1.

(Algebraic geometer's)

Outsider's Survey of

Constant scalar curvature Kähler metrics and stability of algebraic varieties.

with Julius Ross

1. GIT
2. Symplectic reduction
3. Balanced varieties, cscK metrics
4. Bundle analogue
5. Unstable varieties

2.

Geometric Invariant Theory.

$$\begin{array}{ccc} G & \curvearrowright & X \\ \cap\! \mid & & \cap\! \mid \\ SL(N+1,\mathbb{C}) & & \mathbb{P}^N \end{array} \qquad X/G\,?$$

G-action not proper

Quotient not Hausdorff "separated"

GIT removes "unstable" orbits to get projective quotient. Identifies some "semistable" orbits to compactify - "proper"

$$X \leftrightarrow \bigoplus_r H^0(\mathcal{O}_X(r)) \qquad X/G \leftrightarrow \bigoplus_r H^0(\mathcal{O}_X(r))^G$$

G acts on $\mathbb{C}^{N+1}$ so on $\mathcal{O}(-1) \to \mathbb{P}^N$ so on $\mathcal{O}(r) \to X$

3.

$x \in X$ semistable if $\exists f \in H^0(\mathcal{O}_X(r))^G$ s.t. $f(x) \neq 0$

Kodaira "embedding" via $H^0(\mathcal{O}_X(r))^G$ defined $X \dashrightarrow \mathbb{P}((H^0(\mathcal{O}_X(r))^G)^*)$ at x.

Stable $\Leftrightarrow$ $H^0(\mathcal{O}_X(r))^G$ separates orbits at $G.x$ and $\mathrm{Stab}_G(x)$ finite.

$H^0(\mathcal{O}_X(r))$ = fns (deg r polys) on $\tilde{X} \subseteq \mathbb{C}^{N+1}$

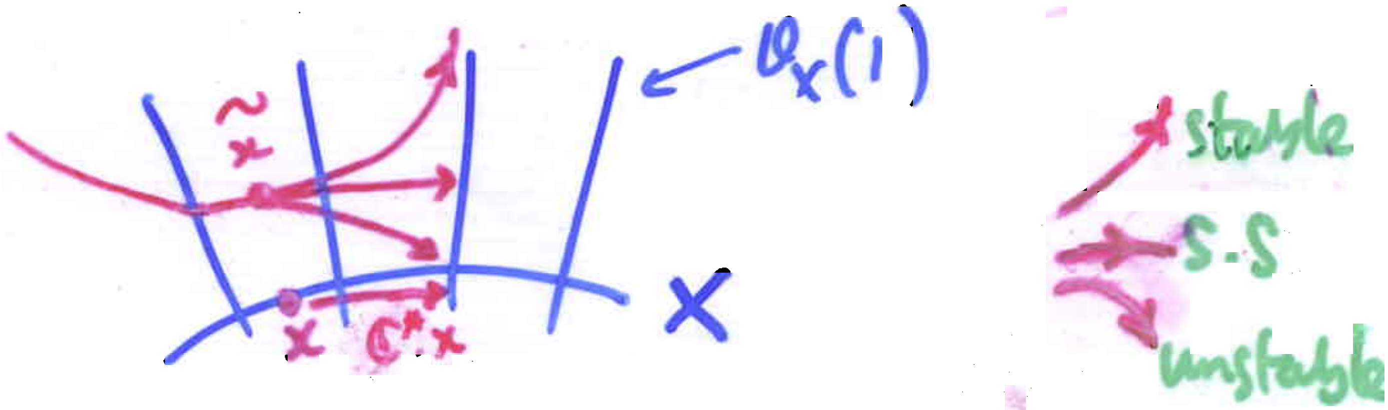
Thm [Mumford] Stable $\Leftrightarrow$ $G.\tilde{x}$ closed in $\mathbb{C}^{N+1}$ and of $\dim = \dim G$

Semistable $\Leftrightarrow$ $0 \notin \overline{G.\tilde{x}}$

Stability "generic" - Zariski open condition

Thm <u>Hilbert-Mumford criterion</u>

Same true iff true $\forall$ 1-param subgps $\mathbb{C}^* \subseteq G$.



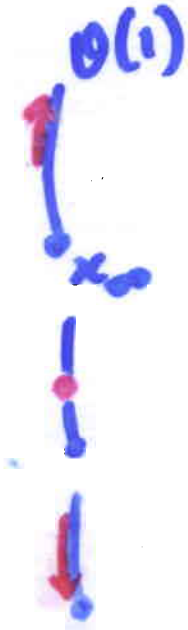
So all reduces to action of $\mathbb{C}^*$ on line over limit point $\lim_{\lambda \to 0} \lambda . x = x_\infty$

(fixed pt of $\mathbb{C}^*$ action in X, so get action on line $\mathcal{O}(-1)\big|_{x_\infty}$)

5.

$$\mathbb{C}^* \longrightarrow \mathrm{Aut}\left(\mathcal{O}(1)_{x_\infty}\right) = \mathbb{C}^*$$

weight $\rho \in \mathbb{Z}$ $\quad \lambda \mapsto \lambda^\rho$



$\rho < 0$ stable

$\rho = 0$ semistable

$\rho > 0$ unstable

So "just" compute this limiting weight $\forall\, \mathbb{C}^* \subseteq G$ 1-p.s.

If < 0 always then stable.

6.

Fundamental example

n points in $\mathbb{P}^1$.

/length n subscheme

od algebraic variety / scheme!

$G = SL(2,\mathbb{C}) \curvearrowright \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$

$\Rightarrow G \curvearrowright S^n(\mathbb{C}^2)^* = \left\{ \begin{array}{l} \text{deg } n \text{ polys} \\ \text{on } \mathbb{C}^2 \end{array} \right\}$

$= H^0(\mathcal{O}_{\mathbb{P}^1}(n))$

$\{n \text{ points}\} = \mathbb{P}(\quad \uparrow \quad)$

roots of deg n poly

Thm n points semistable $\Leftrightarrow$ Each multiplicity $\leq n/2$

stable $\Leftrightarrow$ $< \frac{n}{2}$ pts at any one point

7.

<u>Proof</u> $\mathbb{C}^* \subseteq SL(2,\mathbb{C})$; diagonalise

$$\begin{pmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{pmatrix} \quad \text{w.r.t. } [x:y] \text{ coords on } \mathbb{P}^1.$$

$k > 0$ wlog

Poly $f = \sum_{i=0}^{n} a_i x^i y^{n-i} \xrightarrow[\lambda \to 0]{\text{Scale}} a_i x^i y^{n-i} = f_\infty$

where i = smallest s.t. $a_i \neq 0$.

$f = a_i x^i \left(y^{n-i} + \frac{a_{i+1}}{a_i} x y^{n-i-1} + \dots\right)$

Weight on limit poly is $k(i-(n-i))$

$$W = k(2i-n)$$

So stable $\Leftrightarrow$ $W < 0 \ \forall \mathbb{C}^*$ $\Leftrightarrow$ $i < \frac{n}{2}$

$\Leftrightarrow$ f vanishes to order $< \frac{n}{2}$ at $x = 0$

8.

Symplectic Reduction

Compact group $K = G \cap SU(N+1)$ $\quad \mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$

acts on $\mathbb{P}^N$ preserving J and $g \Rightarrow \omega$.

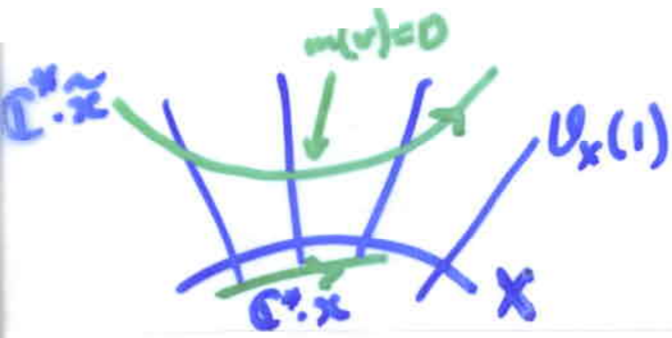
$\Rightarrow \forall\, v \in \mathfrak{k} = LK$, infinitesimal action X_v

is hamiltonian, $X_v \lrcorner\, \omega = d\, m(v)$

$m : X \longrightarrow \mathfrak{k}^*$ moment map

Collection of r hamiltonians; $r = \dim K$

$m(v)$ = derivative down $i(0,\infty) \subseteq \mathbb{C}^*$ orbit of $\log \|\lambda \tilde{x}\|_{\lambda \in \mathbb{C}^*}$ i.e. down JX_v



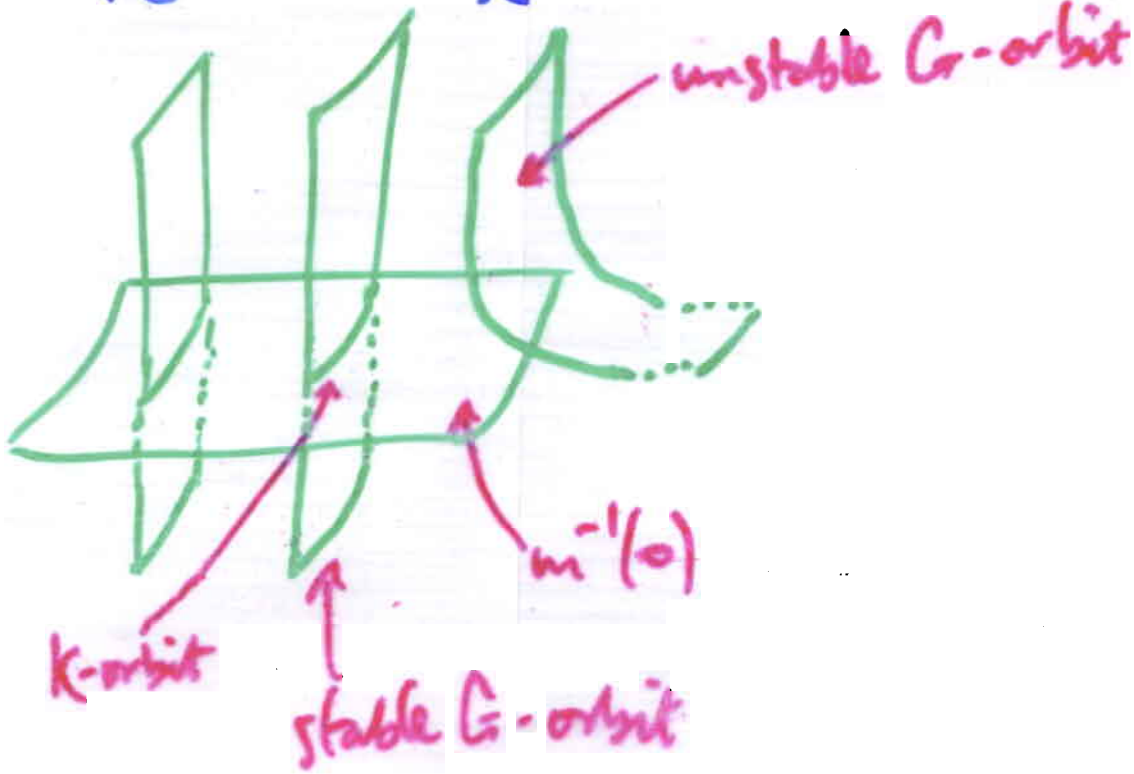
Stable $\Leftrightarrow \|\lambda \tilde{x}\|$ achieves min $\forall\, \mathbb{C}^* \subseteq G$

$\Leftrightarrow m(v) = 0$ on orbit $\forall v$

9. So we find:

Theorem [Kempf-Ness]

$$X/G \cong \frac{m^{-1}(0)}{K}$$



$m^{-1}(0)$ provides slice to $i\mathfrak{k} \subseteq \mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ part of orbit ; K-equivariant

Eg $\quad \dfrac{\mathbb{C}^n \setminus \{0\}}{\mathbb{C}^*} \cong \dfrac{S^{2n-1} = \{z : |z|^2 = a^2\}}{U(1)} \cong \mathbb{P}^{n-1}$

$m = |z|^2 - a^2$

10.

Eg n points in $\mathbb{P}^1$ again.

$$\mathfrak{su}(2) \xleftarrow{m} \mathbb{P}^1 \curvearrowleft SU(2) \subseteq SL(2;\mathbb{C})$$

is $S^2 \subseteq \mathbb{R}^3$ unit sphere

For n points get $S^n\mathbb{P}^1 \xrightarrow[\sum_{i=1}^{n} m_i]{} \mathbb{R}^3$

Sum of n pts $\in \mathbb{R}^3$ = centre of mass

$\Rightarrow$ Zeros of moment map = "<u>Balanced</u>" configs

centre of mass $0 \in \mathbb{R}^3$

Stable $\Leftrightarrow$ $\exists$ $SL(2,\mathbb{C})$ transf. of $\mathbb{P}^1$ s.t. pts are balanced

$\Leftrightarrow$ Mass of each pt $< \frac{n}{2}$ [+ case $\frac{n}{2}$  $n/2$
has dim Stabilizer = 1]

Polarised algebraic varieties (X, L)

$$X \hookrightarrow \mathbb{P}(H^0(X, L^r)^*) = \mathbb{P}^N, \; r \gg 0$$

defines a point in $\mathrm{Hilb} \subseteq \mathrm{Gr} \subseteq \mathbb{P}^M$

by subspace $\subseteq H^0(\mathbb{P}^N, \mathcal{O}(k))$ of deg k

$= S^k H^0_X(L^r)$

polys vanishing on $X \subseteq \mathbb{P}^N$.

i.e. pt. of $\Lambda^{\dim H^0_{\mathbb{P}^N}(\mathcal{I}_X(k))} S^k H^0_X(L^r)$ $\quad k, r \gg 0$

Divide by autos $SL(N+1, \mathbb{C})$ of $\mathbb{P}^N$ to get moduli of polarised varieties.

Choice of line bundle on $\mathrm{Hilb} \Rightarrow$ notion of stability for (X, L)

12.

Moment map for appropriate [ample] line bdl / symplectic structure on Hilb

Have $m: \mathbb{P}^N \longrightarrow \mathfrak{su}(N+1, \mathbb{C})$,

moment map takes $X \subseteq \mathbb{P}^N$ to

$$\int_X m \,\mathrm{vol}_{FS} \in \mathfrak{su}(N+1, \mathbb{C})$$

Centre of mass

Zeros of moment map = "Balanced" varieties $X \subseteq \mathbb{P}^N$

Luo: Balanced $\Rightarrow$ Hilbert-Mumford stable

13.

$r \to \infty$ ($\Rightarrow N \to \infty$) moment map has expansion Catlin, Lu, W.-D. Ruan, Tian, Zelditch involving S = scalar curvature & g_{FS}

Balanced metrics $\overset{\text{"tend}}{\underset{\text{towards"}}{- - \to}}$ cscK metrics

Donaldson:

(X, L) admits cscK metric in $c_1(L)$
$\Rightarrow$ Balanced for $r \gg 0$

(Luo $\Rightarrow$ HM stable) (Chen-Tian $\Rightarrow$ K-stable)

Also partial result in harder converse direction: if $(X, L^r) \subseteq \mathbb{P}^{N(r)}$ balanced for $r \gg 0$ and resulting $\omega_{FS(r)}$ convergent then $\lim_{r \to \infty} \omega_{FS(r)}$ has csc.

Donaldson also gives an infinite dimensional GIT/moment map formalism.

$$\text{Hamiltonian diffeos} \curvearrowright (X, \omega = c_1(L))$$

$$\curvearrowright \{\text{compatible cx. strs. on } X\}$$

Moment map = Scalar curvature − const.

Zeros = cscK metrics

Stability $\leftrightarrow$ cscK metrics suggested by Yau

($L = K_X^{-1}$, cscK = KE case)

and Tian in general.

15.

So have analogue of balanced pts and inf. dim version = cscK metrics.

Missing is alg. geom. description of stability
- Hilbert-Mumford criterion
- $< n/2$ points coincide.

<u>Bundle case</u> Moduli of bdls over (X, L)

similar but easier + worked out.

$$H^0(E(r)) \to E(r) \to 0 \quad \text{on } X$$

$E \otimes L^r$

$$\Rightarrow \mathrm{Map}(X, Gr) \qquad Gr \subseteq \mathbb{P}^N \subseteq \mathfrak{sl}(n+1, \mathbb{C})$$

Can again talk about <u>balanced</u> $X \to Gr$

+ asymptotics as $r \to \infty$ ($\Rightarrow N \to \infty$)

Limiting connection (pulled back via balanced map $X \to Gr$ from trivial/canonical connection on Gr)

$$\begin{tikzcd} \text{Limiting connection} \arrow[d, "\text{X.W. Wang}"', "\text{Stability}"] \\ \text{HYM connection} \end{tikzcd}$$

[Also (Atiyah-Bott) $\exists$ inf. dim picture $\mathcal{G} \curvearrowright \mathcal{A} = \{\text{conns}\}$

Moment map = HYM = $\omega . F_A^{1,1}$

In this case (Gieseker, Maruyama, Simpson) we can also do HM criterion so know what stability of E means in intrinsic algebro-geometric terms.

17.

$$\text{Hilbert poly} \quad h^0(E(r)) = \overset{\operatorname{rk}E\int_X \omega^n}{a_0} r^n + \overset{\int_X c_1(E)\omega^{n-1}}{a_1} r^{n-1} + \dots$$

$$\text{Reduced/monic poly} \quad \widetilde{h^0}(E(r)) = r^n + \frac{a_1}{a_0} r^{n-1} + \dots$$

E stable $\iff$ $\forall$ coherent subsheaves $F \hookrightarrow E$

$$\widetilde{h^0}(F(r)) < \widetilde{h^0}(E(r)) \quad \forall r \gg 0$$

E slope-stable $\iff$ $\dfrac{a_1(F)}{a_0(F)} < \dfrac{a_1(E)}{a_0(E)}$

ie $\mu(F) < \mu(E)$

[Corresponds to a different line bundle on moduli space – Jun Li]

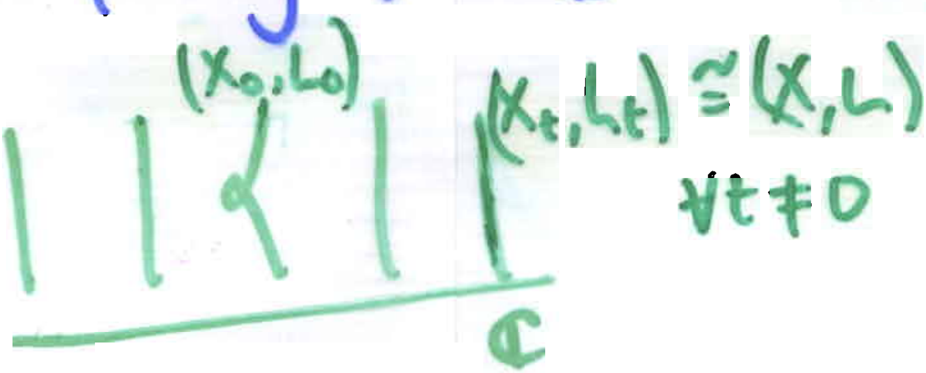
$$\mu = \frac{\int c_1 . \omega^{n-1}}{\text{rank}}$$

18. So bdls destabilised by subsheaves $F \subseteq E$

Does $\mathbb{P}(F) \subseteq \mathbb{P}(E)$ destabilise as varieties?

Can subschemes $Z \subseteq (X, L)$ destabilise?

<u>HM criterion</u> $\mathbb{C}^*$ 1-p.s. action of $X \in \mathrm{Hilb} \subseteq \mathbb{P}^M$ gives flat family $\mathcal{X} \to \mathbb{C}$ $\mathbb{C}^*$-equivariant



Calculate weight of $\mathbb{C}^*$ action on

$$\Lambda^{\max} H^0(X, L^{rk}) \otimes \Lambda^{\max} S^k H^0(X, L^r)^*$$

$w_{r,k} > 0 \Rightarrow$ unstable

19.

$$w_{r,k} = a_{n+1}(r)\, k^{n+1} + a_n(r) k^n + \dots$$

where $a_i(r) = a_{in} r^n + a_{i\,n-1} r^{n-1} + \dots$

<u>Def</u> This 1-p.s. destabilizes iff $w_{r,k} > 0$ in following sense:

- Hilbert-Mumford(r) : $\underline{w_{r,k} > 0}$ $\forall k \gg 0$
- Chow(r) : leading order k-coeff $\underline{a_{n+1}(r) > 0}$
- Asymptotic Chow : $\underline{a_{n+1}(r) > 0}$ $\forall r \gg 0$
- K-stability : leading order term in $a_{n+1}(r)$

$$\underline{a_{n+1,n} > 0}$$

Correspond to different line bdls on Hilb (standard one, Chow, $\text{Chow}_{r\to\infty}$, Paul-Tian)

20.

For simplicity state results for K-stability.

$Z \subseteq (X, L)$ $\quad \omega = c_1(L)$

$$h^0(\mathcal{O}_X(r)) = \overset{\int_X \omega^n}{a_0} r^n + \overset{\int_X c_1(X)\omega^{n-1}}{a_1} r^{n-1} + \dots$$

$$h^0(\mathcal{I}_Z^{xr}(r)) = a_0(x) r^n + a_1(x) r^{n-1} + \dots$$

polys in x

$a_i(0) = a_i$

c s.t. $\mathcal{I}_Z^{cr}(r)$ gen. by sections $r \gg 0$

By analogy with bdls define

$$\mu(Z) = \frac{\int_0^c a_1(x) + \frac{a_0'(x)}{2}\, dx}{\int_0^c a_0(x)\, dx}$$

$Z = \emptyset$ gives $\mu(X) = \dfrac{a_1}{a_0}$

<u>Thm</u> If $\mu(Z) > \mu(X)$ then

(X,L) K-unstable

($\Rightarrow$ asymptotic Chow, HM unstable
$\Rightarrow$ no cscK metric in $c_1(L)$)

<u>Egs</u> 1. $F \subseteq E$ destabilising subbundle
then $\mathbb{P}(F)$ destabilises $(\mathbb{P}(E), \mathcal{L})$
where $\mathcal{L} = \pi^* L^m \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ for $m \gg 0$

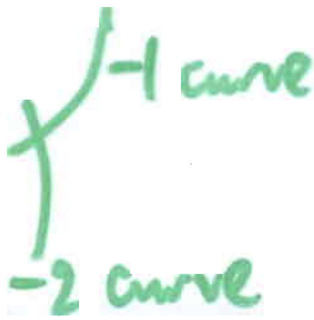
2. -1 curves on del Pezzo surfaces
for appropriate L.

(So $Aut(X)$ reductive $\not\Rightarrow$ cscK metric unless $L = K^{-1}$ (Tian))

3. Example of generic stability specialising to instability.

Move 2 -1 curves together on del Pezzo surface

Limit



(blow up 2 infinitely near points, i.e. a point and then a point on exceptional divisor)

-2 curve destabilises for some polarisation L.

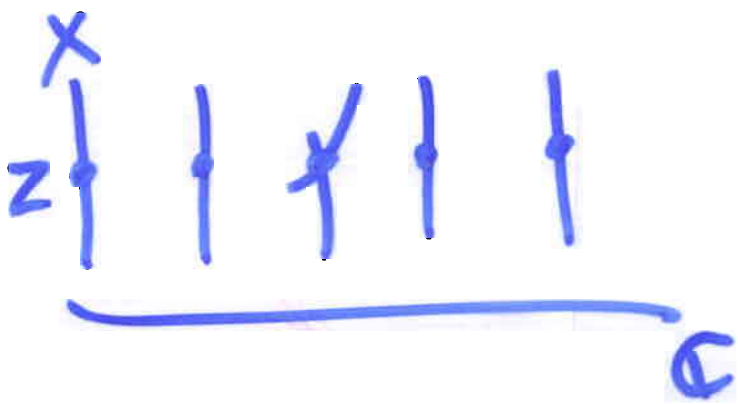
4. $\mathbb{P}^2$ blown up in 1 point.

Automorphism group not reductive $\Rightarrow$ unstable.

Destabilised by the -1 curve.

24.

Proof is by choosing the degen $\mathcal{X} \to \mathbb{C}$, the t.p.s., to be $(X \times \mathbb{C})$ blown up in $Z \times \{0\}$



+ calculating weight.

$\lim_{r,k \to \infty}$ $\Rightarrow$ Sum becomes an integral.

Converse?

Non-projective case: $\exists$ cscK iff $\nexists\, Z \subseteq (X, L)$ st $\mu(Z) \geq \mu(X)$?