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Special Lagrangian T^2-cones in $\mathbb{C}^3$ come in real families of every dim.

joint work w/ Ian McIntosh.

Defn oriented $M^n \subseteq \mathbb{C}^n$ or Calabi-Yau X is special Lagrangian w/ angle θ (θ-SLag) if it is calibrated with respect to the calibration $\varphi_\theta = \mathrm{Re}(e^{i\theta} dz^1 \wedge \ldots \wedge dz^n)$. (or $\mathrm{Re}(e^{i\theta}\Omega)$, Ω = covar const holom vol form)

Equivalently oriented M^n is θ-SLag if all of its oriented tangent planes can be obtained from $\mathbb{R}^n$ by action of $A \in U(n)$, $\det A = e^{i\theta}$.

Recall: $M \subseteq \mathbb{C}^n$ is θ-SLag wrt some θ $\iff$ it is minimal & Lagrangian.

Mirror Symmetry, & SYZ conjecture in particular, motivate us to study SLag 3-folds in Calabi-Yau manifolds & in particular, their singularities.

Local Model/Simplest Case: SLag cones in $\mathbb{C}^3$.

$$
\begin{tikzcd}
\mathrm{Cone}(M^2) = C(M^2) \arrow[r, "\Phi"] & \mathbb{C}^3 - \{0\} & \text{min, Lagrangian} \\
M^2 \arrow[u, phantom, sloped, "\subset"] \arrow[r, "f"] \arrow[dr, "\psi"] & S^5 \arrow[u, phantom, sloped, "\subset"] \arrow[d, "\text{Hopf}"] & \text{min, Legendrian} \arrow[u, phantom, "\Updownarrow"] \\
 & \mathbb{CP}^2 & \text{min, Lagrangian} \arrow[u, phantom, "\Updownarrow"]
\end{tikzcd}
$$

where we're using the standard forms

symp $\omega = \frac{\sqrt{-1}}{2} \sum dz^i \wedge d\bar{z}^i$ on $\mathbb{C}^3$

contact $\alpha = \frac{1}{2} \sum x_i\, dy_i - y_i\, dx_i$ on S^5

symp $\omega_{FS} = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \frac{|z|}{|z_i|}$ on $U_i = \{z_i \neq 0\} \subseteq \mathbb{CP}^2$.

Possibly up to a $\mathbb{Z}_3$-cover:

SLag cones $\phi : C(M^2) \to \mathbb{C}^3 - \{0\}$

$\Longleftrightarrow$ Min Lag $\psi : M^2 \to \mathbb{CP}^2$.

(note that the contact distribution on S^5 is exactly the horiz distrib for the natural connection on $S^5 \to \mathbb{CP}^2$,

$T_z S^5 = \mathbb{R} \cdot iz \oplus H_z$).

Theorem (as stated by McIntosh '02, but ∃ similar work by Sharipov '91, Ma & Ma '01, Hashimoto/Taniguchi/Udagawa '03)

Roughly:

$\psi : T^2 \to \mathbb{CP}^2$ min (Lag) $\overset{1-1}{\longleftrightarrow}$ spectral curve (algebraic curve) $\lambda : X \to \mathbb{CP}^1$ & line bundle $L \to X$ satisfying periodicity (& other) conditions"

More Precisely:

Given minimal (Lag) $\psi : T^2 \to \mathbb{CP}^2$, can construct

or map of $\mathbb{R}^2$ of "finite type"

called the spectral curve

- Alg curve X, genus g ($=2n$)
- degree 3 $\lambda : X \to \mathbb{CP}^1$ totally ramified at $P_0 = \lambda^{-1}(0)$, $P_\infty = \lambda^{-1}(\infty)$
- line bundle $L \to X$, deg $g+2$
- antiholom invol $\rho : X \to X$ covering $\lambda \mapsto \bar{\lambda}^{-1}$
- (holom invol $\mu : X \to X$ " " $\lambda \mapsto -\lambda$)

Turns out that these conditions imply that L is "non-special" ie $h^1(L)=0$ $\therefore h^0(L)=3$ ($h^0-h^1=d+1-g=3$)

so that

- λ has no real branch points
- $\rho^* \bar{L} \otimes L \simeq \mathcal{O}_X(R)$ ramification divisor of λ
- ($\mu^* L \otimes L \simeq \mathcal{O}_X(R)$)
-

"Conversely", given $(X, \lambda, L, \rho, (\mu))$ as above, can invert process & construct min (Lag) $\psi : \underline{\underline{\mathbb{R}^2}} \to \mathbb{CP}^2$.

Need more cdns to ensure ψ doubly-periodic, & then get 1-1 correspondence. More on periodicity cdns later, 1st:

CONSTRUCTION OF SPECTRAL CURVE

$\psi: T^2 \to \mathbb{CP}^2$ min. (Lag)
$\rightsquigarrow$ $\mathbb{C}^*$-family of flat connections
$\rightsquigarrow$ 3-sheeted cover of $\mathbb{C}^*$
$\rightsquigarrow$ spectral curve X, etc.

Take $\psi: T^2 \to \mathbb{CP}^2$ min Lag, w/ lift $f: T^2 \to S^5$ so that $f \wedge N_f$ is $\left(\frac{3\pi}{2}\right)$-SLag (Always OK up to a $\mathbb{Z}_3$ cover)

Reason for this choice of angle comes later.

$\rightsquigarrow$ ψ can be lifted to the 6-sym space

$$FL = \{(p, \Sigma) \in S^5 \times Gr(2, \mathbb{C}^3) : l \subseteq \Sigma\}$$
$$\simeq SU(3)/K, \quad K = \left\{\begin{pmatrix} 1 & & \\ & a & \\ & & a^{-1}\end{pmatrix} \in SU(3)\right\}$$

by $(f, f \oplus f_{\bar{z}})$

FL has order 6 automorphism $\sigma = \mu\nu$ where

$$\nu: SU(3) \to SU(3)$$
$$g \mapsto SgS^{-1}, \quad S = \begin{pmatrix} 1 & & \\ & \varepsilon & \\ & & \varepsilon^2\end{pmatrix} \quad \varepsilon = e^{\frac{2\pi\sqrt{-1}}{3}}$$

Coxeter-Killing aut

$$\mu: g \mapsto T({}^t g^{-1})T^{-1} \quad T = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0\end{pmatrix}$$

Write $\mathfrak{g} = su(3)$, then

$$sl(3, \mathbb{C}) = \mathfrak{g}_{\mathbb{C}} = \bigoplus_{i=0}^{5} \mathfrak{g}_j \quad \mathfrak{g}_j \text{ e'space wrt } (-\varepsilon)^j$$

$$\mathfrak{g}_j = \mathfrak{g}^{\mu}_{j \bmod 2} \cap \mathfrak{g}^{\nu}_{j \bmod 3}$$

& setting

$$\mathfrak{k} = su(3) \cap \mathfrak{g}_0$$
$$\mathfrak{n} = su(3) \cap \bigoplus_{j=1}^{5} \mathfrak{g}_j, \text{ we have the splitting}$$
$$su(3) = \mathfrak{k} \oplus \mathfrak{n}.$$

$f \wedge N_f$ $\frac{3\pi}{2}$-Slag means that the framing

$$F = \left(f, \frac{f_z}{|f_z|}, \frac{-f_{\bar z}}{|f_{\bar z}|}\right) \text{ of } (f, f \oplus f_z)$$

is special unitary.

Set $\alpha = F^{-1} dF$ & decompose:

$$\alpha = \alpha_k + \alpha_n$$

For each $\zeta \in S^1$, define

$$\boxed{\alpha_\zeta = \zeta \alpha_n^{1,0} + \alpha_k + \zeta^{-1} \alpha_n^{0,1}}.$$

Then (see Burstall & Pedit '94 for calculation)

$d + \alpha_\zeta$ is flat $\forall\, \zeta \in S^1$ (& conversely given such flat connections can easily recover f)

α_ζ has symmetries:

$$\nu(\alpha_\zeta) = \alpha_{\varepsilon\zeta}, \quad \mu(\alpha_\zeta) = \alpha_{-\zeta}, \quad -\bar{\alpha}_\zeta^t = \alpha_{\bar\zeta^{-1}}$$

Let $\Lambda_\nu sl(3, \mathbb{C})$ be the loop algebra of maps $X : \mathbb{C}^* \to sl(3, \mathbb{C})$ that are finite Laurent polynomials in $\zeta \in \mathbb{C}^*$

$$X(\zeta) = \sum_{|d| \le n} X_d \zeta^d \quad \text{\& satisfy}$$

$$\nu(X(\zeta)) = X(\varepsilon\zeta)$$

Let $\mathcal{A} = \{ \xi_\zeta : T^2 \to \Lambda_\nu sl(3, \mathbb{C}) : d\xi_\zeta = [\xi_\zeta, \alpha_\zeta]$

ie $(d + \mathrm{ad}_{\alpha_\zeta}) \xi_\zeta = 0$.

For each $z \in T^2$, define

$$X_z - \{P_0, P_\infty\} := \{ (\lambda, [v]) \in \mathbb{C}^* \times \mathbb{C}P^2 \text{ st } \hat{\xi}_\lambda([v]) = [v] \}$$

$\zeta^3 = \lambda$

& compactify to obtain alg curve X_z

(isomorphic $\forall z$ so just call it X)

$\hat{\xi}_\lambda$ here is $\hat{\xi}_{\lambda = \zeta^3} = \mathrm{Ad}_{[1, \zeta^{-1}, \omega^{-2}]} \xi_\zeta$, ftn of λ

(If define $Y_{\mathbb{Z}_3}$ as above using ξ_3. ⑥

H has free action

$\nu : \hat{Y} \to Y$ of order 3:

$(\xi, [v]) \mapsto (\varepsilon\xi, [S[v]])$ $S = \begin{pmatrix} 1 & & \\ & \varepsilon & \\ & & \varepsilon^2 \end{pmatrix}$

quotienting gives spectral curve X.)

Embeddings of X_z in $\mathbb{CP}^1 \times \mathbb{CP}^2$ <u>do</u> vary with z, for each z define

$L_z^* :=$ pullback of tautological bundle over $\mathbb{CP}^2$ by $X_z \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^2$.

Our L is just L_0.

<u>PERIODICITY CONDITIONS</u>

In fact, (of finite type)

$\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ min (Lag) splits

$$\mathbb{R}^2 \xrightarrow{\mathcal{L}'} \mathrm{Pic}^{g+2}(X') \xrightarrow{H} \mathbb{CP}^2$$

where: • X' obtained from X by identifying points in $\lambda^{-1}(1)$ to form a triple node

• so $\mathrm{Pic}^{g+2}(X')$ = deg $g+2$ line bundles on X, together with an identification $L_{O_1} \simeq L_{O_2} \simeq L_{O_3}$ of the fibres over $\{O_1, O_2, O_3\} = \lambda^{-1}(1)$.

• $\mathcal{L}'(0) = L_0$ and

regular diffs on X' = merom diffs on X w/ at most a simple pole at O_1, O_2, O_3.

$$d\mathcal{L}'_0\left(\frac{\partial}{\partial z}\right) = dA'_{P_0}\left(\frac{\partial}{\partial \xi}\right)$$ (where

$$A'_{P_0} : X' \to \mathrm{Jac}(X') = \frac{H_0(\Omega_{X'})^*}{H_1(X - \{O_1, O_2, O_3\}, \mathbb{Z})}$$

Abel-Jacobi map w/ basepoint P_0) $Q \mapsto \left(\omega \mapsto \int_{P_0}^{Q} \omega\right)$

and Ψ is doubly-periodic

$\Updownarrow$

$\mathcal{L}'$ is doubly-periodic

This also tells us how to recover Ψ from $(X, \lambda, L, \mu, \rho)$: just need to describe $(A)|_{\mathcal{L}'(\mathbb{R}^2)}$

- given L'_z ie L_z $\leftarrow$ specified

$$\begin{array}{cc} L'_z & L_z \\ \downarrow & \downarrow \\ X' & X \end{array} \quad w/ \ (L_z)_{O_1} \cong (L_z)_{O_2} \cong (L_z)_{O_3}$$

Symmetry cdns $\Rightarrow$ L_z are "non-special" ($h_1 = 0$)

so $h^0(L_z) = 3$ &

$H^0(L_z) \cong \mathbb{C}^3$

Specify line in $\nearrow$ by

$H^0(L_z(-P_0 - P_\infty)) \leq H^0(L_z)$

& have a map to $\mathbb{CP}^2$ b/c we can identify the different $\mathbb{P}H^0(L_z)$'s using $H^0(L_z) \cong (L_z)_{O_1} \oplus (L_z)_{O_2} \oplus (L_z)_{O_3}$

$\cong (L_z)^3_{O_1}$

$\Rightarrow$ identifying this w/ $\mathbb{C}$ is just a choice of scaling

$\rightsquigarrow$ well-defined $\mathbb{CP}^2$

<u>NOTE</u>: Periodicity cdns have nothing to do with choice of L. If find spectral curve of genus g that satisfies them (not apriori obvious that this is possible!) then can choose L from g ($n = \frac{g}{2}$) dimensional $\mathbb{R}$ family $\rightsquigarrow$ $g-2$ ($n-2$)

min $\rightarrow$

min + Lag

dim'l $\mathbb{R}$ family of maps once factor out choice of $O \in T$?

Mention previous work of Ercolani/Knörrer/Trubowitz & of Jaggy for cmc tori & of —— for tori in S^3

Theorem (—— & McIntosh)

For every g (n) ≥ 1 there are countably many spectral curves of genus g ($2n$) giving rise to non-congruent minimal (Lagrangian) immersions of tori into $\mathbb{CP}^2$

Corollary

For each $g > 2$ ($n > 2$) there are countably many real $g-2$ ($n-2$) real dimensional families of minimal (Lagrangian) immersed tori in $\mathbb{CP}^2$.

($g=0$ get just the spectral curve $\mathbb{CP}^1 \to \mathbb{CP}^1$, $\zeta \mapsto \zeta^3$; which corresponds to the Clifford torus

Slag cone is over Clifford torus: $\mathrm{Im}(z_1 z_2 z_3) = 0$, $|z_1|^2 = |z_2|^2 = |z_3|^2$

torus itself in S^5 is $|z_1|^2 = |z_2|^2 = |z_3|^2 = \frac{1}{3}$

$$\sum_{i=1}^{3} \mathrm{Arg}(z_i) = 0$$

(set $\sum_i \mathrm{Arg}(z_i) = c$, get $n\pi - c$ Slag torus, $n \in \mathbb{Z}$).

Outline of Proof:

$$L^{-1} \otimes \mathcal{L}' : \mathbb{R}^2 \to \mathrm{Prym}_{\mathbb{R}}(X', \mu))$$
$$= \{ E \in \mathrm{Jac}(X') : \mu^* E \simeq E^{-1},\ \rho^* E \simeq \bar{E}^{-1} \}$$
$$\simeq \mathbb{R}^{g+2} \quad (\mathbb{R}^{n+2})$$

($g = 2n$ in min Lag case)

Periodicity Cdn: $\mathcal{A}'_{P_0}(T^{1,0}_{P_0}X')$ to be a rational plane in $\mathrm{Prym}_{\mathbb{R}}(X'(\mu))$.

To make notation simpler, I'll deal with the minimal case (min Lag similar).

$$\frac{\partial}{\partial \varsigma}\Big|_{\varsigma=0} \mathcal{A}'_{P_0}(\varsigma) = \frac{\partial}{\partial\varsigma}\Big|_{\varsigma=0}\left(\omega \mapsto \int_{P_0}^{\varsigma}\omega\right)$$

$$= \left(\omega\mapsto \mathrm{Res}_{P_0}\left(\frac{\omega}{\varsigma}\right) \overset{\text{defn}}{=:} \omega(P_0)\right)$$

Fix basis of regular differentials $\underline{\omega} = \omega^1, \ldots, \omega^{g+2\ell'}$ on X'. Then

$$W_X := \mathcal{L}'(\mathbb{R}^2) = \mathrm{Span}_{\mathbb{R}}(\mathrm{Re}\,\underline{\omega}(P_0), \mathrm{Im}\,\underline{\omega}(P_0))$$

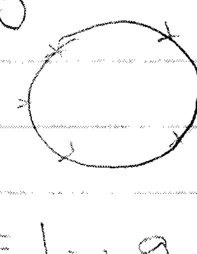
Rationality dense condition; we'll use inverse fn thm to show solutions exist.

Boundary / Degenerate Case:

Let $\mathbb{P}_\varsigma$ be $\mathbb{CP}^1$ w/ coord ς

$$\mathbb{P}_\varsigma \to \mathbb{P}_\lambda$$
$$\varsigma \mapsto \varsigma^3 = \lambda$$

Fix g distinct points $b_1 \ldots b_g$ on unit circle, $b_k = c_k^3$, $c_k = e^{\sqrt{-1}\theta_k}$

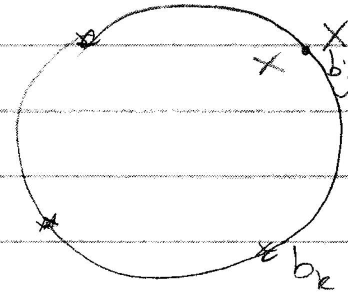


Define X_b from $\mathbb{P}_\varsigma$ by identifying c_k & εc_k, $k = 1 .. g$.

X_b : arith genus g

: geom genus 0

For $t_j \in (-\delta, \delta)$, $\delta > 0$ small, we define a family $X_b(t_j)$ by "pulling apart" node over b_j (w/ branching behaviour "same" as for $t_j = 0$)



Then normalisation $E_j(t_j)$ of $X_b(t_j)$ is an elliptic curve & family X_b is parametrised by $(t_1, \dots, t_g, \theta_1, \dots, \theta_g)$ in $(-\delta, \delta)^{2g} \subseteq \mathbb{R}^{2g}$

Thm (—, McIntosh)
Consider $W : (-\delta, \delta)^{2g} \to Gr(2, \mathbb{R}^{g+2})$
$(t_1 \dots t_g, \theta_1 \dots \theta_g) \mapsto W_X$
There exists
$b = (0, \dots, 0, \theta_1, \dots, \theta_g)$ st dW_b is invertible

(Actually show: setting $t_j = 0$, W is a real-analytic function of $(c_1, \dots, c_g) \in (S^1)^g$; extend to $\mathbb{C}^g$ & show $\lim_{z \to \infty} \det W_X(z, z^2, \dots, z^g) \neq 0$. For Lag, just show $\lim_{z \to \infty} \det W_X(z, \dots z^g, -z, \dots -z^g) \neq 0$

The elliptic curves $E_j(t_j)$ have hyperelliptic realisations & we use these to find an explicit basis for differentials & hence begin the calculations