Minimal and Normal Surfaces

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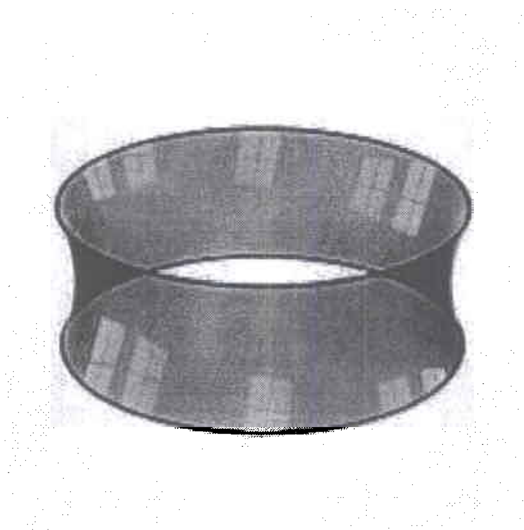
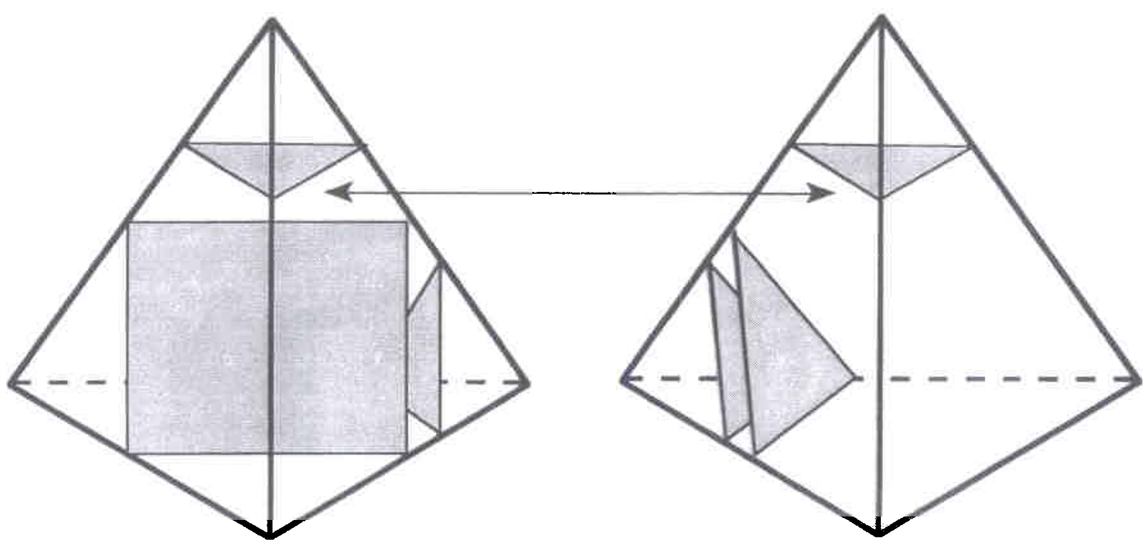




General questions:

1. How hard are the problems of 3-dimensional topology?

e.g.

a) Classify knots

b) Classify 3-manifolds

2. Can the techniques of 3-dimensional topology say something about the relationships between various complexity classes?

(As with applications of the Jaco-Shalen-Johannson decomposition of 3-manifolds to general finitely generated groups).

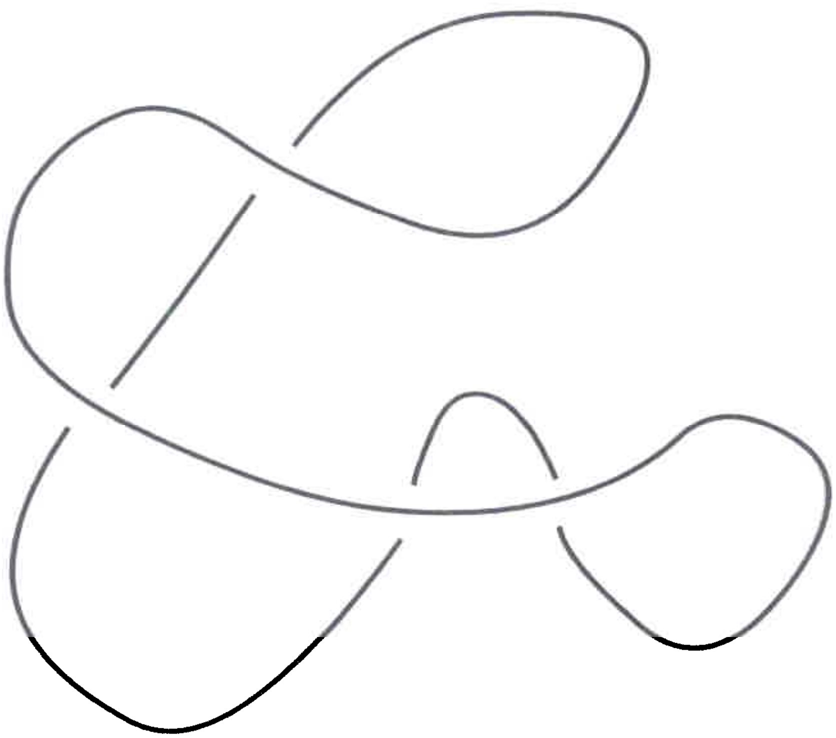
(Maybe)

3. Do the techniques developed in studying the computational complexity of numerous problems have implications in 3-dimensional topology and geometry?

(Yes)

Some motivating questions in topology:

1. Can we find a procedure to decide if a knot is trivial? UNKNOT RECOGNITION or UNKNOTTING



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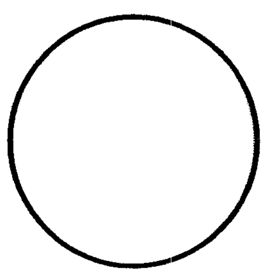
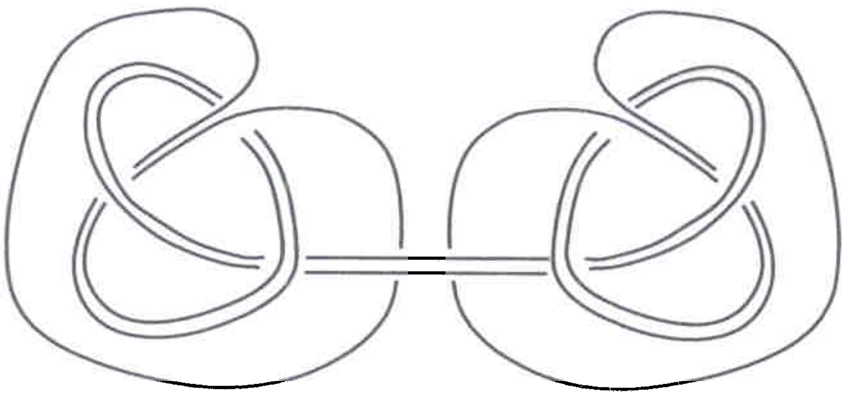


Can we find an algorithm to determine if these knots are the same? (Dehn, 1915)

What about



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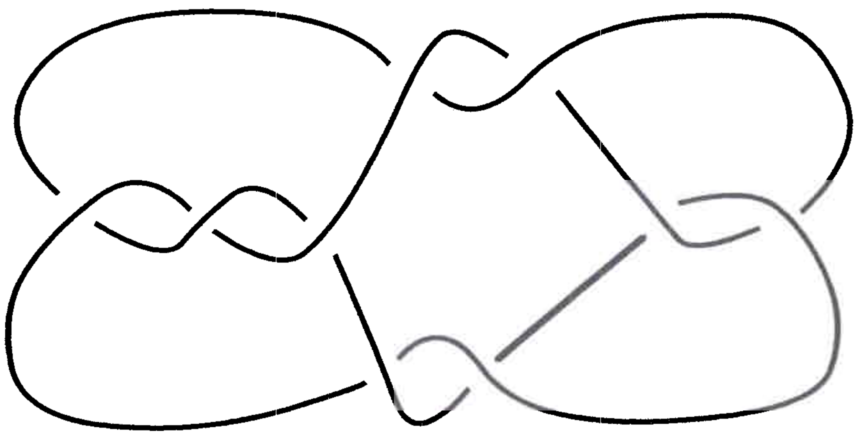

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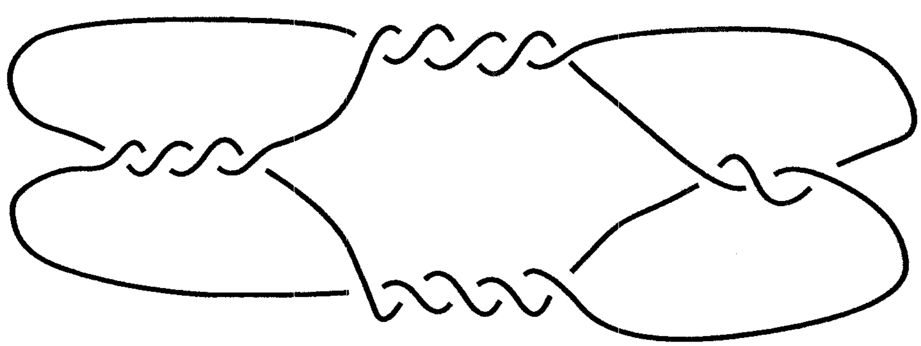
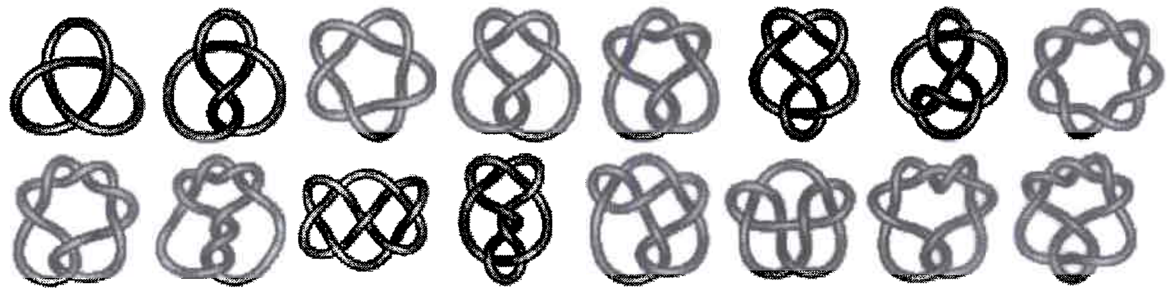


Are these equivalent to  ?

Closely related:

1) Can we tell apart any two knots? (Knot Recognition)

2.) Can we classify all knots?



(Knotplot)

Classification is a consequence of recognition. It is easy to generate all possible knots. The tricky part is to determine if two pictures represent the same knot, leading to duplication.

Generate all knot diagrams:

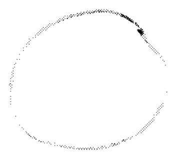
1) Generate all planar graphs, valence-four vertices

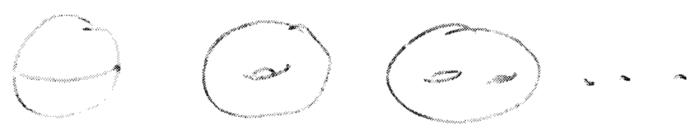
2) Take all overcrossing/undercrossings.

Knot Complements are special cases of Manifolds

*When are two manifolds the same? (homeomorphic)

Dimension 1 & 2: Easy.

1.  Always

2. 

Euler Characteristic Distinguishes Surfaces.

(connected
orientable
triangulated)

Dimension 3: Special cases can be done.
3-sphere, Haken, Lens spaces, ... All ?

(Rubinstein, Thompson, Haken, Matveev, Thurston, Hemion, Jaco, Stocking, Sedgewick, ..)

Dimensions ≥ 4: Many problems are undecidable. No algorithm or procedure can be found that can

a. Classify 4-manifolds (Markov)
or b. Recognize S^5 (Novikov)

Dimension 3 seems the most interesting.

* Problems are just barely solvable.

* Computational complexity of central problems seems to be on the edge between polynomial and exponential.

* Computer programs are already a major tool for 3-manifold research.
(SNAPPEA by Jeff Weeks, Regina (Ben Burton), Knot Simplifier (Dynnikov, Polthier,..)
How far can they go?

Current status of some of these problems:

UNKNOTTING PROBLEM:

Haken (1961): The problem is decidable. Algorithm based on Normal Surfaces.

Other algorithms exist based on geometrization or on variations of Haken's approach. (Thurston, Epstein-Holt-Patterson, Sela, Birman-Hirsch).

Thm: Hass-Lagarias-Pippenger (1997):

1. An algorithm which determines if a knot diagram with n crossings is unknotted runs in time $O(c^n)$

2. The Unknotting Problem is in NP.

Question Is there a polynomial time algorithm for Unknotting?

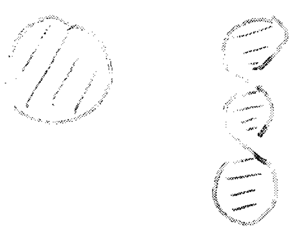
([illegible] Yes)

KNOT EQUIVALENCE PROBLEM:

Can we tell if two knots are the same?

Haken, Hemion, Matveev (1964-2000): A rather complicated procedure, with many cases, will recognize knots. A complexity bound is not currently known. (Partial results by Mijatovic, 2002).

This problem is much harder to analyze then Unknot Recognition. To determine if a knot is equivalent to the unknot, it suffices to check if it is the boundary of an embedded disk.



K is UNKNOT $\Leftrightarrow$ $K = \partial D^2$

$K_1 \neq K_2$: How can we tell?

There is no such shortcut to test if two knots are equivalent. Knot equivalence algorithms go deeply into 3-manifold theory.

Closely related to Knot Recognition:

3-DIMENSIONAL HOMEOMORPHISM PROBLEM:
Instance: A pair of triangulated 3-manifolds, K and L.
Question: Are K and L homeomorphic?

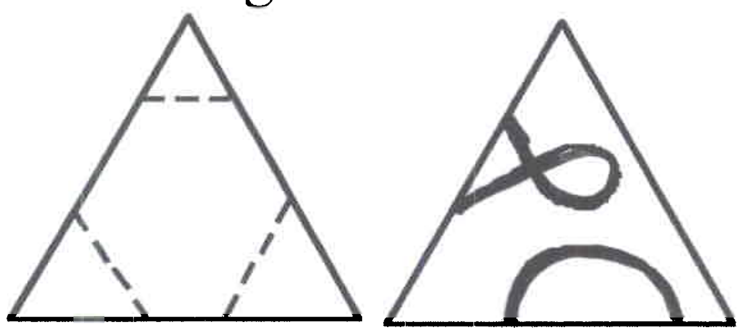
A procedure was discovered by: Haken (& Hemion, Matveev, Thurston, 1965-78). (If the manifold is Haken).

Haken manifolds are a broad class, including all knot complements.

We will outline this approach.

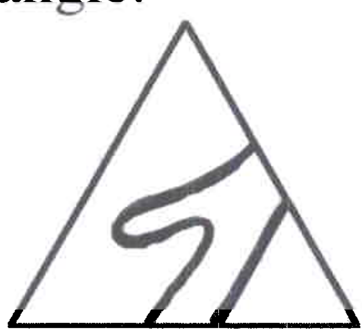
Normal curves in surfaces

Definition: An embedded arc in a triangle is *elementary* if its endpoints lie on distinct edges of the triangle.



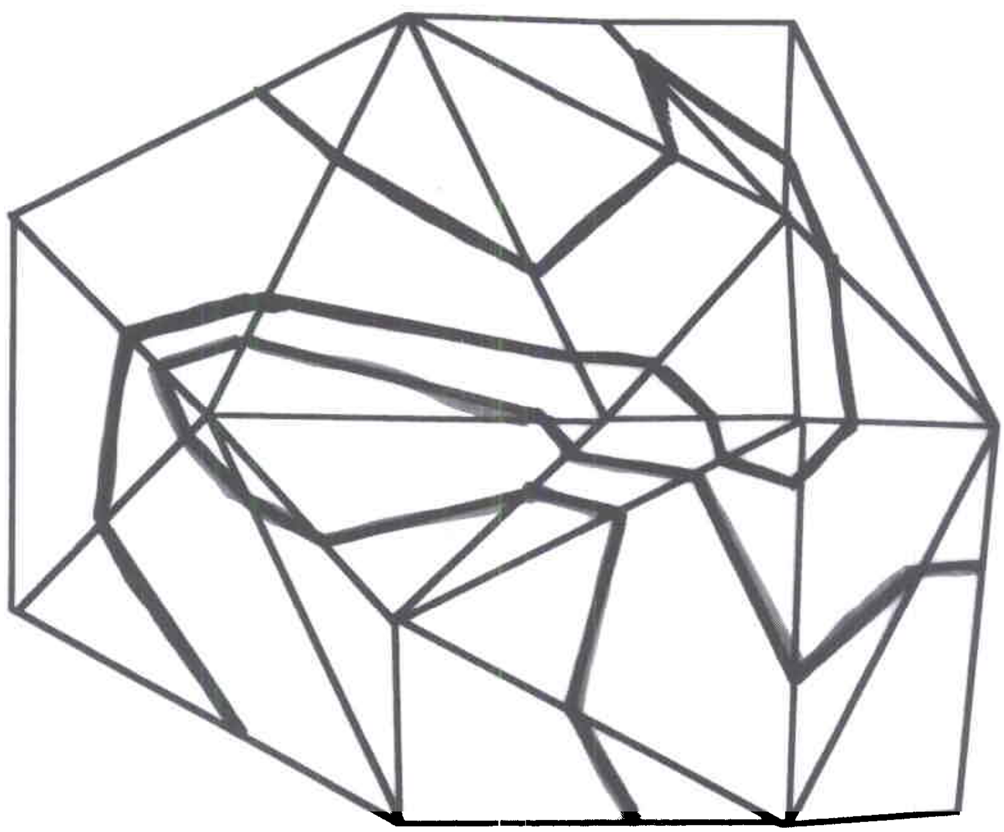
Elementary and non-elementary arcs

There are three types of elementary arcs in a triangle, up to an isotopy preserving the edges of the triangle.

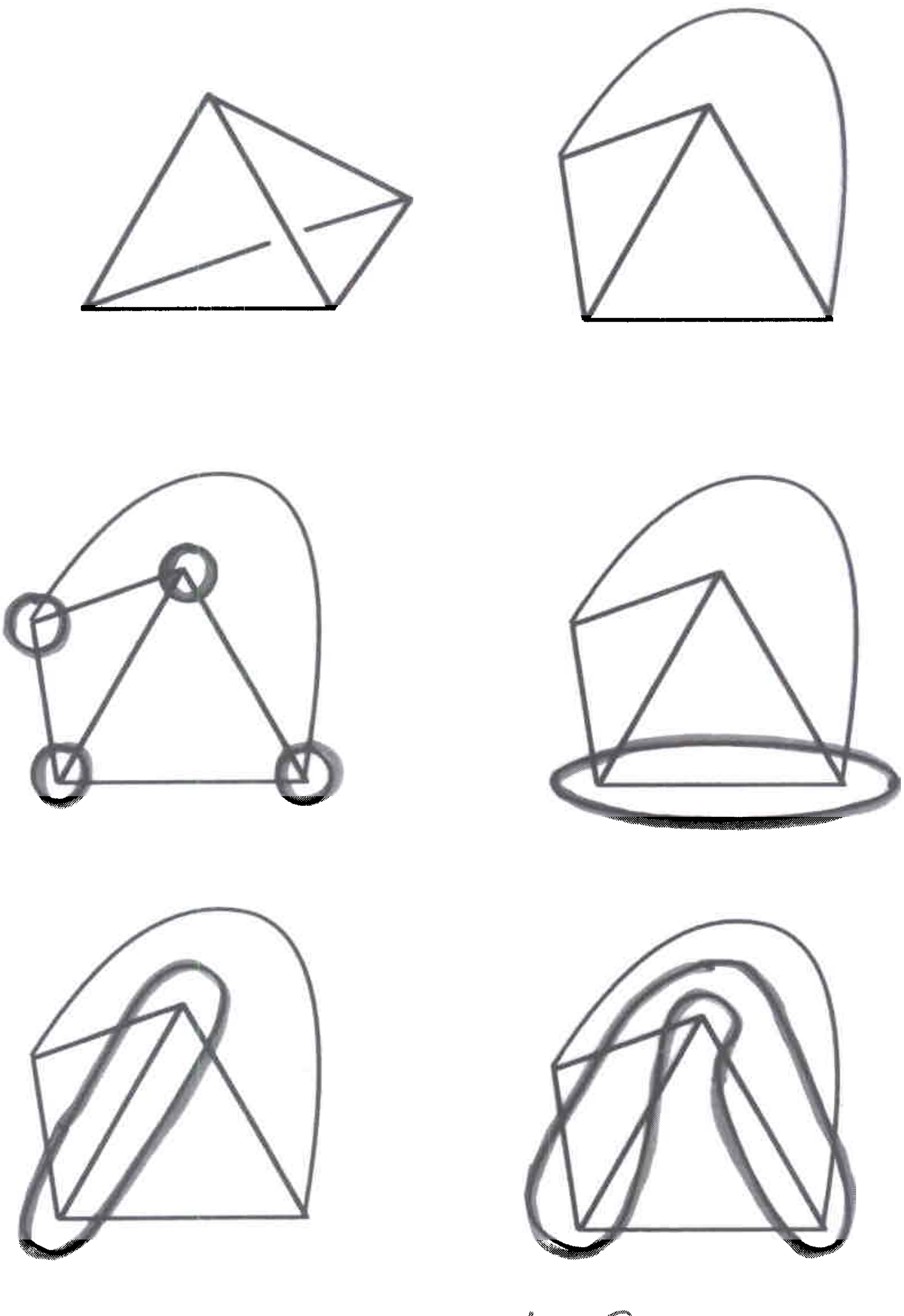


These arcs are the same up to a <u>normal isotopy</u>. There are three types of elementary arc in a triangle.

A curve in a triangulated surface is *normal* if its intersection with every triangle is a collection of elementary arcs.

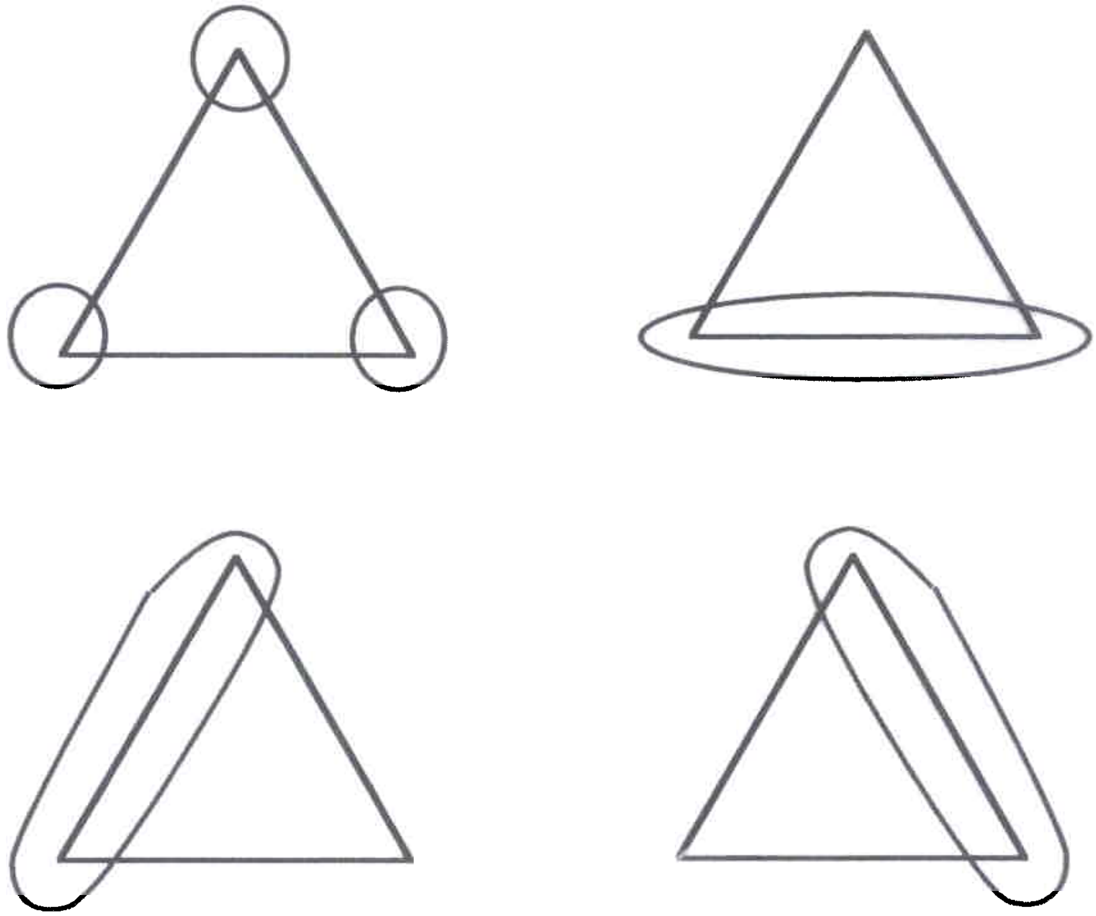


<u>Example:</u> The 2-sphere can be triangulated with four triangles.



Some Normal Curves

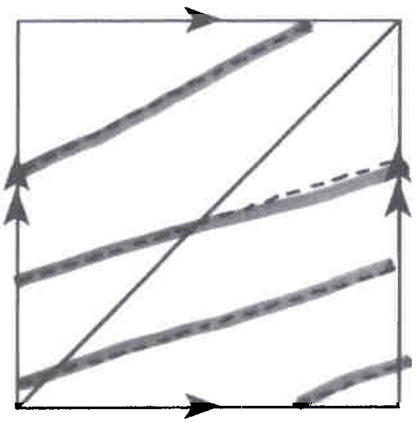
The 2-sphere also admits a pseudo-triangulation with only two triangles.



This time there are only six normal curves possible, three triangles and three quadrilaterals.

This is not a true triangulation, since the structure is not that of a simplicial complex.

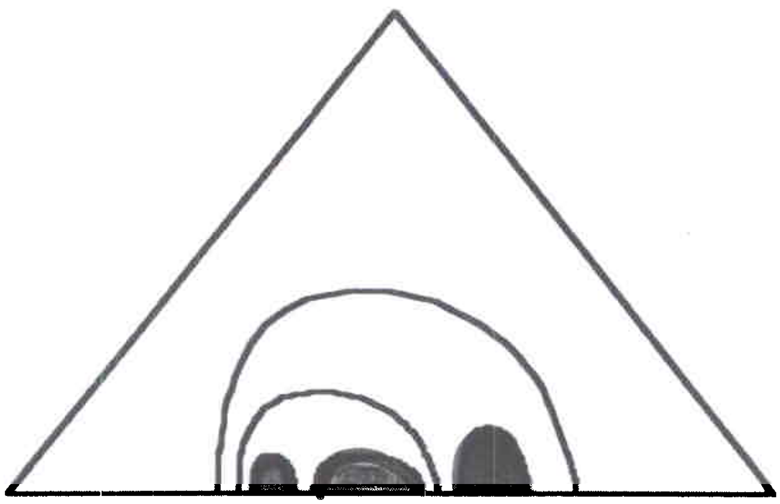
Normal surface theory applies equally well to such “pseudo-triangulations”. They have the advantage of containing fewer triangles.



A torus triangulated with two triangles (pseudo-triangulation) has infinitely many distinct normal curves.

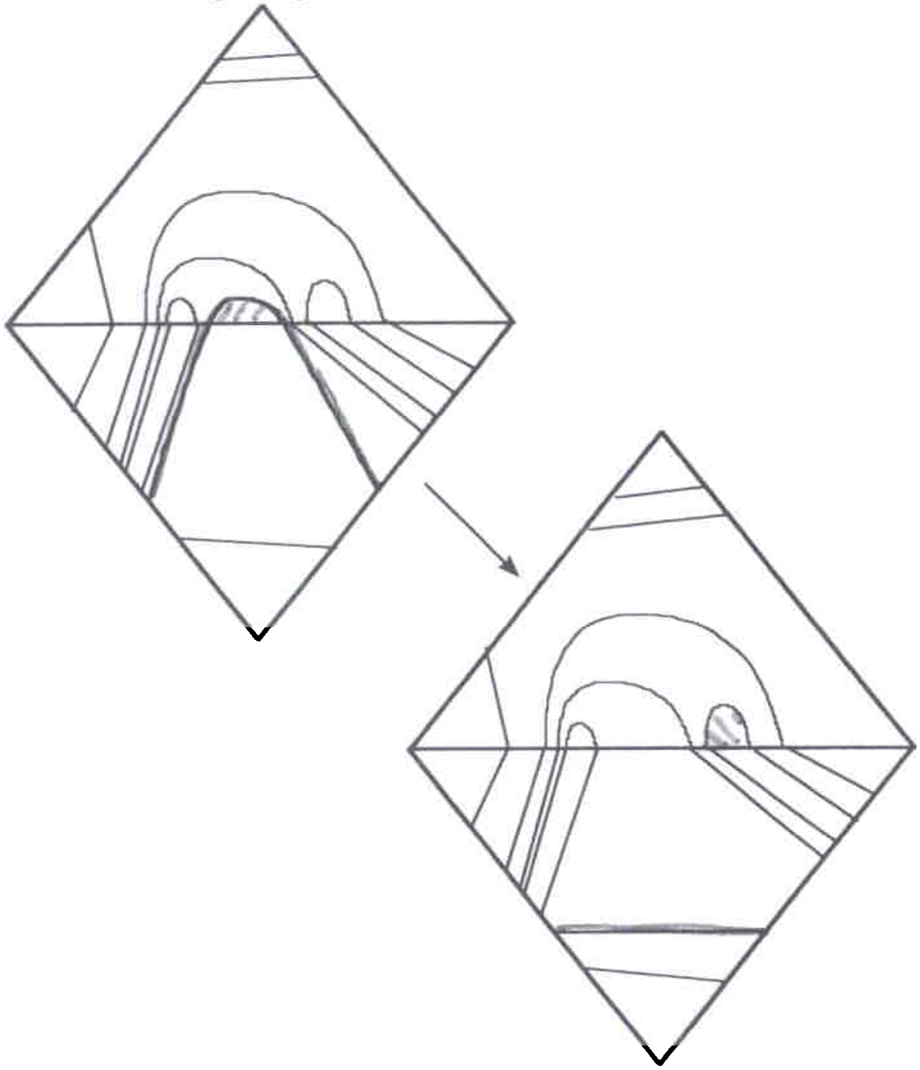
Theorem Any simple closed curve Γ on a surface can be isotoped until it is either normal or lies in a single triangle.

Proof: A non-elementary arc of Γ, if one exists, starts and ends on the same edge of some triangle T. Such an arc, together with a segment of the edge between its endpoints bounds a disk in T.



The arc is *innermost* if this subdisk contains no other arcs of $\Gamma \cap T$.

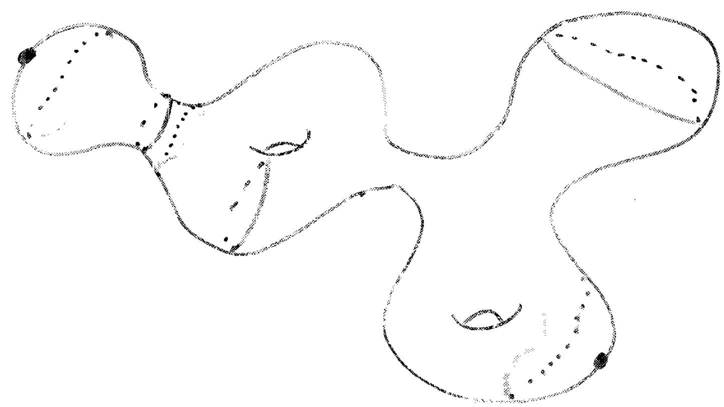
If there is a non-elementary arc, then there is an innermost such arc. Isotop an innermost arc across the innermost subdisk, reducing the number of intersections with the boundary by two.



Continue until the curve is normal or all intersections are eliminated.

Compare this with the following result from differential geometry.

Theorem A connected simple closed curve on an orientable Riemannian surface can be isotoped to a simple geodesic or to a point.

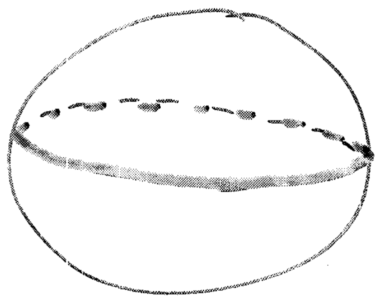


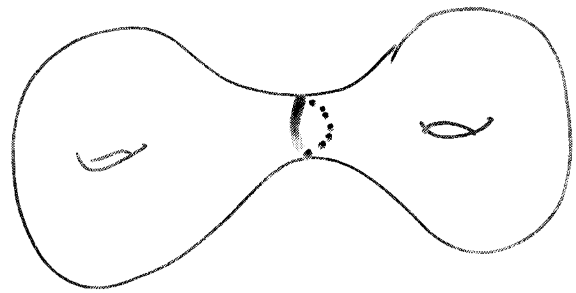
"Shrinking" a curve as fast as possible leads to a geodesic or point. (Grayson)

The proof of this result is much harder. The existence of a geodesic in the homotopy class is not easy, and embeddedness of this geodesic is also a difficult problem.

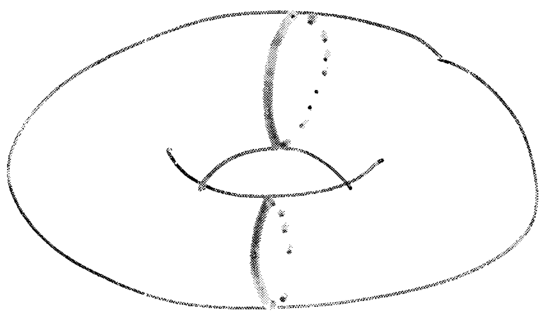
This suggests a connection between geodesics and normal curves. We will pursue this, and the corresponding connection between minimal and normal surfaces.

Geodesics have the property of minimizing length locally.

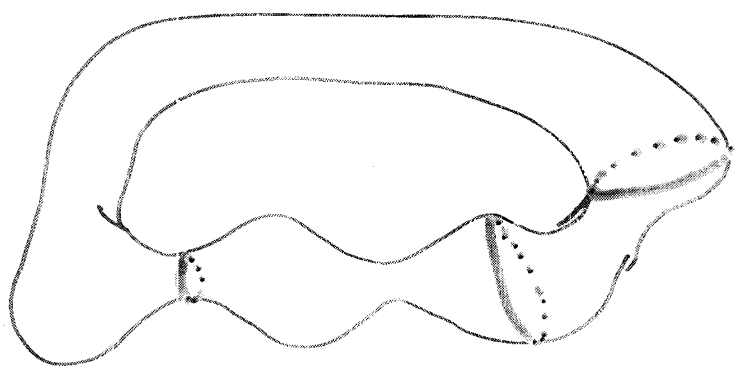




Equator of a sphere.

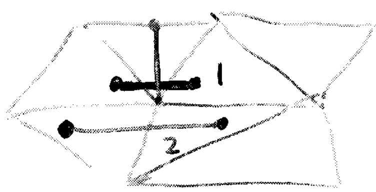


Meridian of a torus

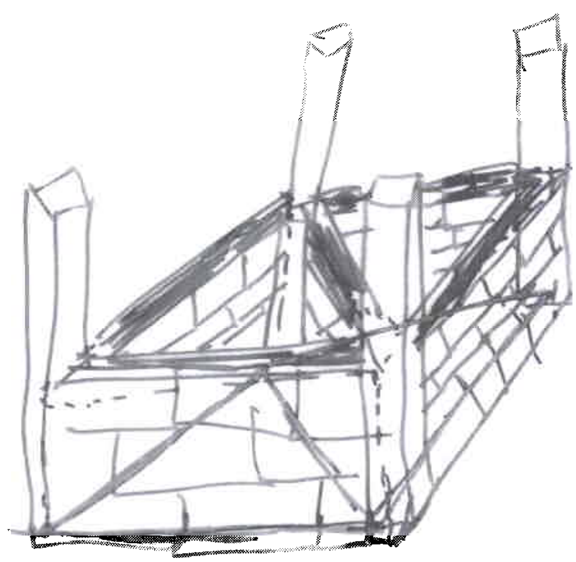


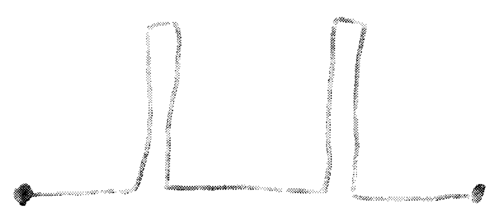
Meridian of a distorted torus. Any metric on a torus gives a length minimizing geodesic in each homotopy class.

In a triangulated surface, we can take metrics that are concentrated at the edges of a triangulation. Along the edges the metric looks like a wall that must be crossed. At vertices the metric has towers. Shortest geodesics cross these "walls" in as few points as possible.



Curve 2 is about 1/3 as big as curve 1.





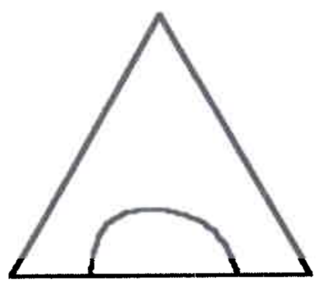
Profile of a shortest curve.

Result: Shortest geodesics in these special metrics are much like normal curves: ``*Normal curves are the geodesics (locally shortest curves) of triangulated (or PL) surfaces.*

<u>Correspondence</u>

<u>Riemannian Surface</u>		<u>Triangulated Surface</u>
Length	$\leftrightarrow$	Weight
Geodesic	$\leftrightarrow$	Normal Curve
Shortest geodesic	$\leftrightarrow$	Least weight curve

The <u>weight</u> of a curve is the number of times it crosses the edges of a triangulation. Minimizing weight by isotopies gives a normal curve.

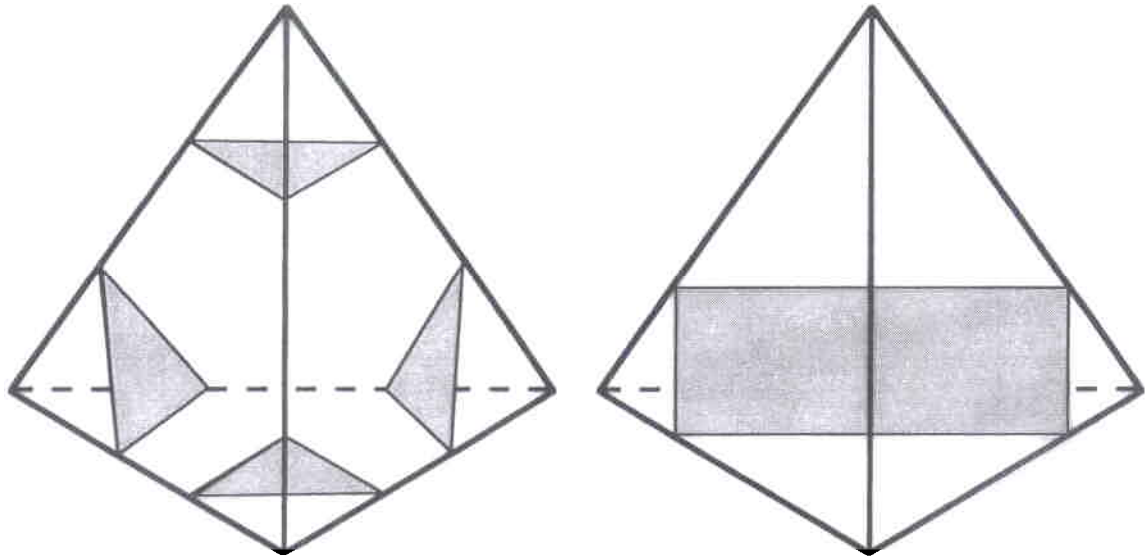


Excessive weight

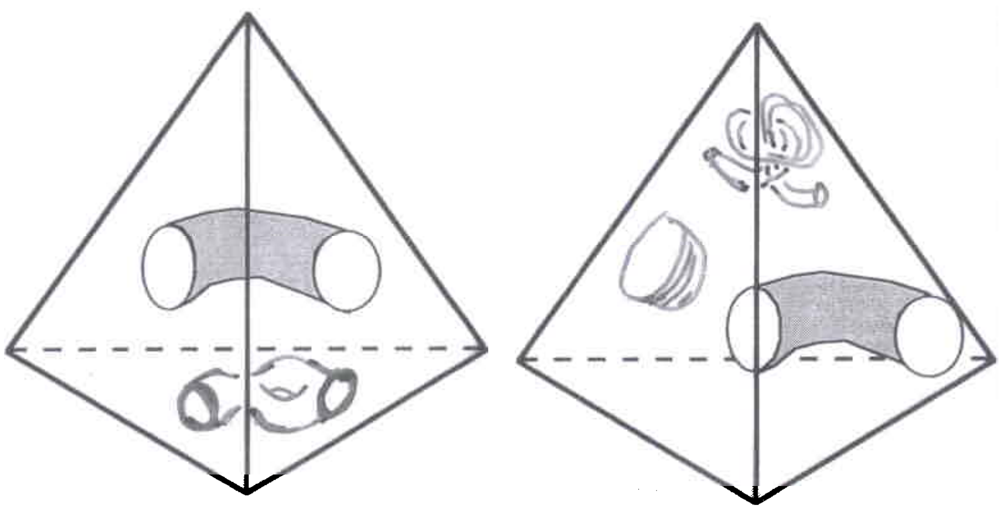
Normal Surfaces in 3-Manifolds

A Key Tool for algorithms in 3-dimensions: Normal Surfaces (Kneser 1929, Haken 1961)

In triangulated 3-manifolds, an attempt to push the surface around until it becomes as simple as possible gives rise to normal surfaces. Normal surfaces are the discrete versions of minimal surfaces. They again form surfaces that minimize intersection with the edges of a triangulation.

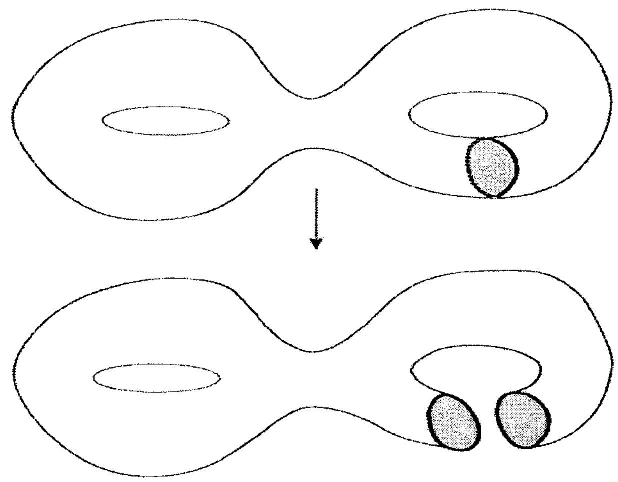


<u>Definition</u>: Normal surfaces are surfaces that intersect each tetrahedron of a triangulation in elementary disks.



No tubes or nonelementary disks are allowed.

Theorem (Kneser) Any embedded surface F in a triangulated 3-manifold can be isotoped (pushed around) and compressed (surgered) until it is either normal or lies in a single tetrahedron.

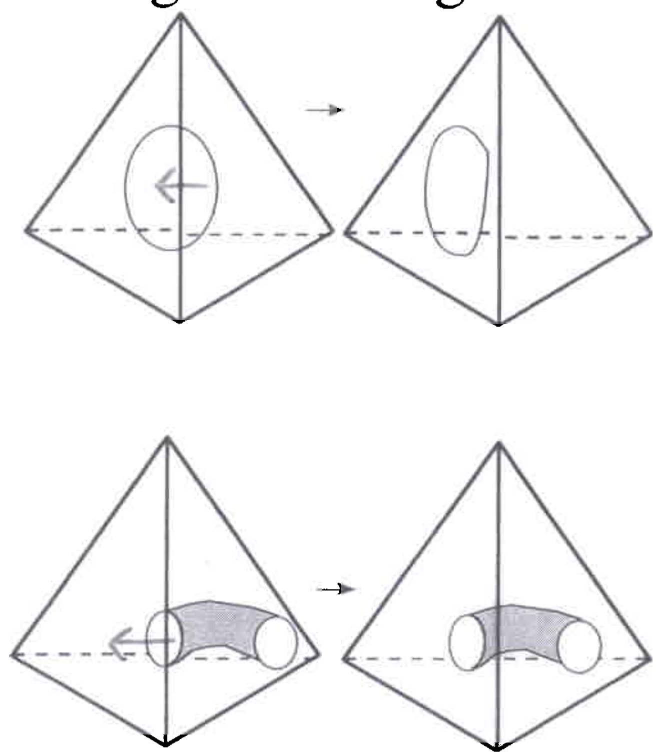


A compression

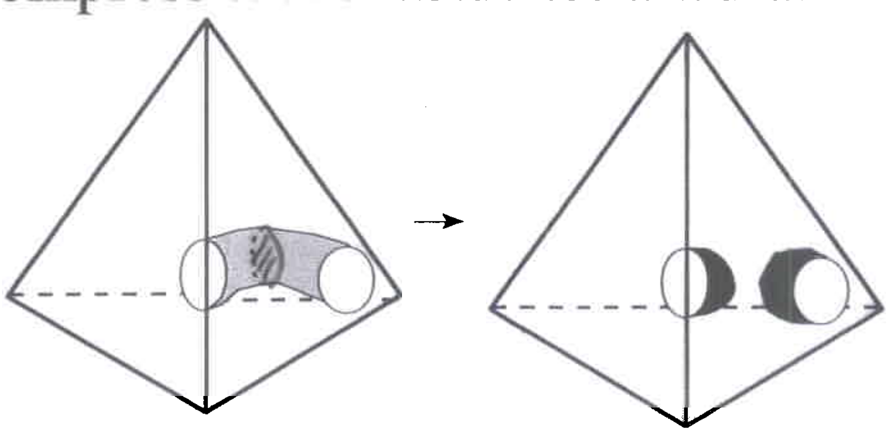
Proof: Start with any surface and look at how it intersects triangular faces of the triangulation of M.

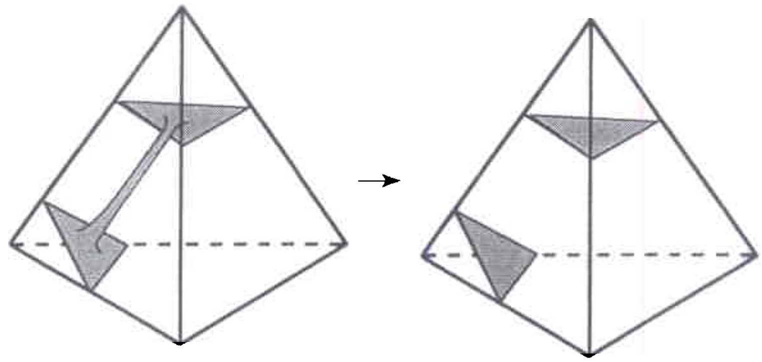
We can push a surface around in M to do two things:

1. Simplify curves of intersection of the surface with edges of triangulation.

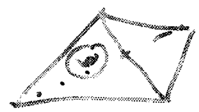


2. Compress tubes inside tetrahedra.



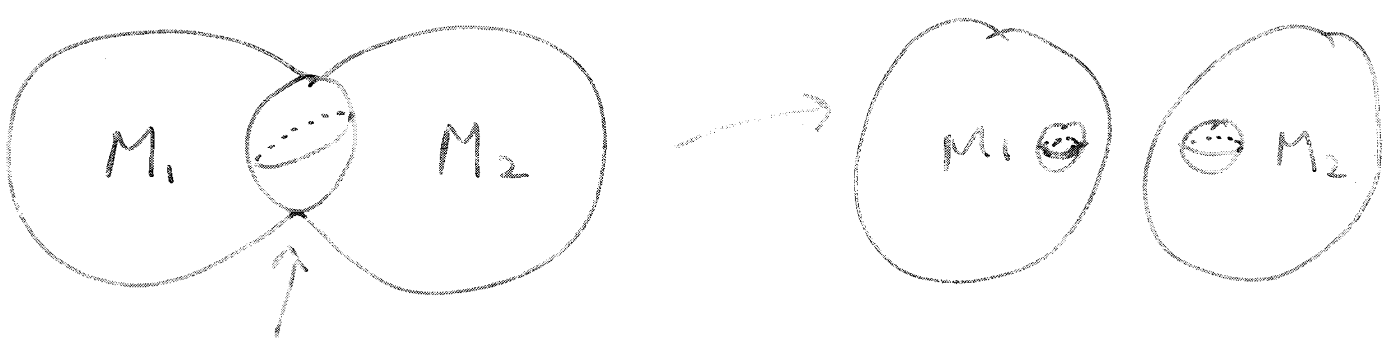


The number of intersections with the edges decreases. When we stop, we either get elementary disks or pieces lying within a tetrahedron.



First important application:

A 3-manifold is <u>prime</u> if it cannot be split into two 3-manifolds by cutting along a 2-sphere (except for a trivial 2-sphere, one bounding a ball).



separating 2-sphere

$$M_1 \# M_2 = (M_1 - \text{Ball}) \cup_{S^2} (M_2 - \text{Ball})$$

Theorem (Kneser Finiteness)
A 3-manifold can be split along 2-spheres into prime pieces.

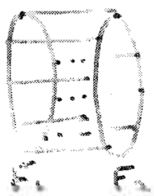
In other words, 3-manifolds have prime decompositions.

Idea of proof: Cut along a 2-sphere. Look at the pieces. Is there another non-trivial 2-sphere in each, not parallel to the first 2-sphere? How many times can this repeat?

If we have a collection of K non-parallel 2-spheres, we can normalize the entire collection simultaneously. The question reduces to,

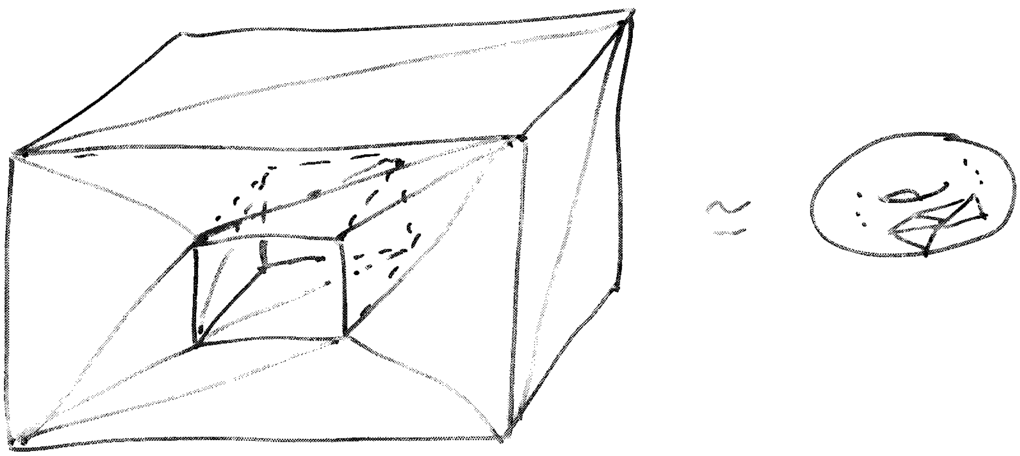
How many non-parallel 2-spheres can be simultaneously embedded in M? Finitely many?
Can we keep splitting forever?

The answer is that if M has t tetrahedra, we cannot have more than 10t normal surfaces, without having two surfaces parallel.

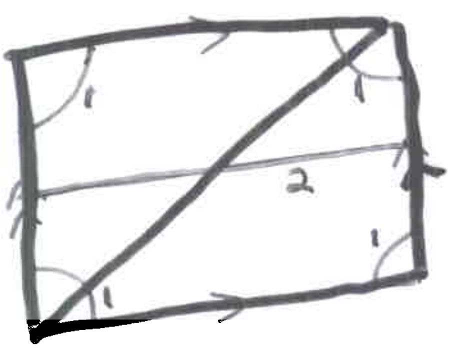


Parallel means F_1 and F_2 are boundary components of $F_1 \times I$.

We will prove something simpler: there are at most 6t non-parallel normal curves on an oriented surface made of t triangles.

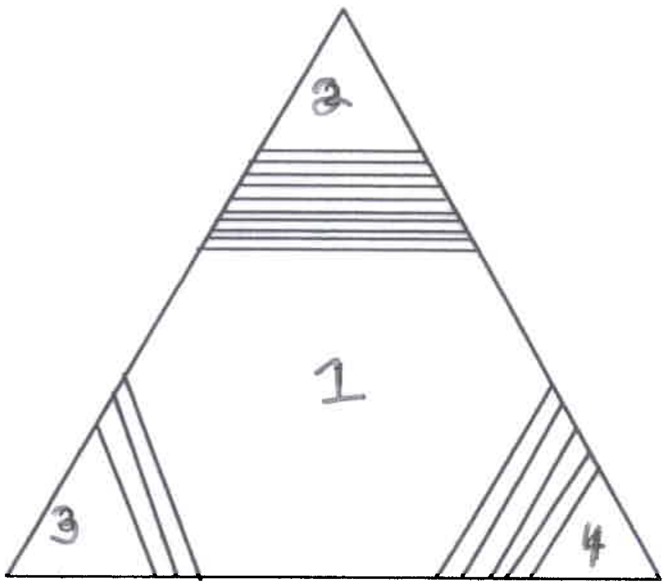


Torus with 32 triangles

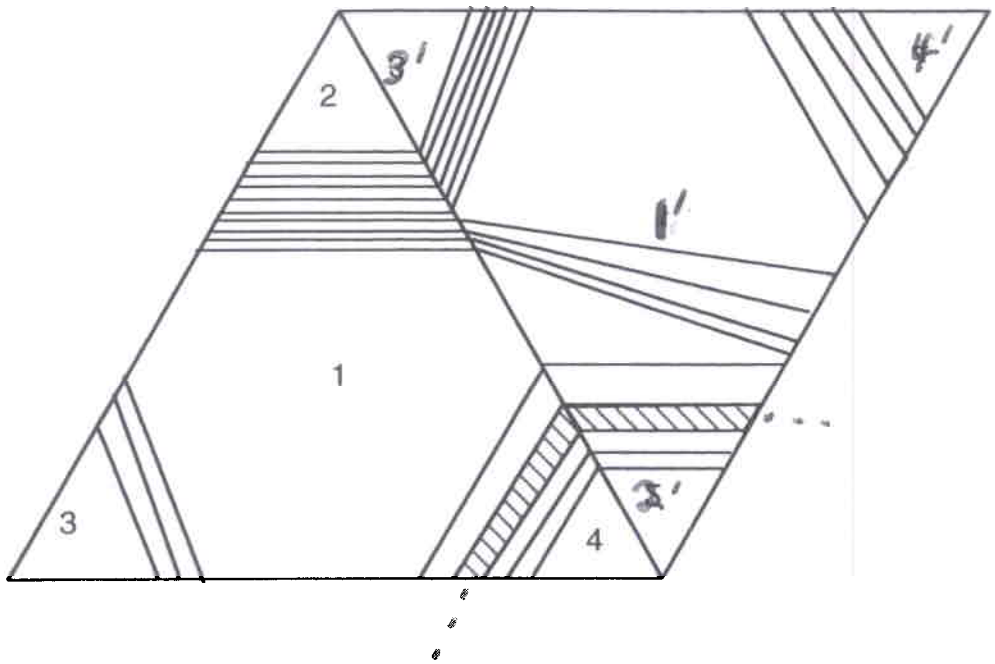


Torus with 2 triangles
(Has two non parallel normal curves)

Suppose we have a collection of k non-parallel normal curves.



The arcs of these normal curves in a triangle cut it into many small quadrilaterals, and four "bad regions". If two curves share a quadrilateral they are parallel in a triangle. If these quadrilaterals continue to form a loop of quadrilaterals, the curves are parallel.

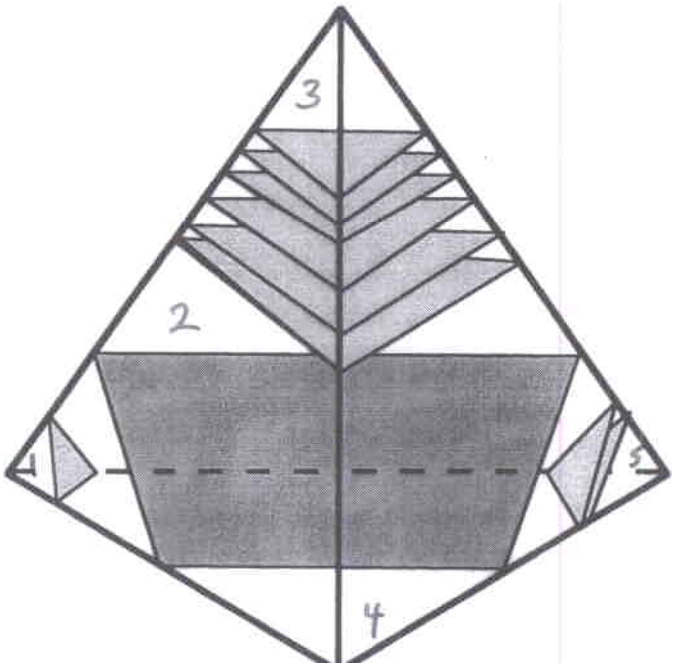


If the curves are not parallel, then they have a bad region between them.

Conclusion: Each curve meets at least one of the 4t bad regions.

In each triangle, the bad regions meet at most 6 curves. Therefore the bad regions meet at most 6t curves. Therefore if there are 6t+1 curves then one does not meet a bad region, and is parallel to a second curve.

Similar reasoning in a 3-manifold shows that there are at most 10t normal surfaces, no pair of which is parallel.



There can be at most 6 bad regions in a tetrahedron. These touch at most 10 surfaces.