S. Kong

1. $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon \cong \mathbb{R}^8$

$(a+b\varepsilon)(c+d\varepsilon) = ac - \bar{d}b + (da + b\bar{c})\varepsilon$

$G_2 = \mathrm{Aut}(\mathbb{O}) \subseteq SO(7)$.

Cross product on $\mathbb{R}^7$

$\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$

$x \times y = \mathrm{Im}(xy)$.

- $x \times y \perp \{x, y\}$
- $\|x \times y\| = \|x \wedge y\|$

$\phi_0(x, y, z) = (x, y \times z)$ 3-form.

$$\phi_0 = dx^{123} + dx^1(dx^{45} - dx^{67}) + dx^2(dx^{46} - dx^{75}) + dx^3(dx^{47} - dx^{56})$$

$G_2 = \{(u, v, u\times v, w, u\times w, v\times w, (u\times v)\times w)\}$

u, v, w orthonormal vectors in $\mathbb{R}^7$
$(u \times v, w) = 0$.

$$G_2 \overset{\text{Bryant}}{=} \{ g \in GL(7,\mathbb{R}) \mid g^* \phi_0 = \phi_0 \}$$

$$SU(3) = \{ g \in G_2 \mid g(e_1) = e_1 \}$$

- $V = \{ u \in \mathbb{R}^7 \mid (u, e_1) = 0 \}$

$$J : V \longrightarrow V$$
$$Ju = e_1 \times u$$

almost complex structure.

$$g \in G_2,\ g(e_1) = e_1 \Rightarrow gJ = Jg.$$
$$g \in U(3).$$

$$SU(3) = \{ (u, v, \overline{u \times v}) \mid u, v \in \mathbb{C}^3 \text{ unitary orthonormal} \}$$

$$\Rightarrow g \in SU(3)$$

- $\phi_0 = dx^1 \wedge \omega + \mathrm{Re}\,\Omega$

$$*\phi_0 = dx^1 \wedge \mathrm{Im}\,\Omega - \frac{1}{2}\omega^2$$

$$\omega = dx^{23} + dx^{45} - dx^{67}$$
$$\Omega = (dx^2 + i\,dx^3) \wedge (dx^4 + i\,dx^5) \wedge (dx^6 - i\,dx^7)$$

kähler form. hol volume form. of J.

2 M^7

3-form ϕ is positive if $\exists\, f : T_pM \to \mathbb{R}^7$ s.t.

$$\phi|_p = f^*\phi_0 .$$

Given ϕ positive, associate

$$g|_p = f^* g_0 .$$

$$\{ G_2\text{-structure} \} \longleftrightarrow \{ \text{positive 3-form} \}.$$

(M, g, ϕ), $\quad g(X \times Y, Z) = \phi(X, Y, Z)$

(M, g, ϕ) is called G_2-manifold if

$$\nabla\phi = 0$$

$$\Leftrightarrow \quad d\phi = d*\phi = 0 .$$

- (Calabi, ~~Gray~~)
  if (M^7, g, ϕ) G_2-structure.
  then any hypersurface $N \subseteq M^7$ admits an almost complex structure.

$$J_p : T_pN \to T_pN$$

$$J_p X = \nu_p \times X$$

where ν_p is the unit normal vector of N

(Calabi- ~~Gray~~). Suppose (M, g, ϕ) is a G_2-manifold.
generalized by Bryant
If J is integrable, then $N \subset M$ is a minimal submanifold.

- If N^6 is a CY 3-fold, then $N \times S^1$, $N \times \mathbb{R}$ is a G_2-manifolds.

In general: (N^6, g_t, J_t) family almost Hermitian.

Ω_t (3,0) form satisfies

$$\frac{\omega_t^3}{3!} = \frac{\sqrt{-1}}{8} \Omega_t \wedge \overline{\Omega_t}$$

Then
$$\phi = dt \wedge \omega_t + \mathrm{Re}\, \Omega_t$$

$$g = dt^2 + g_t$$

defines a G_2-structure on $\mathbb{R} \times M$.

- $$\nabla \phi = 0 \iff \begin{cases} \frac{\partial}{\partial t} \mathrm{Re}\, \Omega_t = d\omega_t \\ \frac{\partial}{\partial t} \omega_t^2 = 2\sqrt{-1}\, d\Omega_t \\ d(\omega_t^2) = 0 \end{cases}$$

3.

(M, J, g) nearly Kähler if $(\nabla_X J)X = 0$.

- $(dw)^{1,2} = 0$ $\quad (\Rightarrow d * w = 0)$
- $(\nabla_X J) Y \perp \{X, Y, JX, JY\}$.
  ($\Rightarrow$ all nearly Kähler 4-manifold are Kähler)
- if $\dim M = 6$ then (Gray - Masumoto)

$$|(\nabla_X J) Y|^2 = \lambda^2 \left(|X|^2 |Y|^2 - (X, Y)^2 - (X, JY)^2 \right),$$

where λ is a constant.

Suppose (M^6, J, g) nearly Kähler but not Kähler.

e_1, e_2 orthonormal ~~frame~~ s.t $(e_1, Je_2) = 0$.

$$|(\nabla_{e_1} J) e_2|^2 = \lambda^2$$

$$e_3 \overset{def}{=} \frac{1}{\lambda} (\nabla_{e_1} J) e_2.$$

$\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ orthonormal frame

ω_i (1,0)-forms.

$$\Longleftrightarrow \begin{cases} (d\omega_1)^{0,2} = -\sqrt{-1}\,\lambda\,\overline{\omega_2}\wedge\overline{\omega_3} \\ (d\omega_2)^{0,2} = -\sqrt{-1}\,\lambda\,\overline{\omega_3}\wedge\overline{\omega_1} \\ (d\omega_3)^{0,2} = -\sqrt{-1}\,\lambda\,\overline{\omega_1}\wedge\overline{\omega_2} \\ (d\omega)^{1,2} = 0 \end{cases}$$

$$\Longleftrightarrow \quad d\omega = 3\lambda\,\mathrm{Re}(\omega_1\wedge\omega_2\wedge\omega_3)$$

$$d(\omega_1\wedge\omega_2\wedge\omega_3) = -2\sqrt{-1}\,\lambda\,\omega^2.$$

4. (M^6, J, g) nearly Kähler, not Kähler

$\omega_1, \omega_2, \omega_3$.

$$g_t = \mathrm{Re}\left(a^2(t)\, \omega_1 \otimes \overline{\omega_1} + b^2(t)\, \omega_2 \otimes \overline{\omega_2} + c^2(t)\, \omega_3 \otimes \overline{\omega_3} \right).$$

ω_t. Kähler form.

$\Omega_t = abc\, \Omega$

ϕ : $\mathbb{R} \times M$.

$$\phi = dt \wedge \omega_t + \mathrm{Re}\, \Omega_t$$

$$g = dt^2 + g_t$$

prop: Suppose.

$$d(\omega_1 \wedge \overline{\omega_1}) = -\frac{2}{3}\sqrt{-1}\, d\omega$$

then $\nabla \phi = 0$ provided $b = c$ and a, b satisfies.

$$\begin{cases} \dfrac{da}{dt} = 2 - \left(\dfrac{a}{b}\right)^2 \\[2ex] \dfrac{db}{dt} = \dfrac{a}{b}. \end{cases}$$

Sol: $b^2 = 2r^2$

$a^2 = 2r^2\left(1 - \left(\frac{m}{r}\right)^4\right)$

$dt = \frac{dr^2}{a}$.

Example: $\mathbb{C}P^3 = \dfrac{Sp(2)}{Sp(1) \times U(1)}$ diagonal.

$$Sp(n) = \{ A \in GL(n, \mathbb{H}) \mid A^* A = I \}$$

θ Maurer - Cartan form.

write

$$\theta = \begin{pmatrix} \alpha_2 i + \alpha_3 j + \alpha_4 k & \frac{\omega_2}{\sqrt{2}} + \frac{\bar{\omega}_3}{\sqrt{2}} j \\ -\frac{\bar{\omega}_2}{\sqrt{2}} + \frac{\bar{\omega}_3}{\sqrt{2}} j, & \alpha_1 i + \omega_1 k \end{pmatrix}.$$

$\alpha_1, \cdots \alpha_4$ real value 1-form.

$\omega_1, \omega_2, \omega_3$ complex valued 1-form.

MC $\Rightarrow$ all assumption satisfied.

$$g = \left(1 - \left(\frac{m}{r}\right)^4\right)^{-1} dr^2 + r^2\left(1 - \left(\frac{m}{r}\right)^4\right)$$

$$\cdot \mathrm{Re}(\omega_1 \bar{\omega}_1) + r^2 \mathrm{Re}(\omega_2 \bar{\omega}_2 + \omega_3 \bar{\omega}_3)$$

#

(Bryant - Salamon)

prop: Suppose

$d(\omega_1 \wedge \bar{\omega}_1) = d(\omega_2 \wedge \bar{\omega}_2) = -\frac{2}{3}\sqrt{-1}\, d\omega$

then $\nabla \Phi = 0$ provided

$$\begin{cases} \dfrac{da}{dt} = \dfrac{b^2+c^2-a^2}{bc} \\[2ex] \dfrac{db}{dt} = \dfrac{a^2+c^2-b^2}{ac} \\[2ex] \dfrac{dc}{dt} = \dfrac{a^2+b^2-c^2}{ab} \end{cases}$$

Nahm equation.

Example. $\dfrac{SU(3)}{T^2}$

Maurer-Cartan $\Theta = \begin{pmatrix} \sqrt{-1}\,\alpha_3, & -\bar{\omega}_1 & \sqrt{-1}\,\omega_2 \\ \omega_1, & \sqrt{-1}\,\alpha_2 & -\bar{\omega}_3 \\ \sqrt{-1}\,\bar{\omega}_2, & \omega_3, & \sqrt{-1}\,\alpha_1 \end{pmatrix}$.

$\omega_1, \omega_2, \omega_3$ defines a nearly Kähler structure

$d(\omega_1 \wedge \bar{\omega}_1) = d(\omega_2 \wedge \bar{\omega}_2) = d(\omega_3 \wedge \bar{\omega}_3)$

$= -\frac{2}{3}\sqrt{-1}\, d\omega$.

Bryant, Bryant - Salamon.

5. $(M, g) \to (N, h)$ Rie submersion

J on M preserves horizontal and vertical.

New $\hat{g}$, $\hat{J}$ on M.

$$\hat{g} = \begin{cases} \frac{1}{2} g & \text{vertical} \\ g & \text{horizontal.} \end{cases}$$

$$\hat{J} = \begin{cases} -J & \text{vertical} \\ J & \text{horizontal.} \end{cases}$$

- if (M, g, J) Kähler, totally geodesic fiber.
  then $(M, \hat{g}, \hat{J})$ nearly Kähler

- Further if $\dim(\text{fiber}) = 2$, then $\exists$ (1,0)-form ~~$\omega_1 \in \Omega^{1,0}$~~
  ω_1 (w.r.t $\hat{J}$) s.t $|\omega_1|_{\hat{g}} = \sqrt{2}$ satisfying

$$d(\omega_1 \wedge \bar{\omega}_1) = -\frac{2}{3}\sqrt{-1}\, d\hat{\omega}$$

- if N^4 is self-dual Einstein, then twistor space $S(N)$ is Kähler and

  $S(N) \to N$ is ~~totally geodesic~~ Rie submersion with totally geodesic fiber.