Volume Minimizing Cycles in Compact Lie Groups

(Talk at MSRI Aug 2003)

I. Introduction:

G cpt simple Lie group with bi invariant metric
$\mathfrak{g}$ Lie alg of G.

- Topology: $H_*(G; \mathbb{Q}) \cong H_*(\text{Product of odd dim. spheres}; \mathbb{Q})$

$$H^*(G, \mathbb{R}) \cong \{\text{bi-invariant forms}\}$$
$$\cong \{Ad\text{-invariant forms on } \mathfrak{g}\}$$

- Geometry:
  - Easy to construct embedding in $\mathbb{R}^N$ (matrices).
  - Symmetric spaces.
  - Totally geod. submflds $\leftrightarrow$ Lie Triple systems. $[[V, V], V] \subset V$ for $V \subset \mathfrak{g}$.

Q: Find all cycles in G which have the smallest volume in their homology classes with coefficient ring $= \mathbb{Z}, \mathbb{Z}_2, \mathbb{R}$.

studied by Fomenko (1972), Dao Chong Thi (1982)
Tasaki (1985) – Ohnita, Pedersen (1988)

Fund. 3-form φ

$$\varphi(X, Y, Z) := \langle [X, Y], Z \rangle \quad \text{for } X, Y, Z \in \mathfrak{g}.$$

<u>Tasaki</u> 1985 : φ calibrates 3-dim'l subalg ($\cong so(3) = su(2)$) subgp $\cong SU(2)$ or $SO(3)$.
corresponding to maximal roots.
(unique up to conjugacy)

$*\varphi$ calibrates cut locus of any point in G.

<u>Ohnita - Tasaki</u> '87 ? : Cycle calibrated by φ is unique up to translation.

<u>R. Bryant</u> (To appear) Moduli space of cycles calibrated by $*\varphi$.

<u>II. Pontryagin cycles in classical Lie gps.</u>

- $G = SO(n)$.

For $x \in \mathbb{R}^n$, $x \neq 0$, $\rho_x : \mathbb{R}^n \to \mathbb{R}^n$ reflection along x
$\rho_x \in O(n)$.

$\{e_1, \cdots, e_n\}$ standard O.N. basis.

$$P_{n-1} := \{\rho_x \rho_{e_1} \mid x \in S^{n-1} \subset \mathbb{R}^n\} \subset SO(n).$$

$\cong \mathbb{R}P^{n-1}$ <u>Pontryagin cycle</u>

If $M, N \subset G$ two cycles

$$M \cdot N := \{ xy \mid x \in M,\ y \in N \}.$$

Pontryagin product.

Thm (Pontrygin, 1939)

$$P_1 \subset P_2 \subset \cdots \subset P_{n-1} \subset SO(n)$$

and their products generates almost all $H_*(SO(n);\mathbb{Z})$ except certain torsion classes.

- $G = SU(n)$

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n, \quad \|z\| = 1, \quad 0 \le t \le 2\pi$$

$$T_{2n-1} := \left\{ e^{-it} \left(Id - (1 - e^{it})\, z\, \bar{z}^t \right) \right\} \subset SU(n)$$

$\Sigma \mathbb{C}P^{n-1}$ special unitary matrix with $(n-1)$ equal eigenvalues

suspension of $\mathbb{C}P^{n-1}$ Pontryagin - Thi cycle

~~Thm~~ Pontryagin (1939): ~~Thm~~

Not totally geod. singular.

$$T_1 \subset \cdots \subset T_{2n-1} \subset SU(n)$$

and their products generates $H_*(SU(n);\mathbb{Z})$

- $G = Sp(n)$ similar. dim of Pont. cycle $= 4n-1$

Q: Are Pont. cycles homol Vol. min. ?

III. Bi-inv. forms corresp. to Pontryagin cycles.

- All bi-inv. forms on $\mathfrak{g}$ are known

  Cartan - Brauer.

- Non-conventional way to define ω_{n-1} on $SO(n)$.

  (L —) $\quad \omega_{n-1} = \omega$

$M_1, \cdots, M_{n-1} \in$ Lie alg of $SO(n)$.

$$\omega(M_1, \cdots, M_{n-1}) := -\int_{x \in S^{n-1} \subset \mathbb{R}^n} \det(x, M_1 x, \cdots, M_{n-1} x)\, dx$$

column vector.

Standard vol elem of S^{n-1}

Ad-inv. on Lie alg. of $SO(n)$.

$\Rightarrow$ Bi-inv. form on $SO(n)$.

$\omega = 0$ if n odd.

$\frac{1}{\|\omega\|^*}\, \omega \Rightarrow$ A. Calibration.

Conj.: ω calibrates P_{n-1} for $n > 4$, n even.

$$\|\omega\|^* = \frac{1}{2^{n-2}} \mathrm{Vol}(S^{n-1}) \cdot \frac{1}{2^{\frac{n-1}{2}}},$$

$*\omega$ calibrates $SO(n-1)$.

$\Rightarrow 1/\sqrt{2}^{(3n-5)}$

Remark. ① False for $n=4$.

$$\begin{array}{c} SU(2)\times SU(2) \\ \downarrow \\ SO(4) \end{array}$$

φ fund. 3-form, ω_3 3-form.

$$\varphi(T_e P_3) = 0 \neq \omega_3(P_3)$$

Both ω_3 and φ calibrate ~~these~~ the 2-copies of $SU(2)$.

P_3 is not vol. min. in $\mathbb{R}$-homol. class

$SO(3) \subset P_3 \subset SO(4)$

② $SU(n) \xhookrightarrow{i} SO(2n)$, $Sp(n) \xhookrightarrow{j} SO(4n)$

$i^* \omega_{2n-1}$ should calibrate T_{2n-1} (True for $SU(3)$)

$j^* \omega_{4n-1}$ ------ Part cycle in $Sp(n)$.

Evidence:

Thm A (L) ω_5 in $SO(6)$ calibrates P_5.

$* \omega_5$ ------ $SO(5)$

Pf: Brutal Force computation.

Thm B (L–) : For all n:

$$P_1 \subset P_2 \subset \cdots \subset P_{n-1}$$

are all volume min. in $\mathbb{Z}_2$-homol. class in $SO(n)$.

Cor : For all ~~odd~~ k, $1 \le k \le n-1$,

P_k ~~are~~ volume min. in $\mathbb{Z}$-homol. class in $SO(n)$.

Rem : ① when $n=4$, $P_3 \subset SO(4)$ is vol. min. in $\mathbb{Z}$-homology class, but not in its $\mathbb{R}$-hom. class.

② min $\mathbb{Z}_2$ ⇒ min. in $\mathbb{Z}$ ⇐ min. in $\mathbb{R}$.

Idea for the pf of Thm B :

$$\begin{array}{ccc}
Cl_n & \hookrightarrow & \mathbb{R}^{N+1} \\
\cup & & \cup \\
Spin(n) & \hookrightarrow & S^N \\
\downarrow & & \downarrow \\
SO(n) & \xhookrightarrow{\text{homothety}} & \mathbb{RP}^N \\
\cup & & \cup \\
P_k & \longrightarrow & \mathbb{RP}^k
\end{array}$$

Construct, totally good.

$k = 1, \cdots, n-1$

Berger, Fomenko (1972)

$\mathbb{RP}^k \hookrightarrow \mathbb{RP}^N$ vol. min. in $\mathbb{Z}_2$-homol. class.

Conj. $SO(n-1)$ is vol. min. in $\mathbb{Z}_2$ homol. class in $SO(n)$.

$$\Uparrow \text{ Poincaré Formula.}$$

Rem: Need

$$\int_{x \in S^{n-1}} |\det(x, M_1 x, \ldots, M_{n-1} x)| \, dx \le \mathrm{Vol}(S^{n-1})$$

$$\frac{1}{2^{\frac{3n-5}{2}}} = 2^{-\frac{3n-5}{2}}$$

for $M_i \in so(n)$ Lie alg.

$\|M_i\| = 1$.

without $|\cdot|$, needed in computing $\|\omega\|^*$.

IV Rigidity of Gauss map

$M^k \hookrightarrow G$.

Gauss map Φ:

$$\begin{array}{ccc} M & \longrightarrow & G_{\mathbb{R}}(\mathfrak{g}) \\ \cup & & \cup \\ x & \longmapsto & (L_{x^{-1}})_* T_x M \subset \mathfrak{g} \end{array}$$

~~Expect Φ If M calibrated by~~

Q. Can $\Phi(M)$ determine M?

G acts on $G_k(\mathfrak{g})$ by adjoint action.

$V \in G_k(\mathfrak{g})$.

$\mathcal{O}(V)$ = adjoint orbit of V

① Expect: If M is calibrated by a bi-inv. form, M often satisfy $\Phi(M) \subset \mathcal{O}(V)$.

Ex: True for fund. 3-form φ, $*\varphi$.

② $\mathbb{CP}^n \hookrightarrow SU(n+1)$ ω_5 in $SO(6)$, $*\omega_5$

Lagrangian submfld in $\mathbb{CP}^n$ are V-submfld for some V

Def: $M \subset G$ is a V-submfld if

$$\Phi(M) \subset \mathcal{O}(V).$$

Q: ① What is the moduli space of V-submflds?

② What geom. properties V-submflds must have?

Totally geod? Minimal? Vol. min.?

Ex: $G = SO(3)$, $SU(2)$.

For any $V \subset \mathfrak{g}$, $\mathcal{O}(V) = G_k(V)$.
dim k

any submfld are V-submfld.

Thm C. If $\mathfrak{h} \subset \mathfrak{g}$ semisimple subalg.

(no requirement on $\mathfrak{g}$ except G has biinv. metric).

$\mathfrak{h}$ - submflds are totally geodesic

complete $\mathfrak{h}$ - submflds = Translations of Lie subgp H.

Rem. ① False if $\mathfrak{h}$ is not semisimple

In general $\mathfrak{h}$ -submfld

$$M \xhookrightarrow{i} G.$$

$$(M, i^*g) \overset{\text{isom}}{\cong} \underbrace{\mathbb{R}^k}_{\text{flat}} \times H'$$

H': subgp with Lie alg $[\mathfrak{h}, \mathfrak{h}]$.

flat: embedding in G could be arbitrary.

② Dim $\mathcal{O}(\mathfrak{h})$ may be much larger than $\dim \mathfrak{h} = \dim M$ for $\mathfrak{h}$-submfld M.

$$\Phi(M) \subset \mathcal{O}(\mathfrak{h}) \implies \Phi(M) = \{pt\}.$$

Ex: $\mathfrak{h} = so(3) \subset so(n)$, $\mathcal{O}(\mathfrak{h}) = SO(n)/SO(3)$ $\dim = \frac{n(n-1)}{2} - 3$

Cor Up to translation $SO(5)$ is the only vol. min cycles in its $\mathbb{R}$-homology class. (cal. by $*\omega_5$)

Thm D: For $n \geq 4$, $m, k \neq 2$

$\mathfrak{g} = so(n)$, $\mathfrak{h} = so(k) \oplus so(m)$.

Then $\mathfrak{h}^\perp$-submanifolds are totally good.

(If complete $\Rightarrow$ Cartan immersion of $G_k(\mathbb{R}^n)$ in $SO(n)$ up to translation).

Cor Up to translation, P_5 is the ~~only~~ unique cycle calibrated by ω_5.

Idea of Pf for Thm C & D

Prop: G any cpt Lie gp.

$V \subset \mathfrak{g}$ $\dim V = k$.

$$n(V) := \{ x \in \mathfrak{g} \mid [x, V] \subset V \}$$

$T_eM = V$

$e \in M$, V-submfld. Then $\exists$ linear map $\tau : V \to n(V)^\perp$ such that the 2nd fund. form of M at e is given by

$$\mathrm{II}(X,Y) = \left(\tfrac{1}{2}[X,Y] + [\tau(X),Y] \right)^{V^\perp}$$

for any $X, Y \in V = T_e M$. $\boxed{\S}$

<u>Observation</u>: II symmetric.

$[\ ,\ ]$ Anti-symm.

If alg. property of V nice. (e.g. Thm C and D)

$\Rightarrow \mathrm{II} \equiv 0$.

<u>Rem for Thm D</u>:

① When m, or $k = 2$, "probably" false.

② Might be able to prove similar result for most symmetric pairs. (Don't have a unified approach).