Generalized Hodge Metrics and BCOV Torsion

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We assume that (X, L) be a polarized Calabi-Yau manifold. And we would like to study the moduli space of (X, L). One reason we are interested in the moduli space is that i~~t~~ is more linear to study. The other reason is that it is related to the Mirrior Symmetry.

The construction of Calabi-Yau moduli can be roughly described as follows: Let $k \gg 0$ such that L^k is very ample. Then X is embedded into $\mathbb{CP}^N$ for $N = \dim H^0(X, L^k) - 1$. Let $\cancel{\mathcal{X}}^{Hilb}$ be the Hilbert Scheme of X. That is, $\mathcal{M}$ is the parameter space of all subvarietres of $\mathbb{CP}^N$ having the same Hilbert polynomial. For example, Let X be a degree $(n+1)$ hypersurface of $\mathbb{CP}^n$. Then the Hilbert Scheme of X is the projective space of $a_{i_0 \cdots i_{n+1}}$, Where

$$\sum a_{i_0 \cdots i_{n+1}} Z_0^{i_0} \cdots Z_n^{i_{n+1}} = 0$$

defines the hypersurface X. The Hilbert Scheme of X, in this case, is also a projective space $\mathbb{CP}^m$. Of course, for two different $(n+1)$-polynomials, it is possible that after a change of element of $SL(n+1, \mathbb{C})$, there are the same. So the Calabi-Yau moduli is defined ~~th~~ to be

$$\mathcal{M} = \text{Hilb}^{\mathbb{CP}^m} / SL(N, \mathbb{C})$$

Like Richard said yesterday, if we don't remove certain bad points, the quotient is not even Hausdorff. The correct notation is the stability. Let Hilb^s be the set of stable points. Then

$$\mathcal{M} = \text{Hilb}^s / SL(N, \mathbb{C})$$

exists and is called the Calabi-Yau moduli. In fact, $\mathcal{M}$ is a quasi-projective variety ~~and~~ (Viehweg and Donaldson, more recently).

In order to understand Mirror Symmetry, we need to know ~~th~~ how $\mathcal{M}$ is compactified. Unfortunately, from GIT we know nothing about the asymptotic behavior of the moduli space at infinity.

We will use differential geometric method to study the moduli space $\mathcal{M}$. First, by Tian's smoothness (Todorov) theorem, the local universal deformation space is ~~unobstructed~~ smooth. $\mathcal{M}$ may not be smooth due to the presence of discrete Aut group of X (i.e. $\text{Aut}(X)$ is a finite group). But the singularities are at worst quotient singularities, and the local universal deformation space is the uniformization of the neighborhood of the moduli space.

Globally, the compactification $\bar{\mathcal{M}}$ of $\mathcal{M}$ may be very singular along the infinity. However, since we don't know how exactly $\mathcal{M}$ is compacified, it loses nothing by blowing up $\bar{\mathcal{M}} - \mathcal{M}$ so that by the Hironaka Theorem, we can assum that $\bar{\mathcal{M}}$ is smooth and $\bar{\mathcal{M}} - \mathcal{M}$ is a divisor Y of normal crossing.

So far, we collected ~~all~~ the algebraic geometry properties of $(\mathcal{M}, \bar{\mathcal{M}})$. Next, we will study the differential geometry of $\mathcal{M}$.

- Local differential geometry

We will study the special case: X is a Calabi-Yau three fold, simply-connected. Then we know that in the Hodge groups

$$H^{1,0} = H^{0,1} = 0, \quad H^{2,0} = H^{0,2} = 0, \quad H^2(X, \mathbb{C}) = H^{1,1}$$

The only interesting groups would be

$$H^{3,0}, H^{2,1}, H^{1,2}, H^{0,3} \subset H^3(X, \mathbb{C})$$

(a). $\dim H^{3,0} = \dim H^{0,3} = 1$, because there are nonwhere zero $(3,0)$ forms

(b). $\dim \mathcal{M} = \dim H^1(X, TX) = \dim H^{2,1}$ by Serre duality.

We define $Q(\omega, \eta) = \sqrt{-1} \int \omega \wedge \eta$ on $H^3(X, \mathbb{C})$

Then we have

①. $Q(\omega,\eta)=0$ if ω,η are of not complementary type $\Leftrightarrow$ $\omega\in H^{p,q}$, $\eta\in H^{p',q'}$ but $p+p'=3$ or $q+q'\neq 3$

②. $Q(\omega,\omega)>0$, $\omega\in H^{3,0}$, $\omega\neq 0$

$Q(\eta,\eta)<0$, $\eta\in H^{2,1}$, $\eta\neq 0$.

Idea of Griffiths: Let D be the space of all $\{H^{p,q}\}$ with the above properties. D is called the classifying space.

Natural map

$$p: \mathcal{M}\longrightarrow D, \quad X\longmapsto \text{Hodge structure of } X$$
$$\{H^{p,q}\}_{p+q=3}\subset H^3(X,\mathbb{C})$$

①. p is holomorphic; $p(\mathcal{M})$ is a submanifold of D.

②. p is an immersion; $\dim\mathcal{M}=\dim H^{2,1}$

③. Griffiths transversality:

$$\partial H^{3,0}\subset H^{3,0}\oplus H^{2,1}$$
$$\partial H^{2,1}\subset H^{3,0}\oplus H^{2,1}\oplus H^{1,2}$$
$$\partial H^{1,2}\subset V=H^3(X,\mathbb{C})$$

Horizontal distribution is the distribution s.t. the above are satisfied. It is not integrable but

$$p_*(T\mathcal{M})\subseteq \underbrace{\text{Distribution}}_{\text{Horizontal}}$$

In general, if a distribution is not integrable, it is rather difficult to ~~find~~ identify the integral submanifold. However, Bryant and Griffiths found all the local integrable submanifolds. They all determined by a holomorph function.

Weil-Petersson metric and the Hodge metric

① Let $\Omega \in H^{3,0}$ be a holomorphic section. Then

$$\omega_{WP} = -\sqrt{-1}\,\partial\bar{\partial}\log Q(\Omega,\bar{\Omega})$$

②. (Lu) Let ω_D be the invariant Hermitian metric of D. Then

$$\omega_H = p^*\omega_D$$

is called the Hodge metric.

Theorem: (Lu) With the above notations we have

①. ω_H is a kähler metric;

②. $Ric(\omega_H) \leq -\alpha < 0$, $Hol(\omega_H) \leq -\alpha < 0$, $Bisec(\omega_H) \leq 0$

③. $2\omega_{WP} \leq \omega_H$

④ For Calabi-Yau 3-folds

$$\cancel{\omega_H = 2\omega(m+3)\omega_H + Ric(}$$

$$\omega_H = (m+3)\,\omega_{WP} + Ric(\omega_{WP})$$

where $m = \dim M$

⑤ (Lu-Sun) For Calabi-Yau 4-folds,

$$\omega_H = 2(m+2)\,\omega_{WP} + 2\,Ric(\omega_{WP})$$

6. (Folk Theorem) For K3

$$\omega_H = 2\,\omega_{WP}$$

$\Rightarrow$ ω_{WP} is K-E

$\Rightarrow$ ω_{WP} is the invariant Hermitian metric

In what follows is my joint work with H. Fang.

BCOV torsion: Bershadsky - Cecotti - Ooguri - Vafa

First, we define the determinant of the Laplacian: Let $\Delta_{p,q}$ be the Hodge Laplacian of X. Let $\xi_1, \xi_2, \cdots$ be the sequence of eigenvalues (non-zero). In order to define $\prod_{i=1}^{\infty} \xi_i$, ~~we~~

we define the ζ-function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{\xi_i^s}$$

~~By Weyl theorem, $\xi_i \sim C\,(\mathrm{vol}(X))$~~

By Weyl theorem, $\zeta(s)$ exists for $Re(s) \gg 0$. It turns out that $\zeta(s)$ can be extended as a meromorphic function on $\mathbb{C}$. Formally

$$-\zeta'(0) = \prod_{i=1}^{\infty} \xi_i$$

Thus we can define

$$\det \Delta'_{p,q} = -\zeta'(0)$$

The BCOV torsion of a Calabi-Yau manifold is

$$T = \prod_{1 \le p,q \le n} \left(\det \Delta'_{p,q}\right)^{(-1)^{p+q}pq}$$

There is a good reason we are interested in the ~~que~~ quantity. In fact, physicists believed that the asymptotic behavior of T on the moduli space is related to the Mirror Symmetry.

It is impossible the compute the $\det \Delta'_{p,q}$ directly. The following is standard: let

$$E = \bigoplus_{p=1}^{n} (-1)^p \, p \, \Omega^p(\mathcal{X}/\mathcal{M})$$

where $\mathcal{X}$ is the total space. From E, we can construct a line bundle

$$\lambda = \bigwedge_{0 \le p,q \le n} \left(\det\left(H^{p,q}(Z, E, \bar\partial)\right)\right)^{(-1)^{p+q}p}$$

over $\mathcal{M}$. Since Z is a Calabi-Yau manifold, there is a natural L^2-metric of λ. The Quillen metric is given by

$$\|\cdot\|_Q = \|\cdot\|_{L^2} \cdot T$$

The curvature of the Quillen metric is given by the Riemann-Roch-Grothendieck Theorem.

Lemma :

$$\mathrm{Ric}(\|\cdot\|_Q) = \left(\int_Z Td(T^{1,0}\mathcal{X}/\mathcal{M})\, Ch(E)\right)^{(1,1)}$$

This is the RRG Theorem.

Corollary

$$\mathrm{Ric}(\|\cdot\|_Q) = \frac{1}{12}\chi_Z\, \omega_{WP}$$

Our observation is

Theorem

$$\mathrm{Ric}(\|\cdot\|_{L^2}) = -\,\omega_H$$

Thus we have

Theorem Let (X, L) be a Calabi-Yau moduli space.

$$-\omega_H + \partial\bar\partial \log T = \frac{1}{12}\chi_Z\, \omega_{WP}$$

~~Not~~ In general, we can define the generalized Hodge metric. Thus the formula would be

Theorem

$$\sum_{k=1}^{n}(-1)^k\, \omega_H + \partial\bar\partial \log T = \frac{1}{12}\chi_Z\, \omega_{WP}$$

It is interesting to see the analytic torsion T is related to the Hodge and the Weil-Petersson metrics, thus to the variation of Hodge structure.

Shafarovich-type question: Given a complex manifold, is it possible to be the parameter space of a family of Calabi-Yau manifolds?

The following results partially answers the question

Theorem Assume that X is a Calabi-Yau threefold. Assume that $\chi > -24$. Then there exists no complete curve in M. Hence, there exists no projective subvariety of M. In particular, M is not compact.

The other application

Theorem.

$$\left| \partial\bar\partial \log T \right| \leq C\,\omega_p$$

where ω_p is the Poincaré metric.

Proof of $\mathrm{Ric}(\|\ \|_{L^2}) = -\omega_H$

Proof: We assume Z is a Calabi - Yau threefold

$$\mathrm{Ric}(\|\cdot\|_{L^2}) = \sum_{p+q=n} (-1)^n\, p\, \mathrm{Ric}(H^{p,q}(Z, E, \bar\partial))$$

$$= (-1)^n \left(\mathrm{Ric}(H^{1,2}) - 2\,\mathrm{Ric}(H^{2,1}) + 3\,\mathrm{Ric}(H^{3,0})\right)$$

$$= (-1)^n \left(3\,\omega_{WP} - \mathrm{Ric}(H^{1,2})\right)$$

$$= (-1)^n \left((n+3)\,\omega_{WP} + \mathrm{Ric}(\omega_{WP})\right]$$

$n=3 \Rightarrow$

$$\mathrm{Ric}(\|\ \|_{L^2}) = -\,\omega_H$$