and non-congr. minimal Legendrian surfaces in another → integrable system.

e) 2). $g = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}$ ← second orbit

a). $G = \mathbb{Z}_3$: L → I-special Lagrangian J-holo surface in $\mathbb{C}^4$
(L is given by 2 holom. eqs)

b). $G = D_3$: L is ruled I-special Lagr. J-holo surface in $\mathbb{C}^4$

→ large family of SL in this case.
$\{I, J, K\}$ - hyperkähler str. on $\mathbb{C}^4$

If G has at least sym. $\mathbb{Z}_2$ → C has 6 parameters.
(small symmetry) (large number of param).
Complicated analysis, not done since space of fixed traceless cubics involve a large number of parameters.