Examples of Associative, Co-associative, and Cayley Submanifolds

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- For this talk, I am going to focus on techniques we used in finding examples of Cayley submflds. Similar techniques can be applied to identify associative or co-associative submflds.

(Associative, Co-associative &)

Q: What are Cayley Manifolds?

(Associative) (3-dim'l)

Ans: Cayley manifolds are 4-dim'l sub-mflds
(G_2 mfld) (M^7)
of a Spin_7 mfld M^8 calibrated by
(associative form φ)
Cayley form Φ.

(Associative, Co-associative &)

$\Rightarrow$ Cayley manifolds are volume-minimizing

Outline of the talk

1. Calibrated Geometry
2. Cayley forms
3. $Spin_7$-manifolds M^8
4. Cayley manifolds $\mathbb{C}^4$
5. Techniques used to find Cayley mflds
6. Examples of Cayley manifolds
7. Similar Techniques have been applied to find SLag mflds, associative mflds, and co-associative mflds.
8. Work in Progress

1. Calibrated Geometry

(M, g) given

- Calibrations : closed differential forms on M
- Comass of a calibration φ
- Calibrated planes
- Calibrated manifolds, cycles, φ-current

Key: Any Calibrated cycle $N \subset M$ is of absolutely minimal Volume in its homology class.

The comass of a differential form

Let φ be a differential form on a Riemannian manifold M of dimension n.

Its ***comass*** $\|\varphi\|^*$ is given by

$$\boxed{\|\varphi\|^* = \sup\{\|\varphi_x\|^* : x \in M\},}$$

where φ_x is the corresponding p-form on the tangent space $T_x M$. And

$$\boxed{\|\varphi_x\|^* = \max\{\varphi(\xi) : \xi \; is\; a\; p-plane\; in\; T_x M\}.}$$

(A p-plane is technically a unit, simple p-vector.)

Calibrations.

A ***calibration*** is just a closed differential p-form φ on a Riemannian manifold M, used to prove p-cycles volume minimizing.

φ ***calibrates*** a p-plane ξ in the tangent space to M at x if $\varphi_x(\xi)$ achieves the maximum value $\|\varphi\|^*$; φ ***calibrates*** a p-cycle S if φ calibrates (almost) all of its tangent planes.

The principal observation is:

> ***A p-cycle calibrated by some closed differential p-form must be mass-minimizing in its homology class.***

General Case:

Remark: In general, the volume minimizing cycle may not be a smooth submfld at all.
We need:
(1) generalize the submfld of dim p in above example to p-current.
(2) generalize the notion of volume of a submfld to that of mass of a current.

DEFN: A *p-current* (in the sense of de Rham) is a continuous functional on the space of smooth p-forms on M in the C^∞ topology.

mean: Singular Lipschitz chain.

In particular, a classical p-cycle defines a p-current by integration.

Fact 1: If S is a smooth p-chain, and U_S is the p-current defined by integrati[on] of p-form over S, i.e. $U_S(\varphi) = \int_S \varphi$, then

$$\boxed{\text{mass}(U_S) = \text{vol}(S)}$$

Thus, the notion of mass for currents is the generalization of that of volume for classical p-chain.

Fact 2: If we restrict ourselves to current[s] on M of finite mass whose bdries also have finite mass (i.e so called normal currents), then their homology coincides with the real homology of M.

It is easy to check:

Mass is a norm on homology:

It is linear on rays and

it is subadditive.

DEFN: Given a smooth p-form φ on M

$\underset{\sim}{\text{Comass}(\varphi)} :\triangleq \max\{\text{comass}(\varphi_x) \mid x \in M\}$, where

$$\text{comass}(\varphi_x) :\triangleq \max\left\{ \varphi_x(v) \;\middle|\; \begin{array}{l} v \text{ is a decomposable} \\ p\text{-vector of Norm 1} \\ \text{in tangent space } T_xM \end{array} \right\}$$

DEFN: Given a p-current U on M,

its mass is defined by

$$\underset{\sim}{\text{mass}(U)} = \underset{\text{sup}}{\cancel{\max}} \left\{ U(\varphi) \;\middle|\; \begin{array}{l} \varphi = \text{smooth } p\text{-form} \\ \text{of comass } 1 \end{array} \right\}$$

DEFN: The Mass of a real homology class is the minimal mass of any closed current in that class.

Regularity Results

- **Thm** (Morrey) Let $N \subset M$ be a φ-calibrated cycle. Assume M is of class C^k. Then if N is C^1 then N is C^k.

- **Thm** (Almgren) Let $T =$ a φ-current of dim p. Then $\mathrm{Supp}(T)$ is a C^ω submfld (with $\mathbb{Z}$-multi) except a singular set of codim 2.

i.e. $$\mathrm{Supp}(T) - \mathrm{Supp}(dT) = \underset{\substack{\uparrow \\ \text{a } C^\omega \text{ mfld} \\ \text{of dim } p}}{N} \amalg \underset{\substack{\uparrow \\ \mathcal{H}\text{-dim } p-2}}{\Sigma}$$

Technique Issues

- How to pick a calibration when M is given?

- How to find the comass of φ? How hard is it?

- How to identify φ-calibrated submflds, cycles or currents?

- How to prove uniqueness theorems?

- How to prove a minimal mfld being minimizing?

Let's discuss these issues by considering following examples.

Examples

1. (M, J) Kähler mfld

Calibrations $\begin{cases} \omega = \text{Kähler form} \quad (d\omega = 0) \qquad \omega(v,w) = \langle Jv, w\rangle \\ \frac{1}{p!}\omega^p \text{ has comass } 1 \text{ (by Wirtinger's inequality)} \end{cases}$

Calibrated mflds $\begin{cases} \text{holomorphic curves} \quad (\text{calibrated by } \omega) \\ \text{Complex mfld of dim } p \quad (\text{calibrated by } \frac{1}{p!}\omega^p) \end{cases}$

(M, g) Calabi-Yau

2. $\mathbb{C}^n$: $\quad z = (z_1, \cdots, z_n)$, $z = x + iy$, $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$

Calibrations $\begin{cases} Re(\Omega) = \text{special Lagrangian form.} \quad \Omega = dz_1 \wedge \cdots \wedge dz_n \\ Im(\Omega). \text{ Both have comass } 1. \end{cases}$

Calibrated mflds $\begin{cases} \text{Special Lagr. submflds w/ phase} = 0 \quad (\text{by } Re(\Omega)) \\ \text{Special Lagr. submflds w/ phase} = \pi/2 \quad (\text{by } Im(\Omega)) \end{cases}$

3. $G_k\mathbb{R}^n$ = the set of oriented k-planes throu 0 in $\mathbb{R}^n$.

Calibration: P_1 = first pontryagin form

$\|P_1\|^* \triangleq \text{comass}(P_1)$ see below.

Calibrated cycles: see below.

e.g.

$\|P_1\|^* = 3/2$ $\quad G_k\mathbb{R}^n$ $\quad S^4$

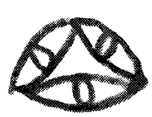
$\|P_1\|^* = 3/2$ … $\quad G_4\mathbb{R}^8$ $\quad S^4 \cong \mathbb{H}P^1$

$\|P_1\|^* = \sqrt{3}/6$ $\quad G_3\mathbb{R}^6$

$\|P_1\|^* = 4/3$ $\quad G_3\mathbb{R}^7$

diffused curent (not integrable)

$\|P_1\|^* = 1$ $\quad G_2\mathbb{R}^4$



each is a suspension of $\mathbb{R}P^3$

$S^2(\frac{1}{\sqrt{2}}) \times S^2(\frac{1}{\sqrt{2}})$

For more details, please see

H. Gluck, D. Mackenzie, F. Morgan, Volume-minimizing cycles in Grass. mflds. Duke Mathematical Journal, vol. 79, 1995, No. 2. pgs. 335–404.

W. Gu, The stable 4-dimensional geometry of the real Grass. mflds. Duke Mathematical Journal, vol. 93, No 1, 1998. pgs. 155–178.

D. Grossman, W. Gu, Uniqueness of Volume-Minimizing Submflds Calibrated by the first Pontryagin form. Transactions of AMS Vol 353

4. In 1976, Lawson & Osserman discovered an very interesting minimal "surface", which is 4-dim'l but not C^1, by solving the non-parametric minimal surface system.

- It is the cone on the graph of a hopf map:

  Fix a unit vector $u \in \mathrm{Im}\,\mathbb{H}$, $|u|=1$. Consider the hopf map

$$H: S^3 \longrightarrow S^2\left(\tfrac{\sqrt{5}}{2}\right)$$

$$q \longmapsto \frac{\sqrt{5}}{2}\,\bar{q}uq \quad , \forall q \in \mathbb{H}, |q|=1$$

  Define the cone by $h(q) = \|q\|\, H\!\left(\frac{q}{\|q\|}\right) = \frac{\sqrt{5}}{2}\frac{1}{\|q\|}\bar{q}uq$.

- Q: Is this minimal "surface" minimizing? (absolute)

- Ans: Yes! It is calibrated by the co-associative form!

- The fcn h is Lipschitz soln to $D(h) = \sigma(y)$ which is not C^1.

How to show

■ **Certain uniqueness results ?**

e.g. By associating to the family of calibrated planes a Pfaffian system on the symmetric group $SO(k+n)$, an analysis of which yields a uniqueness result; namely, that any connected submanifold of $G_k R^{k+n}$ calibrated by the first Pontryagin form is contained in one of these 4-spheres.

2. Cayley Form

- The four form $\Phi \in \Lambda^4 \mathbb{O}^*$ defined by

$$\Phi(x, y, z, w) = \langle x, y \times z \times w \rangle$$

for all $x, y, z, w \in \mathbb{O}$ is called the **Cayley form** (or Cayley Calibration) on $\mathbb{O}$.

$\mathbb{O} \triangleq$ the set of Cayley numbers.

- Φ is Closed, Alternating, and has comass 1.

Normed Algebra

$\left[\begin{array}{l}\text{i.e. Algebra}/\mathbb{R} \text{ with Unit} \\ \quad \|u \cdot v\| = \|u\|\|v\| \end{array}\right] \Rightarrow$ **Division Algebra**

Complexification (Cayley–Dickson Process)

$$\mathbb{R} \longrightarrow \mathbb{C} \longrightarrow \mathbb{H} \longrightarrow \mathbb{O}$$

New algebra $\mathbb{B} = \mathbb{A} \oplus \mathbb{A}$

$* \quad (a, b) \cdot (c, d) = (ac - \bar{d}b,\ da + b\bar{c})$

$a, b, c, d \in \mathbb{A}$

Classification: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Cross Products

Let $x, y, z \in \mathbb{O}$

Define

- $x \times y \triangleq -\frac{1}{2}(\bar{x}y - \bar{y}x)$

  $\mathrm{Re}(x \times y) = 0 \Rightarrow \left[\begin{array}{ccc} \mathrm{Im}\,\mathbb{O} \times \mathrm{Im}\,\mathbb{O} & \longrightarrow & \mathrm{Im}\,\mathbb{O} \\ (x, y) & \longmapsto & x \times y \ (= \mathrm{Im}\,\bar{y}x) \end{array}\right]$

- $x \times y \times z \triangleq \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x))$

Propositions

- $x \times y$ alternating $\qquad x \times y \times z$ alternating
- $|x \times y| = |x \wedge y| \qquad |x \times y \times z| = |x \wedge y \wedge z|$
- $(x \times y) \perp x$, $(x \times y) \perp y \qquad (x \times y \times z) \perp x, y, z$
- $(xu) \times (yu) = u(x \times y)\bar{u} \qquad (xu) \times (yu) \times (zu) = (x \times y \times z)u$

  $u \in \mathrm{Im}\,\mathbb{O}$, $|u| = 1$ (so $\bar{u} = -u$)

  $\vdots$

$$\Phi = \frac{1}{2}\,\omega\wedge\omega + \mathrm{Re}(\Omega)$$

J_e = complex structure on $\mathbb{O} \cong \mathbb{C}^4$.

ω = associated Kähler form.

$$\begin{tikzcd}
\mathbb{H}\oplus\mathbb{H} \arrow[r, leftrightarrow, bend left, "\text{identified}"] \arrow[d, equal] & \mathbb{R}^4\oplus\mathbb{R}^4 \arrow[d, equal] \\
\mathbb{O} \arrow[r, phantom, "\cong"] & \mathbb{C}^4
\end{tikzcd}$$

$$\Phi = \overbrace{e_{1234} - e_{1278} - e_{1638} - e_{1674} - e_{1265} - e_{1375} - e_{14\ldots}}^{1^*\wedge\varphi}$$
$$\underbrace{+ e_{5678} - e_{5634} - e_{5274} - e_{5238} + e_{3478} + e_{2468} + e_{2367}}_{*\varphi}$$

$$\Phi = \underset{\substack{\uparrow\\ \text{associative}\\ \text{form}}}{1^*\wedge\varphi} + \underset{\substack{\nwarrow\\ \text{coassociative}\\ \text{form}}}{(*\varphi)}$$

(M,g)	Kähler manifold	Calabi-Yau manifold	G_2 manifold	Spin(7) manifold	
Holonomy groups $\subseteq$	$U(m)$	$SU(m)$	G_2	Spin(7)	
Form Preserved by Hol(g)	ω_0 Kähler form	Ω_0 holomorphic volume form	φ_0 $*\varphi_0$ associative and coassociative form	Φ_0 Cayley form	
Calibrations	ω $\omega^k/k!$	$\mathrm{Re}\,\Omega$	φ $*\varphi$	Φ	
Name of calibrated manifolds	holomorphic curve Complex mflds of dim k in (M,J)	Special Lagrangian mflds	associative 3-mflds Coassociative 4-mflds	Cayley 4-folds	

3. Spin_7 Manifolds

- Call a Riemannian mfld (M,g) a Spin_7-mfld if $\mathrm{Hol}(g) \subseteq \mathrm{Spin}_7$.

- Examples: $\mathbb{R}^8$, T^8, $\overbrace{T^8/\Gamma}^{\text{the resolution}}$ (Γ is a finite gp of automorphisms of T^8 preserving a flat spin(7)-structure (Ω_0, g_0) on T^8.)

Different ways to Describe Spin_7

- $Spin_7 = \{g \in SO(8) : g^*\Phi = \Phi\}$.

- $$\mathrm{Spin}_7 = \left\{ g \in SO_8 \,\middle|\, g(uv) = g(u)\overline{\chi}_g(v) \right\} \qquad \forall u, v \in \mathbb{O}$$

where $\overline{\chi}_g$: $\mathrm{Spin}_7 \to SO(\mathrm{Im}\,\mathbb{O}) \cong SO_7$ is defined by $\overline{\chi}_g(v) = g(g^{-1}(1) \cdot v)$ for all $v \in \mathbb{O}$, which is the standard double cover of SO_7 by Spin_7.

- $\mathrm{Spin}_7 = \left\{ (g_1, g_2) \in SO(8) \times SO(8) \,\middle|\, g_1(xy) = g_1(x) g_2(y) \right\}$

- $\mathrm{Spin}_7 = \left\{ (g_1, g_2, g_3) \in SO(8) \times SO(8) \times SO(8) \,\middle|\, g_1(xy) = g_2(x) g_3(y) \right\}$ (Cartan triality)

- $\mathrm{Spin}_7 =$ the subgp of SO_8 generated by $S^6 \equiv \{R_u : u \in \mathrm{Im}\,\mathbb{O},\ |u| = 1\}$

Theorem *The action of $Spin_7$ on $G(\Phi)$ is transitive with isotropy subgroup $K = SU(2) \times SU(2) \times SU(2)/\mathbb{Z}_2$. Thus $G(\Phi) \cong Spin_7/K$.*

$G(\Phi) \triangleq$ the set of Cayley planes

We define an action $(q_1, q_2, q_3) \in K$ as following:

$$\mathbb{H} \oplus \mathbb{H} \longrightarrow \mathbb{H} \oplus \mathbb{H}$$
$$a + be \longmapsto q_3 a \bar{q}_1 + (q_2 b \bar{q}_1) e$$

4. Cayley Manifolds

- Call a 4-mfld a Cayley manifold if all its tangent planes are Cayley planes.

  [A Cayley plane is a plane where Φ achieves the comass 1.]

- The geometry of Cayley submanifolds includes several other geometries.

Call them trivial Cayley Manifolds

(1) *A submanifold M which lies in* $\mathrm{Im}\,\mathbb{O} \subset \mathbb{O}$ *is Cayley if and only if M is coassociative.*

(2) *A submanifold M of $\mathbb{O}$ of the form $\mathbb{R} \times N$, where N is a submanifold of* $\mathrm{Im}\,\mathbb{O}$, *is Cayley if and only if N is associative.*

(3) *Fix a unit imaginary quaternion $u \in S^6 \subset \mathrm{Im}\,\mathbb{O}$. Consider the complex structure J_u and let $\mathbb{O} \cong \mathbb{C}^4$. Each complex surface in $\mathbb{O}$, with the reverse orientation, is a Cayley submanifold.*

(4) *In addition to choosing one of the distinguished complex structures J_u (as in (3)) choose a quaternion subalgebra $\tilde{\mathbb{H}}$ of $\mathbb{O}$ orthogonal to u and identify $\mathbb{R}^4 \subset \mathbb{C}^4$ with $\tilde{\mathbb{H}} \subset \mathbb{O}$. Each special Lagrangian submanifold of $\mathbb{C}^4 \cong \mathbb{O}$ is a Cayley submanifold.*

Why are Cayley Manifolds Interesting?

- There has been new interest recently in the geometry of Cayley cycles. Following SYZ, the roles of exceptional geometries in mirrow symmetry were first investigatived by K. Becker, M. Becker, D. Morrison, H. Ooguri, Y. Oz and Z. Yin.

- From physics point of view, the authors showed that the Cayley cycles in $Spin_7$ holonomy eight-mflds preserves half of the space time supersymmetry They discovered that while the holomorphic and special Lagrangian cycles in Calabi-Yau 4-folds preserve half of space-time super-symmetry, the Cayley submflds are novel as they preserve only one quarter of it.

- They also conjectured what kind of roles Cayley cycles will play in mirror symmetry for Calabi-Yau 4-folds (in contrast to the roles of holomorphic and special Lagrangian cycles in mirror symmetry of CY 3-folds).

- Proposed the problem of finding explicit examples of Cayley cycles to demonstrate their conjectured phenomenon

Goal: Find explicit examples of (nontrivial) (associative, co-associative &) Cayley manifolds.

Especially, those are complete!

Recall:

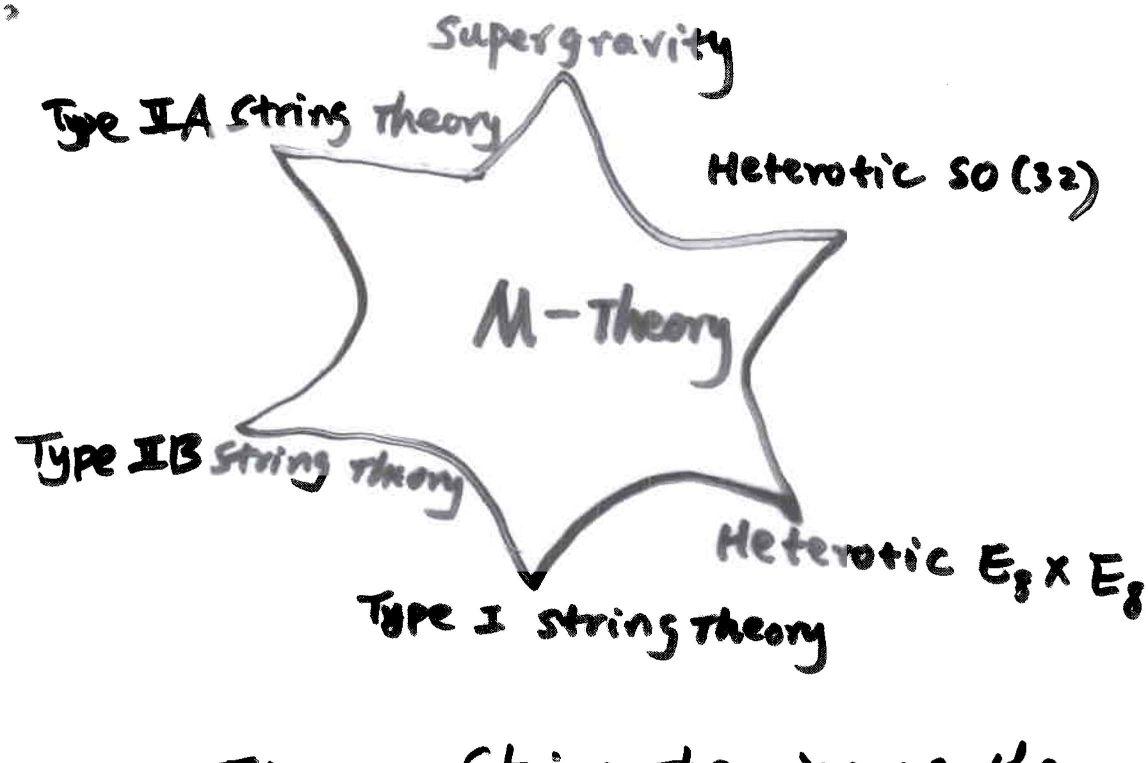


Figure: String theories as the limits of M-theory

e.g. (want) a family of associative cycles $\xrightarrow{\text{asymtc}}$ $\mathbb{R} \times SLG$ (or $S^1 \times SLG$)

(want) a family of associative cycles $\xrightarrow{\text{approach}}$ associative cones

Theorem *Suppose* $f : \Omega \subset \mathbb{H} \to \mathbb{H}$ *is* C^1. *The graph of* f *is a Cayley manifold if and only if* f *satisfies the differential equations*

$$\begin{cases} Df = \sigma f \\ \delta f = 0 \end{cases}$$

Where: The *Dirac operator* D is defined on f as

$$Df = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\hat{\imath} - \frac{\partial f}{\partial x_3}\hat{\jmath} - \frac{\partial f}{\partial x_4}\hat{k}$$

The *first order Monge-Ampere operator* on f is defined as

$$\sigma f = \left(\frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right) + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right)\hat{\imath} -$$

$$\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_4}\right)\hat{\jmath} + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}\right)\hat{k}$$

and a third operator is defined by

$$\delta f = \mathrm{Im}\left[\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right)\hat{\imath}\right] +$$

$$\mathrm{Im}\left[\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_4}\right)\hat{\jmath} + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_4} - \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}\right)\hat{k}\right]$$

Recall:

For a compact Kähler mfld, with first Chern class zero, there is unique Ricci flat metric in each Kähler class.

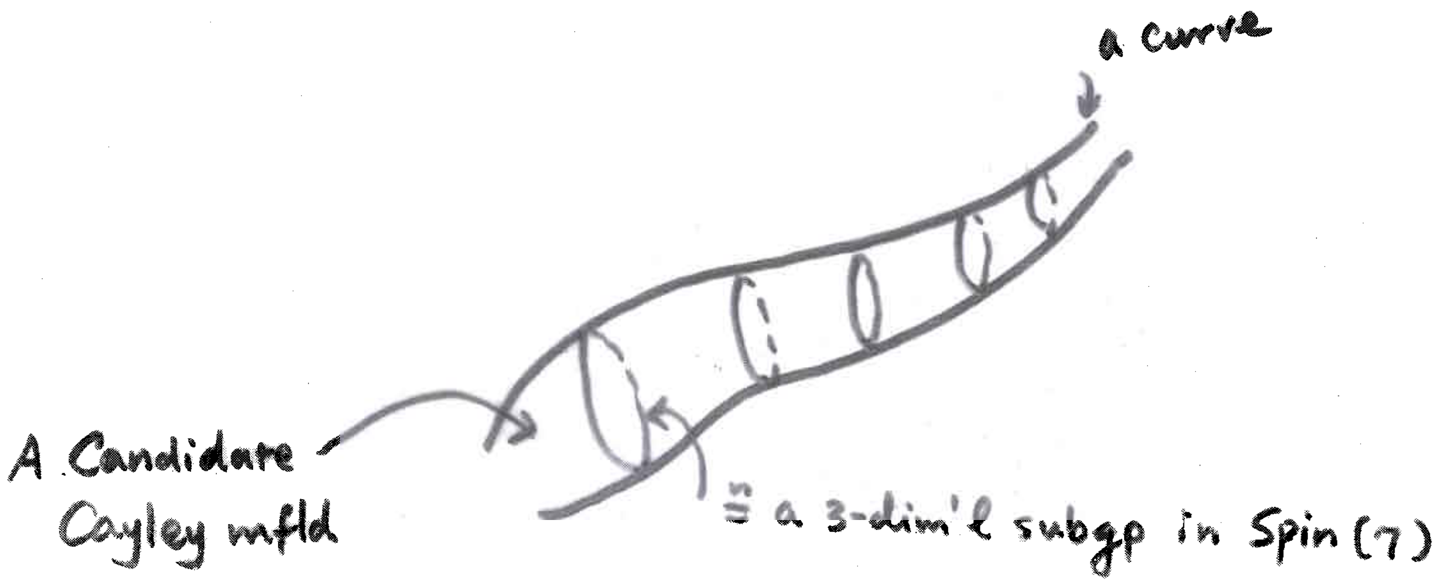
Yau reduced problem to solve the complex Monge - Ampère eqn:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z \partial \bar{z}}\right) = e^F \det(g_{i\bar{j}})$$

5. Techniques we used to find Cayley Manifolds

(Add symmetry Restrictions to Simplify the PDE)

We are going to fix a nonzero vector ϵ in $\mathbb{H}$ and seek a curve in the plane $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}\epsilon \oplus \mathbb{R} \subset \mathbb{O}$ which is going to sweep out a candidate Cayley manifold under the action of some 3-dimensional subgroup of Spin_7. Then the PDE will be simplified to an ODE or to one with less variables.



- Example of 3-dim'l subgps we used:

$$(1,1,q) = \left\{ (1,1,q) \;\middle|\; q \in Sp_1 \right\} \subset \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2} = K \subseteq Spin_7$$

Similarly we denote the other key subgroups we are going to consider as (1,q,1), (1,q,q), (q,1,1), (q,1,q), (q,q,1), and (q,q,q).

- In fact, we considered all subgps in K, and showed that these can be reduced to consider the key subgps above.

Further Simplification of PDE

Theorem *Let M be a 4-submanifold of $\mathbb{O} \cong \mathbb{R}^8$, symmetric under the action of $< g > \subset Spin_7$, the subgroup generated by g, where the action is defined above. Let $\alpha \in M$. Let ξ_α be an oriented tangent 4-plane of M at α in $\mathbb{O}$. Similarly let $\xi_{g(\alpha)}$ be the oriented tangent 4-plane of M at $g(\alpha) \in M$ in $\mathbb{O}$ with orientation inherited from ξ_α via g. Then,*

$$\Phi(\xi_{g(\alpha)}) = \Phi(\xi_\alpha) \tag{14}$$

where Φ is the Cayley calibration.

Corollary *Let $M = \{f(x) + x\mathbf{e} \mid x \in \mathbb{H}\}$ be a graph over $\mathbb{H}\mathbf{e}$ that is symmetric under the action of the group generated by $(q_1, q_2, q_3) \in K \cong \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$. Then $(Df - \sigma f)(q_2 x \overline{q_1}) = 0$ and $\delta f(q_2 x \overline{q_1}) = 0$ if and only if $(Df - \sigma f)(x) = 0$ and $\delta f(x) = 0$.*

6. Examples of Cayley Manifolds

Theorem [redacted] *For any $c \in \mathbb{R}$ fixed, the graph*

(1)
$$M_c = \left\{ s\frac{x}{|x|} + x\mathrm{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - s^3|x| = c \right\}$$

is a Cayley manifold in $\mathbb{R}^8$.

Theorem [redacted] *For $c, k, s \in \mathbb{R}$ constants, and $\varphi : \mathbb{H} \to \mathbb{R}$, the graph*

(2)
$$M_{c,k,s} = \left\{ x\varphi(x) + s\varphi(x) + k + x\mathrm{e} \;\middle|\; \forall x \in \mathbb{H}, \varphi(x) - \varphi^3(x) = \frac{c}{(|\mathrm{Im}\, x|^2 + (s + \mathrm{Re}\, x)^2)^2} \right\}$$

is a Cayley Manifold in $\mathbb{R}^8$.

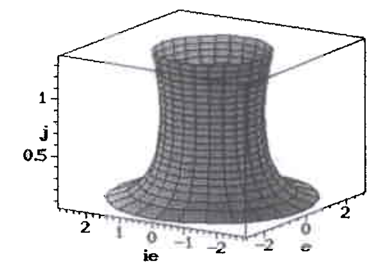


FIGURE 3. A slice of the Cayley manifold of Theorem 2.2 in $\{e, ie, j\}$ space. Here again $x_4 = 0$ and $x_1^2 + x_2^2 = x_3^2$.

2.3. **The Proof of Theorems 2.1 and 2.2.** Before proving Theorems 2.1 and 2.2, we will introduce two lemmas.

Lemma 2.3. *Assume that* $M = \left\{ f(x) + x\mathbf{e} \,\middle|\, x \in \mathbb{H} \right\} \subset \mathbb{O}$ *is the graph of a function* $f : \mathbb{H} \to \mathbb{H}$. *Then* M *is invariant under the action (26) of* Sp_1 *if and only if*

$$f(x) = \frac{x}{|x|} f(|x|) \tag{29}$$

PROOF. If M is invariant under the action (26) above, then for each $q \in Sp_1 \subset \mathbb{H}$ and each $x \in \mathbb{H}$, the point $qf(x) + (qx)\mathbf{e}$ also belongs to M. Thus

$$f(qx) = qf(x) \tag{30}$$

for all $q \in Sp_1$ and all $x \in \mathbb{H}$. Now by replacing x by $|x|$ and q by $\frac{x}{|x|}$ in equation (30) recover equation (29). Now consider a function characterized by equation (29). Plugging in qx yields

$$f(qx) = \frac{qx}{|qx|} f(|qx|) = q\frac{x}{|x|} f(|x|) = qf(x) \tag{31}$$

for all $q \in Sp_1$ and all $x \in \mathbb{H}$. Thus we obtain equation (30), and the graph of f is invariant under action (26) □

By Theorem 1.6, for f symmetric under the action (26), we have $(Df - \sigma f)(qr) = 0$ if and only if $(Df - \sigma f)(r) = 0$. Thus it is enough to compute $Df - \sigma f$ at $x = |x|$.

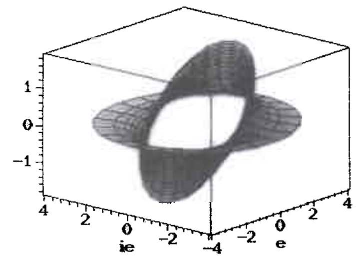


FIGURE 1. A slice of the Cayley manifold of Theorem 2.2 in $\{e, ie, real\}$ space. Here $x_3 = x_4 = 0$.

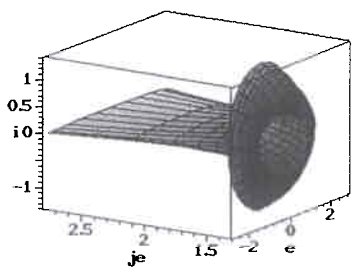


FIGURE 2. A slice of the Cayley manifold of Theorem 2.2 in $\{e, je, i\}$ space. Here $x_4 = 0$, and $x_1^2 + x_2^2 = x_3^2$.

Theorem 2.1. *For any $\varepsilon \in \operatorname{Im}\mathbb{H}$ fixed, for any $c \in \mathbb{R}$ fixed, the graph*

$$M_{\varepsilon,c} = \left\{ cx\varepsilon + x\mathbf{e} \;\middle|\; x \in \mathbb{H} \right\} \tag{27}$$

is a Cayley manifold in $\mathbb{R}^8$.

Theorem 2.2. *For any $k \in \mathbb{R}$ fixed, for any $c \in \mathbb{R}$ fixed, the graph*

$$M_{k,c} = \left\{ ks\frac{x}{|x|} + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - k^2 s^3 |x| = c \right\} \tag{28}$$

is a Cayley manifold in $\mathbb{R}^8$.

The first three figures depict slices of one member of the second family of Cayley Manifolds. The specific Cayley Manifold is $M = \{s\frac{x}{|x|} + x\mathbf{e} \mid x \in \mathbb{H}, |x|^3 s - s^3|x| = 5\}$. These figures begin to show the intricate structure of this family of manifolds.

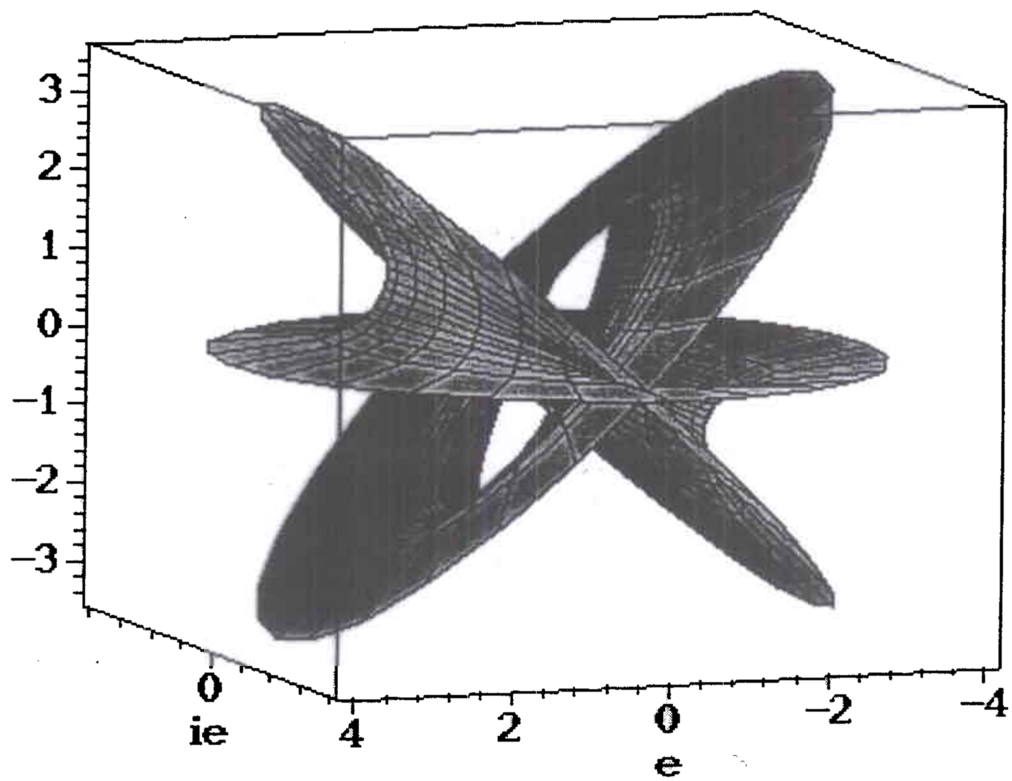


Figure 6.1: A slice of the Cayley manifold of Thm 6.2.2 in $\{1, i, e\}$ space

- We classified all subgroup of dim 3 in
  $K = SU(2) \times SU(2) \times SU(2) / \mathbb{Z}_2$
- We used each such subgp as an action to add symmetry Restrictions to simplify the PDE.
- We solved simplified PDEs or ODEs in each case
- We proved uniqueness of obtained list of Cayley cycles in this content.

Theorem 6.5. *Let M be the graph of a function, $f : \mathbb{H} \to \mathbb{H}$, symmetric under the action (78), so that f obeys Theorem 6.4, i.e. $f(x) = x\varphi(\mathrm{Re}\, x, |\mathrm{Im}\, x|) + \nu(\mathrm{Re}\, x, |\mathrm{Im}\, x|)$ for $x \in \mathbb{H}$. Then M is Cayley if and only if*

$$(1-\varphi^2(r,s))\left(s\frac{\partial\varphi}{\partial r}(r,s) - r\frac{\partial\varphi}{\partial s}(r,s) - \frac{\partial\nu}{\partial s}(r,s)\right) = 0 \tag{95}$$

$$\left(r\frac{\partial\varphi}{\partial s}(r,s) + \frac{\partial\nu}{\partial s}(r,s) - s\frac{\partial\varphi}{\partial r}(r,s)\right)\varphi(r,s) = 0 \tag{96}$$

and

$$\begin{aligned}
&4\varphi(r,s) - 4\varphi^3(r,s) + \frac{\partial\nu}{\partial r}(r,s) + r\frac{\partial\varphi}{\partial r}(r,s) + s\frac{\partial\varphi}{\partial s}(r,s) \\
&-3\varphi^2(r,s)\frac{\partial\nu}{\partial r}(r,s) - 3r\varphi^2(r,s)\frac{\partial\varphi}{\partial r}(r,s) - 3s\varphi^2(r,s)\frac{\partial\varphi}{\partial s}(r,s) \\
&+2s\varphi(r,s)\frac{\partial\nu}{\partial r}(r,s)\frac{\partial\varphi}{\partial s}(r,s) - 2s\varphi(r,s)\frac{\partial\varphi}{\partial r}(r,s)\frac{\partial\nu}{\partial s}(r,s) = 0
\end{aligned} \tag{97}$$

for $r, s \in \mathbb{R}^+, r = \mathrm{Re}\, x, s = |\mathrm{Im}\, x|$.

- Each member in above family is complete!
- In certain cases, they represent certain kinds of cone.
- A family of Cayley submflds $\xrightarrow[\text{to}]{\text{approach}}$ a cone
- In the case of a cone, then the solution is a Lipschitz solution to Cayley eqns which is not C^1.

7. Similar Techniques used to find Special Lagrangian mflds & Associative mflds

- With Ian Weiner
  - We used similar techniques to find many interesting examples of associative mflds that are invariant under 1-parameter subgp of G_2.
  - Most of these examples are new.
  - We examined all possible 1-para. subgps of $SO(4)$.
  - We proved uniqueness theorems.

Recall:

- $G_2 = \{ g \in GL_8(\mathbb{R}) \mid g(xy) = g(x)g(y),\ \forall x, y \in \mathbb{O} \}$
- $G_2 = \{ g \in O(7) \mid g^*\varphi = \varphi \}$
- $G(\varphi) = G_2 / SO(4)$

8. Work in Progress

- Cayley cycles in $\mathbb{T}^8$
- Systematic way to construct Cayley cycles in $\mathbb{T}^8/\Gamma$
- $\mathfrak{M}$ (Cayley cycles in $Spin_7$-mfld M, $\partial M = G_2$ with bdry an associative cycle in G_2)
- Regularizing Singularities

By removing singularity, then adding a Lawlor neck

Defn: Given any two n-dim'l oriented linear subsps ξ and η of $\mathbb{R}^{2n}$, there exist characterizing angles $\theta_1, \cdots, \theta_n$, together with an O.N. basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$ s.t.

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_{n-1} \le \pi/2 \quad , \quad \theta_{n-1} \le \theta_n \le \pi - \theta_{n-1}$$

where $\xi = e_1 \wedge \cdots \wedge e_n$

$$\eta = (\cos\theta_1 e_1 + \sin\theta_1 e_{n+1}) \wedge \cdots \wedge (\cos\theta_n e_n + \sin\theta_n e_{2n})$$

Angle Thm: (Lawlor–Nance) A pair of oriented planes (ξ, η) is minimizing iff the characterizing angles between ξ & η satisfy $\sum_{i=1}^{n} \theta_i \ge \pi$.

- We say ξ and η satisfy the angle criterion if the equality $\sum_{i=1}^{n} \theta_i = \pi$ holds.

- If ξ and η satisfy the angle criterion, then there $\exists$ a family of Lawlor-necks asymptotic to the union of these two planes.

*We have used above to Cayley planes.

- By remove singularity then doing connected sum

Let M_1 & M_2 be Cayley submflds of a compact Spin_7 mfld X^8 that intersect transversely at a single pt. $M_1 \cup M_2$ can be thought of as a single Cayley submfld of X with a single isolated singularity.

Q: Can we regularize $M_1 \cup M_2$ (in certain sense)?

Hope: There $\exists$ a family of Spin(7) structure X_α on X and family of Cayley submflds M_α of X_α s.t.

$$\begin{cases} M_\alpha \to M_1 \cup M_2 \\ \text{Spin7 structure } X_\alpha \to \text{Spin7 structure of } X \end{cases}$$

Regularization seems exist

① When $X = T^4_{\mathbb{C}}$, M_1 is flat and intersection of M_1 & M_2 satisfies a certain angle criterion

② When $[M_1]$ is not a multiple of $[M_2]$ in $H_4(X)$.

Q: Can we do above without deform Spin_7 struc?