002

Instantons

&

Lagrangians

in

Manifolds w/ V.C.P.

[Vector Cross Product].

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Joint work with Jae Hyouk Lee.

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- Vector Cross Product (V.C.P.)
  - Kähler / Symplectic Manifold

    $\partial\left(\begin{array}{c}\text{Holomorphic}\\ \text{Curve}\end{array}\right) \subset$ Lagrangian Submanifold.
  - G_2 - manifold

    $\partial\left(\begin{array}{c}\text{Associative}\\ \text{submanifold}\end{array}\right) \subset$ Coassociative submanifold.

- $\mathbb{C}$ - V. C. P.
  - Calabi - Yau Manifold.

    $\partial(\mathrm{SLag}_\theta) \subset$ Complex Hypersurface and $\mathrm{SLag}_{\theta+\pi/2}$.
  - Hyperkähler Manifold.

    $\partial\left(\begin{array}{c}\text{Holomorphic}\\ \text{Curve}\end{array}\right) \subset$ Complex Lagrangian submanifold.

§ Symplectic Geometry

[Review].

(M^{2n}, ω) $\qquad \omega \in \Omega^2(M)$.

$$\begin{cases} d\omega = 0 \\ \omega > 0. \end{cases}$$

Example. $\quad M = \mathbb{C}^n$

$$\omega = dx^1 \wedge dy^1 + \ldots\ldots + dx^n \wedge dy^n.$$

metric

$$g = (dx^1)^2 + (dy^1)^2 + \ldots\ldots + (dx^n)^2 + (dy^n)^2$$

complex str.

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j} \qquad \forall j.$$

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complex structure.
(1-fold V.C.P.)

$\omega \in \Omega^2(M)$ $\qquad$ $J : T_M \to T_M, \; J^2 = -1.$

via metric

$\omega(u, v) = g(Ju, v).$

(M, ω, J, g) Almost Kähler mfd.

$$(M^{2n}, g, \omega, J)$$

Definition: $A^2 \subset M$

Holomorphic Curve / Instanton

if A preserved by J.

[$\xLeftrightarrow[\text{Wirtinger}]{}$ A calibrated by ω, i.e. $\omega|_A = \nu_A$]

[$\Longrightarrow$ Absolute minimum area.]

$$\mathrm{Area}(A) = \int_A [\omega].$$

(topological)

Remark: Boundary value problem

$$\partial A \subset C \subset M$$

$$\omega|_C = 0.$$

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$$(M^{2n}, g, \omega, J)$$

Definition: $C^n \subset M^{2n}$

<u>Lagrangian</u> submanifold if

$$\omega|_C = 0$$

$$\dim C = \frac{1}{2} \dim M.$$

Remark:

$\#\{$ instantons: $A^2 \subset M \}$

(i) $\partial A = \phi$ $\rightsquigarrow$ Gromov-Witten Invariants

(ii) $\partial A \subset C$ $\rightsquigarrow$ Fukaya-Floer Category.

§. Vector Cross Product.

Example : $(\mathbb{R}^3, \times)$.

Example : (M^{2n}, J).

Definition : (M^m, g) Riemannian manifold.
[Gray].

$$\chi : \Lambda^r T_M \longrightarrow T_M. \qquad \text{r-fold. VCP}$$

if. (i) $\chi(v_1, \ldots\ldots, v_r) \perp v_1$

(ii) $v_1, \ldots\ldots, v_r$: orthonormal

$\Rightarrow |\chi(v_1, \ldots\ldots, v_r)| = 1$

(i)⇔(i)' $\varphi(v_1, \ldots\ldots, v_r, v_{r+1}) \triangleq \langle \chi(v_1, \ldots, v_r), v_{r+1} \rangle$.

then $\varphi \in \Omega^{r+1}(M)$.

(iii) $d\varphi = 0$.

[$\widetilde{\text{(iii)}}$ $\nabla \varphi = 0$ Integrable VCP]

(M^m, g) w/ r-fold VCP.

$$\varphi \in \Omega^{r+1}(M) \qquad \chi : \Lambda^r T_M \to T_M$$

$$\varphi(v_1, \dots, v_r, v_{r+1}) = \langle \chi(v_1, \dots, v_r), v_{r+1} \rangle.$$

Def$^{\underline{n}}$. $A^{r+1} \subset M^m$ <u>instanton</u>

if A preserved by χ

[$\overset{\text{Lemma.}}{\Longleftrightarrow}$ A calibrated by φ]

Def$^{\underline{n}}$. $C^k \subset M^m$ <u>Lagrangian</u>

if $\begin{cases} \varphi|_C = 0 \\ \dim C = k = \dfrac{n+r-1}{2}. \end{cases}$

Remark: $\dim C > \frac{n+r-1}{2} \Rightarrow \varphi|_C \neq 0.$

§ (Unparametrized) Loop Space Interpretation.

$(M, g) \qquad \varphi \in \Omega^{r+1}(M).$

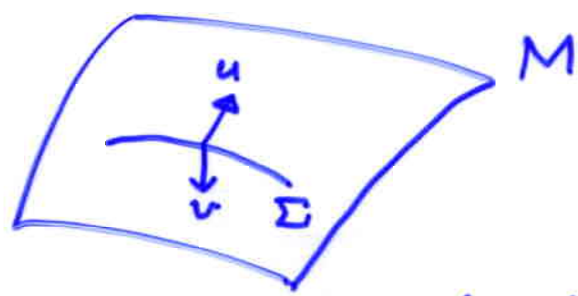
Fix ANY Σ^{r-1}

$$\mathcal{L}_\Sigma M \triangleq \mathrm{Map}(\Sigma, M)_{\mathrm{embed}} / \mathrm{Diff}^+(\Sigma).$$

Transgression $\rightarrow$

$$\omega_{\mathcal{L}_\Sigma M} = \int_\Sigma \varphi \quad \in \Omega^2(\mathcal{L}_\Sigma M).$$

That is,



$u, v \in \Gamma(N_{\Sigma/M}).$

$$\omega_{\mathcal{L}_\Sigma M}(u, v)$$

$$= \int_\Sigma \iota_{u \wedge v} \varphi$$

Theorem. (M^n, g) $\quad \varphi \in \Omega^{r+1}(M)$

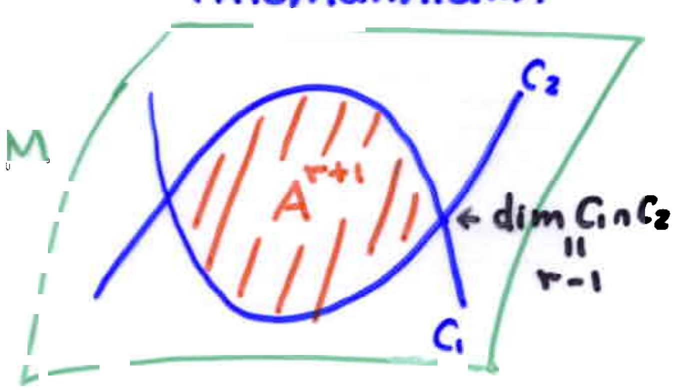
$$\rightsquigarrow \quad \omega_{\mathcal{L}_\Sigma M} \in \Omega^2(\mathcal{L}_\Sigma M).$$

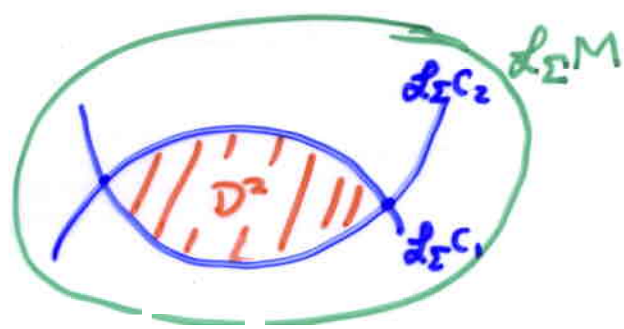
(1) φ : r-fold VCP on M $\iff$ $\omega_{\mathcal{L}_\Sigma M}$: 1-fold VCP on $\mathcal{L}_\Sigma M$.
(i.e. Symplectic).

(2) $C \subset M$ $\iff$ $\mathcal{L}_\Sigma C \subset \mathcal{L}_\Sigma M$
φ-Lagrangian $\qquad$ Lagrangian
$(\dim C = (n+r-1)/2)$.

(3) $\underset{\substack{\shortparallel \\ A^{r+1}}}{D^2 \times \Sigma} \subset M \iff D^2 \subset \mathcal{L}_\Sigma M$

instanton (Riemannian). $\qquad$ instanton (i.e. holomorphic curve).





§ Examples / Classification.

1° Kähler / Symplectic Manifolds.

$(r = 1)$.

$$(M^{2m}, \chi, \varphi) = (M, \underset{\text{cpx. str.}}{J}, \underset{\text{sympl. form.}}{\omega})$$

- Instanton
- Lagrangian

(usual def$^{\underline{n}}$.).

2° Volume form $(r = n-1)$

(M^n, g) $\quad \varphi = \nu_M \in \Omega^n(M)$

$$= \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

- Instantons $\quad A^n \underset{\text{open}}{\subseteq} M^n \quad$ (domain in M)

- Lagrangians $\quad C^{n-1} \subset M^n$

  $\parallel$

  Hypersurfaces (

3° G_2-manifold $(r=2)$.

(M^7, g) $\quad \varphi =: \Omega \in \Omega^3(M^7)$

e.g. $M^7 = X^6 \times S^1$

Calabi-Yau 3-fold.

$$\Omega_M = \mathrm{Re}\,\Omega_X + \omega_X \wedge d\theta$$

When $M^7 = \mathrm{Im}\,\mathbb{O}$

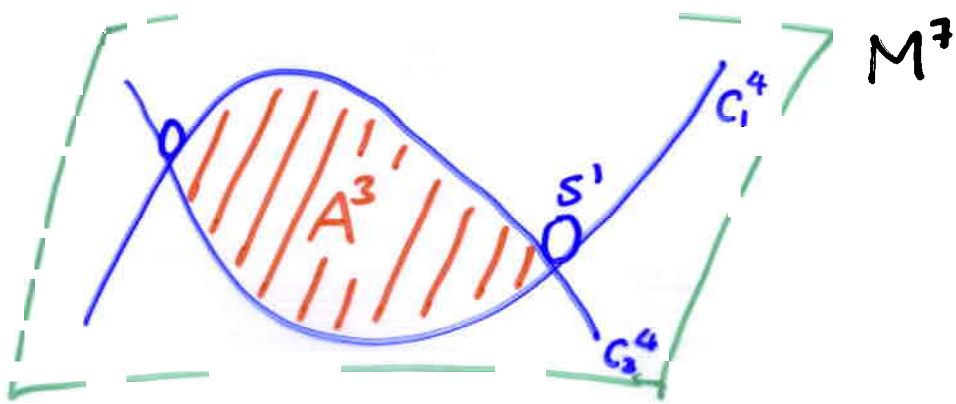
$$\chi(u, v) =: u \times v$$
$$= \mathrm{Im}(u \cdot v)$$

(M^7, g, Ω) : G_2-mfd.

$\Omega(u,v,w) = g(u \times v, w)$.

- Instanton $A^3 \subset M$ preserved by $\times$
  $\shortparallel$ (calib. by Ω).
  Associative

- Lagrangian $C^4 \subset M$ $\Omega|_C = 0$
  $\shortparallel$ (calib. by $*\Omega$).
  Coassociative.



Remark: Count A^3.
when $C_1 \sim C_2$. (L. - X. W. Wang).

$\# A^3$

$\approx$ # holo. curves in C_1^4 (+ bubbles)

$\overset{\text{Taubes}}{\approx}$ Seiberg-Witten Inv. of C_1.

4°. Spin(7)-manifold $(r=3)$

M^8.

e.g. $M^8 = Z^7 \times S^1$

$\uparrow$ G_2-mfd.

$\varphi_M = \Omega_Z \wedge d\theta + *\Omega_Z$.

Example: $M = \mathbb{O}$

$$u \times v \times w = \chi(u, v, w) = \frac{1}{2}[u(\bar{v}w) - w(\bar{v}u)]$$

- Instanton $A^4 \subset M^8$ preserved by $\times$ (calib. by φ).
  $\|$
  Cayley

- Proposition: $\nexists$ φ-Lagrangian in any Spin(7)-manifold.

Remark: No other VCP — classification by Brown-Gray.

In fact, we can also allow

$\underline{r = 0}$. i.e.

(M^n, g) $\qquad \varphi \in \Omega^1(M)$

$$\begin{cases} d\varphi = 0 \\ |\varphi| = 1. \end{cases}$$

Eg.

$f : M \longrightarrow S^1$ Riemannian submersion.

$\varphi = f^*(d\theta).$

- Instanton $A^1 \subset M$
  ||
  Gradient Flow Line.

- Lagrangian ?

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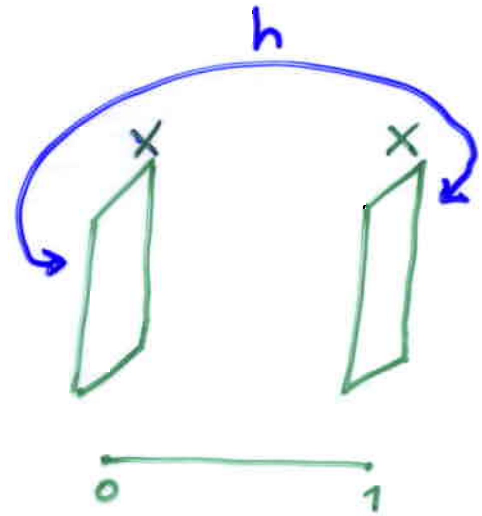
$$f : M^n \longrightarrow S^1$$

$\Longleftrightarrow$. $\quad M = X \times [0,1]/\sim$

$h : X \xrightarrow[\text{isometry}]{} X$

(Mapping Cylinder).

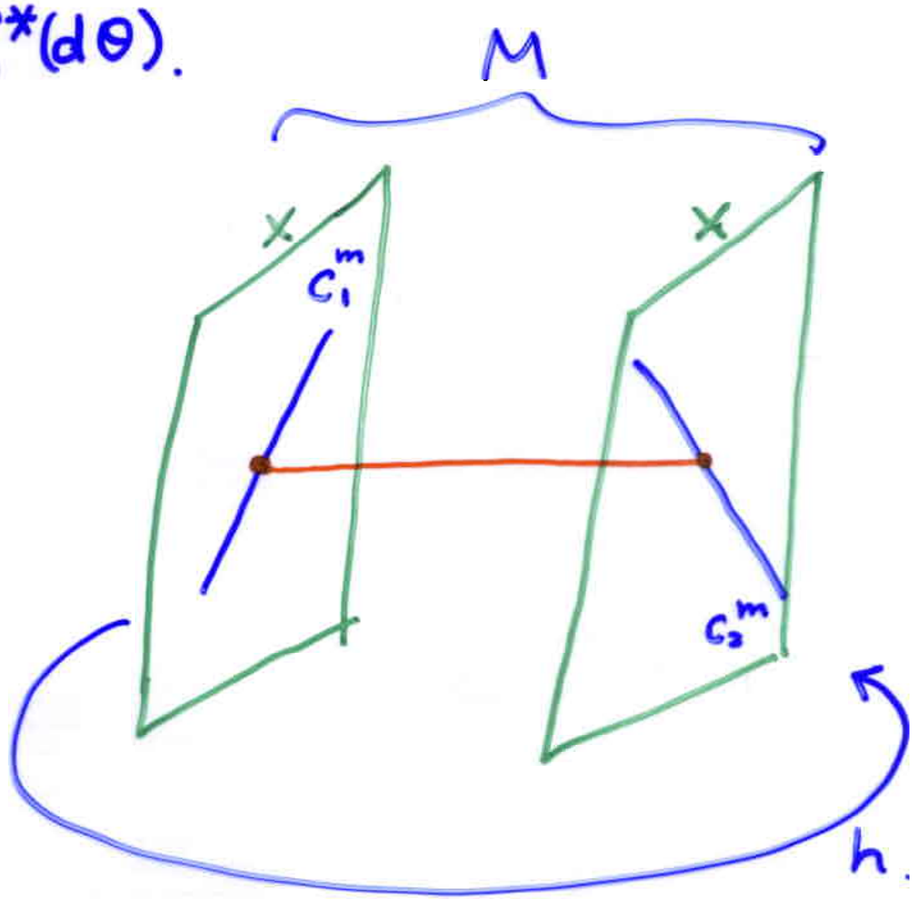


• Lagrangian.

$$C^{m} \subset X^{2m} \times \{t\} \quad \exists t.$$

[$r=0$ Continue ....].

$$\begin{array}{ccc} M^{2m+1} & & \varphi \in \Omega^1(M). \\ f \big\downarrow & & \| \\ S^1 & & f^*(d\theta). \end{array}$$



$\#$ Instantons bounding $C_1 \cup C_2$

$$= \sum_{k=-\infty}^{\infty} \left(\# C_1 \cap h^k(C_2) \right) t^k .$$

§ Complex V.C.P.

Definition: (M^{2n}, g, J) Kähler.

$\varphi \in \Omega^{r+1,0}(M)$, $d\varphi = 0$.

satisfying

$$| \iota_{u_1 \wedge \cdots \wedge u_r} \varphi | = 1$$

for any orthonormal $u_1, \ldots, u_r \in T^{1,0}M$.

is called a <u>Complex Vector Cross Product</u>.

Classification Theorem (Lee-L.)

(M^{2n}, g, J), $\varphi \in \Omega^{r+1,0}(M)$

r-fold & $\mathbb{C}$. V.C.P.

$\Rightarrow$ (1). Calabi-Yau. $(r = n-1)$.

φ = holomorphic Volume form.

(2) Hyperkähler $(r = 1)$.

φ = holomorphic Symplectic form.

Remark: $\nexists$ $\mathbb{C}$-analog. of G_2 or Spin(7) type V.C.P.

Remark: $d\varphi = 0 \iff \nabla\varphi = 0$

for $\mathbb{C}$-VCP.

(M^{2n}, J, g), $\varphi \in \Omega^{r+1,0}(M)$.

$\mathbb{C}$ VCP.

Def$^{\underline{n}}$ (i) $A^{r+1} \subset M$ <u>Instanton</u>

if Calibrated by $\mathrm{Re}(e^{i\theta}\varphi)$

for some 'phase' θ.

(ii) $C^{2k} \subset M^{2n}$ <u>Lagrangian</u>

$k = (n+r-1)/2$ (of cx. type).

if $\varphi|_C = 0$.

Remark: $C \subset M$ Lagr. $\Rightarrow$ complex submfd.

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Def. Lagrangian of Real Type

(M^{2n}, g, J)

$\rightsquigarrow$

$\omega \in \Omega^2(M)$ | $\varphi \in \Omega^{n+1,0}(M)$

$C^n \subset M^{2n}$.

(i) $\omega|_C = 0$

(ii) $\mathrm{Re}(e^{i\theta}\varphi)|_C = 0$

real type

$C^{2k} \subset M^{2n}$

$\varphi|_C = 0$

complex type

Remark: Both are good boundary value for Instantons.
(Calib. by $\mathrm{Re}(e^{i\theta}\varphi)$).

Example $[r=1]$: Hyperkähler.

$$(M^{4m},\ g\ ,\ J\ ,\ \omega=\omega_J)$$

$$\varphi \in \Omega^{2,0}(M)$$

$$\varphi = \omega_I + i\,\omega_K.$$

- Instanton. $A^2 \subset M$ calib. by $Re(e^{i\theta}\varphi)$.
  ||
  J_θ-holomorphic curve.
  $J_\theta = \cos\theta\ I + \sin\theta\ K.$

- Lagr. of Cx. Type. $C^{2n} \subset M^{4n}$. $\varphi|_C = 0$
  ||
  J- Complex Lagr.

- Lagr. of Real Type. $C^{2n} \subset M^{4n}$ $\omega|_C = 0$, $Re(e^{i\theta}\varphi)|_C = 0$.
  ||
  $J_{\theta+\pi/2}$-complex Lagr.

Example $[r = n-1]$ <u>Calabi-Yau</u>

$$(M^{2m}, g, J, \omega).$$

$$\varphi \in \Omega^{n,0}(M) \qquad \text{holo. vol. form.}$$

- Instanton $A^m \subset M^{2m}$ calib. by $\mathrm{Re}(e^{i\theta}\varphi)$.

  ||

  SLag w/ phase $= \theta$.

- Lagr. of $\mathbb{C}$-type $C^{2m-2} \subset M^{2m}$ $\varphi|_C = 0$.

  ||

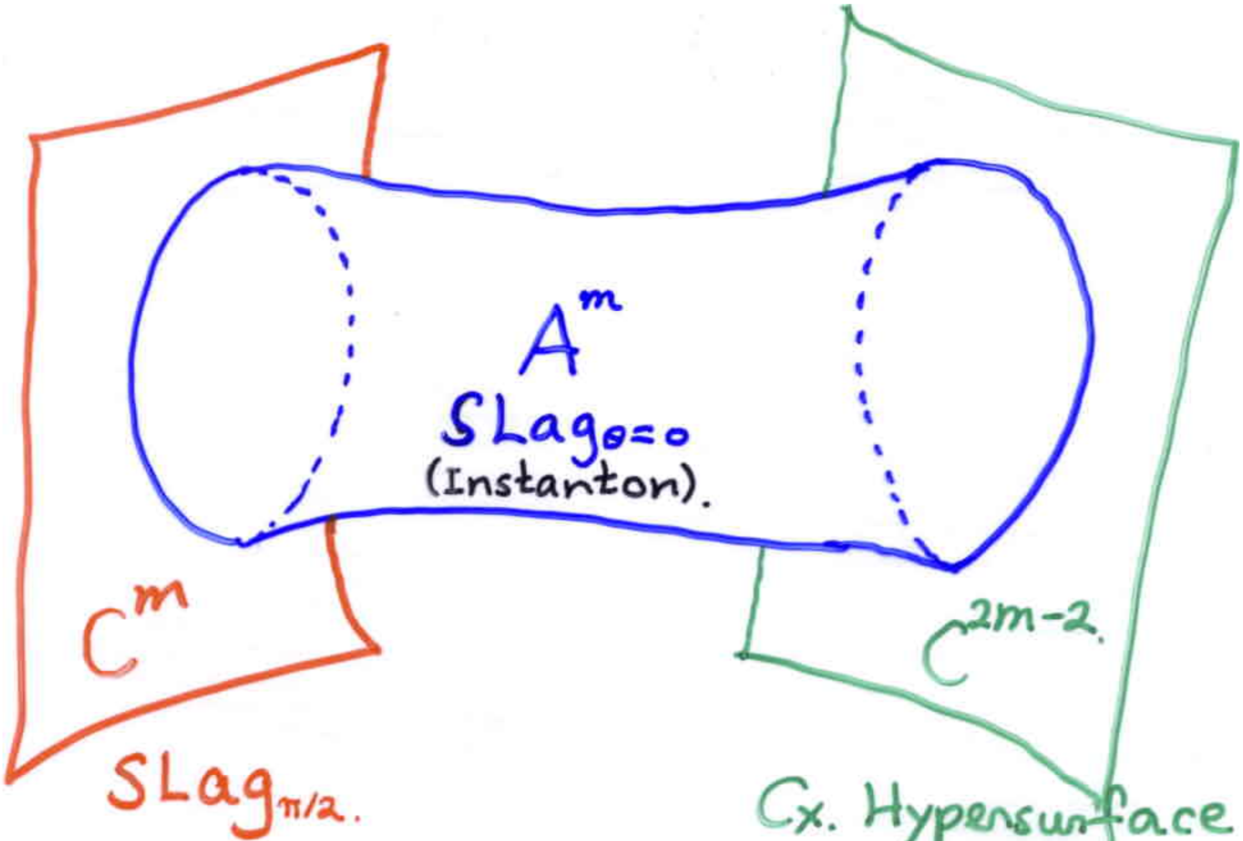
  Complex Hypersurface.

- Lagr. of $\mathbb{R}$-type $C^m \subset M^{2m}$ $\omega|_C = 0$, $\mathrm{Re}(e^{i\theta}\varphi)|_C = 0$

  ||

  SLag. w/ phase $= \theta + \frac{\pi}{2}$.

Remark: M^{2m} : C.Y.



(Lagr., $\mathbb{R}$-type)

(Lagr. $\mathbb{C}$-type).

$SLag_0$ with boundary lying on $SLag_{\pi/2}$ (Dirichlet) or Cx. Hypersurface (Neumann).

— Schoen's school.

(A. Butscher, W.Y. Qiu).

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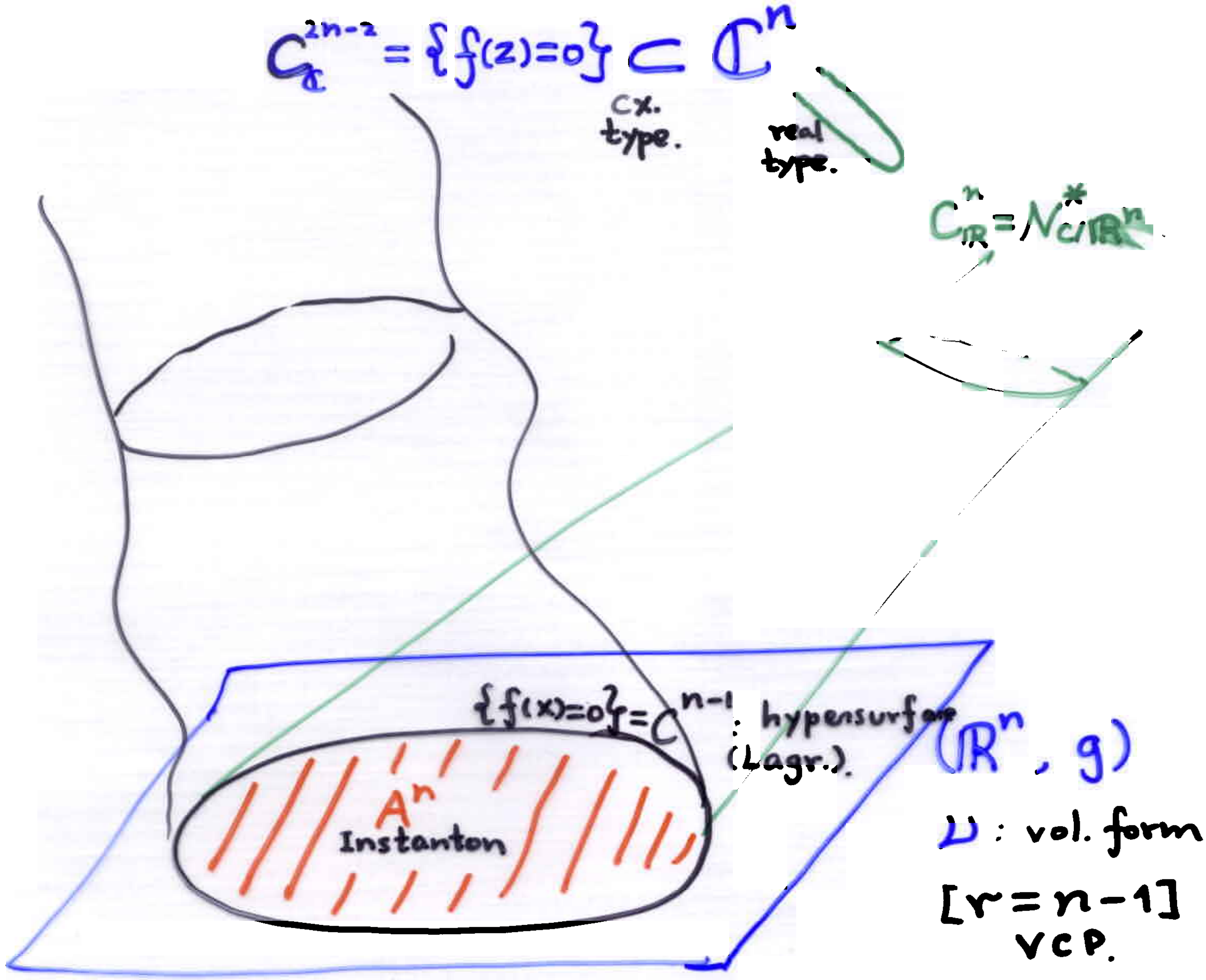
Remark: Motivation of Lagr. of $\mathbb{R}$ vs $\mathbb{C}$ type. (Complexification).



§ Loop Space Interpretations of $\mathbb{C}$. VCP.

Recall: $[r = n-1]$

(M^n, g) , $\nu \in \Omega^n(M)$. $(n-1)$-fold.

$\rightsquigarrow$

$\mathcal{L}_{\Sigma^{n-2}} M$, $\omega_{\mathcal{L}_\Sigma M} \in \Omega^2(\mathcal{L}_\Sigma M)$

$\| $

$\dfrac{\mathrm{Map}(\Sigma^{n-2}, M^n)}{\mathrm{Diff}(\Sigma)}$

1-fold.
(symplectic).

PLAN

M^{2n} : Calabi-Yau. (n-1)-fold & VCP.

(M, g, J, ω), $\varphi \in \Omega^{n,0}(M)$.

$$\frac{\mathrm{Map}(\Sigma^{n-2}, M)}{\mathrm{Diff}(\Sigma)} \quad : \text{ Hyperkähler}$$

Symplectic Quotient ??

- For ω / M induces a symplectic str.

  ω_{Map} on $\mathrm{Map}(\Sigma, M)$. Need

  FIX. $\nu_\Sigma \in \Omega^{n-2}(\Sigma)$. i.e.

$$\omega_{\mathrm{Map}}(u, v) = \int_\Sigma \omega(u, v) \; \nu_\Sigma .$$

- Problem: ω_{Map} is preserved

  only by $\mathrm{Diff}(\Sigma, \nu_\Sigma)$,

  not $\mathrm{Diff}(\Sigma)$.

Moment Map (Donaldson, Hitchin).

$$\mathrm{Diff}(\Sigma, \nu_\Sigma) \curvearrowright \mathrm{Map}(\Sigma, M) \xrightarrow{\mu} \Omega^1(\Sigma)/d\Omega^0(\Sigma).$$

$$\mu(\Sigma \xhookrightarrow{f} M) = \alpha$$

where $d\alpha = f^*\omega$.

$$\frac{\mu^{-1}(0)}{\mathrm{Diff}(\Sigma, \nu)} \quad : \quad \text{Symplectic (NOT Hyperkähler)}$$

$$\big\| $$

$$\frac{\mathrm{Map}(\Sigma, M)}{\mathrm{Diff}(\Sigma, \nu)}.$$

<u>Want</u> $\cdot /\!/ \mathrm{Diff}(\Sigma)$

AIM : $(M^{2n},\ \omega \in \Omega^{1,1},\ \varphi \in \Omega^{n,0})$: C.Y.

Construct Hypenkähler " $\dfrac{\mathrm{Map}(\Sigma^{n-2}, M)}{\mathrm{Diff}(\Sigma)}$ "

$$\tilde{\varphi} := \tilde{\omega}_I - i\tilde{\omega}_K \overset{\text{def}}{=\!=} \int_\Sigma \varphi$$ (2,0)-form on $\mathrm{Map}(\Sigma, M)$.

$\rightsquigarrow$ I, K : $T(\mathrm{Map}) \circlearrowleft$

BUT $\begin{cases} I^2 \neq -1 \neq K^2 \\ \tilde{\omega}_I,\ \tilde{\omega}_K \ \text{: degenerate.} \end{cases}$

$\tilde{\omega}_I$-Sympl. Reduction of $\mu^{-1}(0)$

$$\mu^{-1}(0) = \{\Sigma \hookrightarrow M \text{ isotropic}\}.$$

$\|$

$\tilde{\omega}_K$-Sympl. Reduction of $\mu^{-1}(0)$

is Hypenkähler ?

$$(\tilde{\omega}_I,\ \underset{\tilde{\omega}_J}{\omega_{Map}},\ \tilde{\omega}_K)$$

$$\mathrm{Map}(\Sigma, M) /\!/ \mathrm{Diff}(\Sigma).$$

M^{2n} Calabi-Yau. $\rightsquigarrow$ $\widetilde{\mathcal{L}}_\Sigma M$ Hyperkähler

$\widetilde{\mathcal{L}}_\Sigma M =$ " $\mathrm{Map}(\Sigma^{n-2}, M) /\!/ \mathrm{Diff}(\Sigma)$ ".

$\omega \in \Omega^2(M) \rightsquigarrow \widetilde{\omega}_J \in \Omega^2(\widetilde{\mathcal{L}}_\Sigma M)$

$\varphi \in \Omega^{n,0}(M) \rightsquigarrow \widetilde{\varphi} \in \Omega^{2,0}(\widetilde{\mathcal{L}}_\Sigma M)$, $\widetilde{\varphi} = \widetilde{\omega}_I - i\,\widetilde{\omega}_K$

holo. volume form — holo. sympl. form.

$A^n \subset M$, $A^n = D^2 \times \Sigma$ $\longleftrightarrow$ $D^2 \subset \widetilde{\mathcal{L}}_\Sigma M$

instanton (i.e. $SLag_{\theta+\pi/2}$) — instanton (i.e. holo. curve).

$C^{2n-2} \subset M$ $\longleftrightarrow$ $\widetilde{\mathcal{L}}_\Sigma C \subset \widetilde{\mathcal{L}}_\Sigma M$

Lagr. $\mathbb{C}$-type. (i.e. Cx. hypersurface). — J-complex Lagr.

$C^n \subset M^{2n}$ $\longleftrightarrow$ " $\widetilde{\mathcal{L}}_\Sigma C$ " $\subset \widetilde{\mathcal{L}}_\Sigma M$

" $\widetilde{\mathcal{L}}_\Sigma C$ " $= \dfrac{\mathrm{Map}(\Sigma, C) \cap \overbrace{\mu^{-1}(0)}^{\text{automatic}}}{\mathrm{Diff}(\Sigma)}$

Lagr. $\mathbb{R}$-type (i.e. $SLag_\theta$). — J_θ-complex Lagr.

Comparisons :

(1) Vector Cross Product

$r=$	$n-1$	1	2	3
VCP	Oriented	Kähler	G_2-mfd	Spin7-mfd
$\mathbb{C}$-VCP	Calabi-Yau	Hyperkähler		

(2) Geometry / Normed Algebras $\mathbb{A}$.

$\mathbb{A}=$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{A}$-mfd	Manifold	Kähler	Quaternionic Kähler	Spin7-mfd
Special $\mathbb{A}$-mfd	Oriented	Calabi-Yau	Hyperkähler	G_2-mfd.