TROPICAL CONVEXITY

MIKE DEVELIN AND BERND STURMFELS

Abstract. The notions of convexity and convex polytopes are introduced in the setting of tropical geometry. Combinatorial types of tropical polytopes are shown to be in bijection with regular triangulations of products of two simplices. Applications to phylogenetic trees are discussed.

1. Introduction

The *tropical semiring* $(\mathbb{R}, \oplus, \odot)$ is the set of real numbers with the arithmetic operations of *tropical addition*, which is taking the minimum of two numbers, and *tropical multiplication*, which is ordinary addition. Thus the two arithmetic operations are defined as follows:

$$a \oplus b \; := \; \min(a, b) \qquad \text{and} \qquad a \odot b \; := \; a + b.$$

The n-dimensional space $\mathbb{R}^n$ is a semimodule over the tropical semiring, with tropical addition

$$(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) \quad = \quad (x_1 \oplus y_1, \ldots, x_n \oplus y_n),$$

and tropical scalar multiplication

$$c \odot (x_1, x_2, \ldots, x_n) \quad = \quad (c \odot x_1, c \odot x_2, \ldots, c \odot x_n).$$

The semiring $(\mathbb{R}, \oplus, \odot)$ and its semimodule $\mathbb{R}^n$ obey the usual distributive and associative laws.

The purpose of this paper is to propose a tropical theory of convex polytopes. Convexity in arbitrary idempotent semimodules was introduced by Cohen, Gaubert and Quadrat [4] and Litvinov, Maslov and Shpiz [12]. We provide a combinatorial refinement of their approach which is consistent with the recent developments in tropical algebraic geometry (see [14], [17], [18]). The connection to tropical methods in representation theory (see [11], [15]) is less clear and deserves further study.

There are many notions of discrete convexity in the computational geometry literature, but none of them seems to be quite like tropical convexity. For instance, the notion of *directional convexity* studied by Matoušek [13] has similar features but it is different and much harder to compute with.

A subset S of $\mathbb{R}^n$ is called *tropically convex* if the set S contains the point $a \odot x \,\oplus\, b \odot y$ for all $x, y \in S$ and all $a, b \in \mathbb{R}$. The *tropical convex hull* of a given subset $V \subset \mathbb{R}^n$ is the smallest tropically convex subset of $\mathbb{R}^n$ which contains V. We shall see in Proposition 4 that the tropical convex hull of V coincides with the set of all tropical linear combinations

$$(1) \qquad a_1 \odot v_1 \,\oplus\, a_2 \odot v_2 \,\oplus\, \cdots \,\oplus\, a_r \odot v_r\,, \qquad \text{where } v_1, \ldots, v_r \in V \text{ and } a_1, \ldots, a_r \in \mathbb{R}.$$

Any tropically convex subset S of $\mathbb{R}^n$ is closed under tropical scalar multiplication, $\mathbb{R} \odot S \subseteq S$. In other words, if $x \in S$ then $x + \lambda(1, \ldots, 1) \in S$ for all $\lambda \in \mathbb{R}$. We will therefore identify the tropically convex set S with its image in the $(n-1)$-dimensional *tropical projective space*

$$\mathbb{TP}^{n-1} \quad = \quad \mathbb{R}^n/(1, \ldots, 1)\mathbb{R}.$$

Basic properties of (tropically) convex subsets in $\mathbb{TP}^{n-1}$ will be presented in Section 2. In Section 3 we introduce tropical polytopes and study their combinatorial structure. A *tropical polytope* is the tropical convex hull of a finite subset V in $\mathbb{TP}^{n-1}$. Every tropical polytope is a finite union of convex polytopes in the usual sense. The following main result will be proved in Section 4:

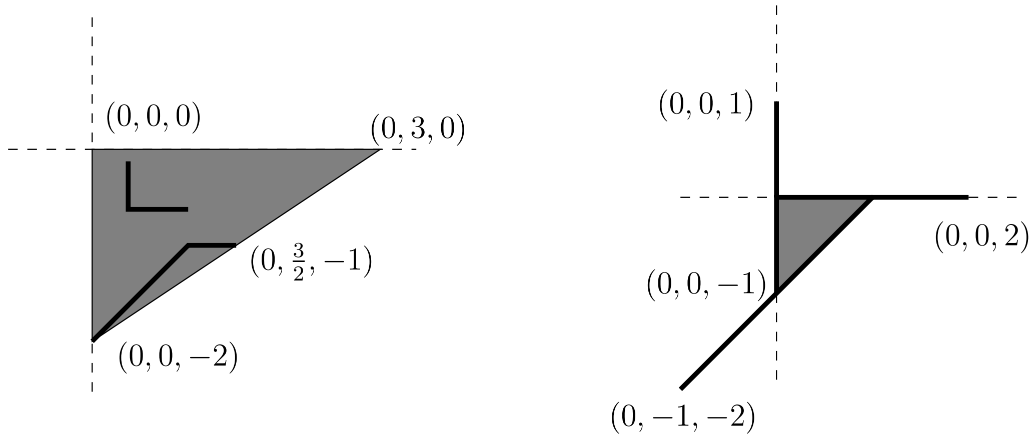

FIGURE 1. Tropical convex sets and tropical line segments in $\mathbb{TP}^2$.

Theorem 1. *The combinatorial types of tropical polytopes with r vertices in $\mathbb{TP}^{n-1}$ are in bijection with the regular polyhedral subdivisions of the product of two simplices $\Delta_{n-1} \times \Delta_{r-1}$.*

This implies a remarkable duality between tropical $(n-1)$-polytopes with r vertices and tropical $(r-1)$-polytopes with n vertices. Another consequence of Theorem 1 is a formula for the f-vector of a generic tropical polytope. In Section 5 we discuss applications of tropical convexity to phylogenetic analysis, extending known results on spaces of trees (see e.g. [2], [7], [8], [9] and [18]).

2. Tropically convex sets

We begin with two pictures of tropical convex sets in the tropical plane $\mathbb{TP}^2$. A point $(x_1, x_2, x_3) \in \mathbb{TP}^2$ is represented by drawing the point with coordinates $(x_2 - x_1, x_3 - x_1)$ in the paper plane. The triangle on the left hand side in Figure 1 is tropically convex but it is not a tropical polytope because it is not the tropical convex hull of finitely many points. The thick edges indicate two tropical line segments. The picture on the right hand side is a *tropical triangle*, namely, it is the tropical convex hull of the three points $(0, 0, 1)$, $(0, 2, 0)$ and $(0, -1, -2)$ in the tropical plane $\mathbb{TP}^2$. The thick edges represent the tropical segments connecting any two of these three points.

We next show that tropical convex sets enjoy many of the features of ordinary convex sets.

Theorem 2. *The intersection of two tropically convex sets in $\mathbb{R}^n$ or in $\mathbb{TP}^{n-1}$ is tropically convex. The projection of a tropically convex set onto a coordinate hyperplane is tropically convex. The ordinary hyperplane $\{x_i - x_j = l\}$ is tropically convex, and the projection map from this hyperplane to $\mathbb{R}^{n-1}$ given by eliminating x_i is an isomorphism of tropical semimodules. Tropically convex sets are contractible spaces. The Cartesian product of two tropically convex sets is tropically convex.*

Proof. We prove the statements in the order given. If S are T are tropically convex, then for any two points $x, y \in S \cap T$, both S and T contain the tropical line segment between x and y, and consequently so does $S \cap T$. Therefore $S \cap T$ is tropically convex by definition.

Suppose S is a tropically convex set in $\mathbb{R}^n$. We wish to show that the image of S under the coordinate projection $\phi : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $(x_1, x_2, \dots, x_n) \mapsto (x_2, \dots, x_n)$ is a tropically convex subset of $\mathbb{R}^{n-1}$. If $x, y \in S$ then we have the obvious identity

$$\phi\big(c \odot x \,\oplus\, d \odot y\big) \quad = \quad c \odot \phi(x) \,\oplus\, d \odot \phi(y).$$

This means that ϕ is a homomorphism of tropical semimodules. Therefore, if S contains the tropical line segment between x and y, then $\phi(S)$ contains the tropical line segment between $\phi(x)$ and $\phi(y)$ and hence is tropically convex. The same holds for the induced map $\phi : \mathbb{TP}^{n-1} \to \mathbb{TP}^{n-2}$.

Most ordinary hyperplanes in $\mathbb{R}^n$ are not tropically convex, but we are claiming that hyperplanes of the special form $x_i - x_j = k$ are tropically convex. If x and y lie in that hyperplane then $x_i - y_i = x_j - y_j$. This last equation implies the following identity for any real numbers $c, d \in \mathbb{R}$:

$$(c \odot x \oplus d \odot y)_i - (c \odot x \oplus d \odot y)_j \quad = \quad \min(x_i + c, y_i + d) - \min(x_j + c, y_j + d) \quad = \quad k.$$

Hence the tropical line segment between x and y also lies in the hyperplane $\{x_i - x_j = k\}$.

Consider the map from $\{x_i - x_j = k\}$ to $\mathbb{R}^{n-1}$ given by deleting the i-th coordinate. This map is injective: if two points differ in the x_i coordinate they must also differ in the x_j coordinate. It is clearly surjective because we can recover an i-th coordinate by setting $x_i = x_j + k$. Hence this map is an isomorphism of $\mathbb{R}$-vector spaces and it is also an isomorphism of $(\mathbb{R}, \oplus, \odot)$-semimodules.

Let S be a tropically convex set in $\mathbb{R}^n$ or $\mathbb{TP}^{n-1}$. Consider the family of hyperplanes $H_l = \{x_1 - x_2 = l\}$ for $l \in \mathbb{R}$. We know that the intersection $S \cap H_l$ is tropically convex, and isomorphic to its (convex) image under the map deleting the first coordinate. This image is contractible by induction on the dimension n of the ambient space. Therefore, $S \cap H_l$ is contractible. The result then follows from the topological result that if S is connected, which all tropically convex sets obviously are, and if $S \cap H_l$ is contractible for each l, then S itself is also contractible.

Suppose that $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ are tropically convex. Our last assertion states that $S \times T$ is a tropically convex subset of $\mathbb{R}^{n+m}$. Take any (x, y) and (x', y') in $S \times T$ and $c, d \in \mathbb{R}$. Then

$$c \odot (x, y) \oplus d \odot (x', y') \quad = \quad \big(c \odot x \oplus d \odot x' \, , \, c \odot y \oplus d \odot y' \big)$$

lies in $S \times T$ since S and T are tropically convex. □

We next give a more precise description of what tropical line segments look like.

Proposition 3. *The tropical line segment between two points x and y in $\mathbb{TP}^{n-1}$ is the concatenation of at most $n - 1$ ordinary line segments. The slope of each line segment is a zero-one vector.*

Proof. After relabeling the coordinates of $x = (x_1, , \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we may assume

$$y_1 - x_1 \leq y_2 - x_2 \leq \cdots \leq y_n - x_n. \tag{2}$$

The following points lie in the given order on the tropical segment between x and y:

$$\begin{array}{rcl}
x \;=\; (y_1 - x_1) \odot x \oplus y & = & \big(y_1,\, y_1{-}x_1{+}x_2, y_1{-}x_1{+}x_3, \ldots, y_1{-}x_1{+}x_{n-1}, y_1{-}x_1{+}x_n\big) \\
(y_2 - x_2) \odot x \oplus y & = & \big(y_1,\, y_2,\, y_2{-}x_2{+}x_3, \ldots, y_2{-}x_2{+}x_{n-1}, y_2{-}x_2{+}x_n\big) \\
(y_3 - x_3) \odot x \oplus y & = & \big(y_1,\, y_2,\, y_3,\, \ldots, y_3{-}x_3{+}x_{n-1}, y_3{-}x_3{+}x_n\big) \\
\cdots\cdots & & \cdots\cdots \quad \cdots\cdots \\
(y_{n-1} - x_{n-1}) \odot x \oplus y & = & \big(y_1,\, y_2,\, y_3,\, \ldots, y_{n-1}, y_{n-1}{-}x_{n-1}{+}x_n\big) \\
y \;=\; (y_n - x_n) \odot x \oplus y & = & \big(y_1,\, y_2,\, y_3,\, \ldots,\, y_{n-1},\, y_n\big).
\end{array}$$

Between any two consecutive points, the tropical line segment agrees with the ordinary line segment, which has slope $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$. Hence the tropical line segment between x and y is the concatenation of at most $n - 1$ ordinary line segments, one for each strict inequality in (2). □

This description of tropical segments shows an important feature of tropical polytopes: their edges use a limited set of directions. The following result characterizes the *tropical convex hull.*

Proposition 4. *The smallest tropically convex subset of $\mathbb{TP}^{n-1}$ which contains a given set V coincides with the set of all tropical linear combinations (1). We denote this set by* $\operatorname{tconv}(V)$.

Proof. Let $x = \bigoplus_{i=1}^r a_i \odot v_i$ be the point in (1). If $r \leq 2$ then x is clearly in the tropical convex hull of V. If $r > 2$ then we write $x = a_1 \odot v_1 \oplus (\bigoplus_{i=2}^r a_i \odot v_i)$. The parenthesized vector lies the tropical convex hull, by induction on r, and hence so does x. For the converse, consider any two

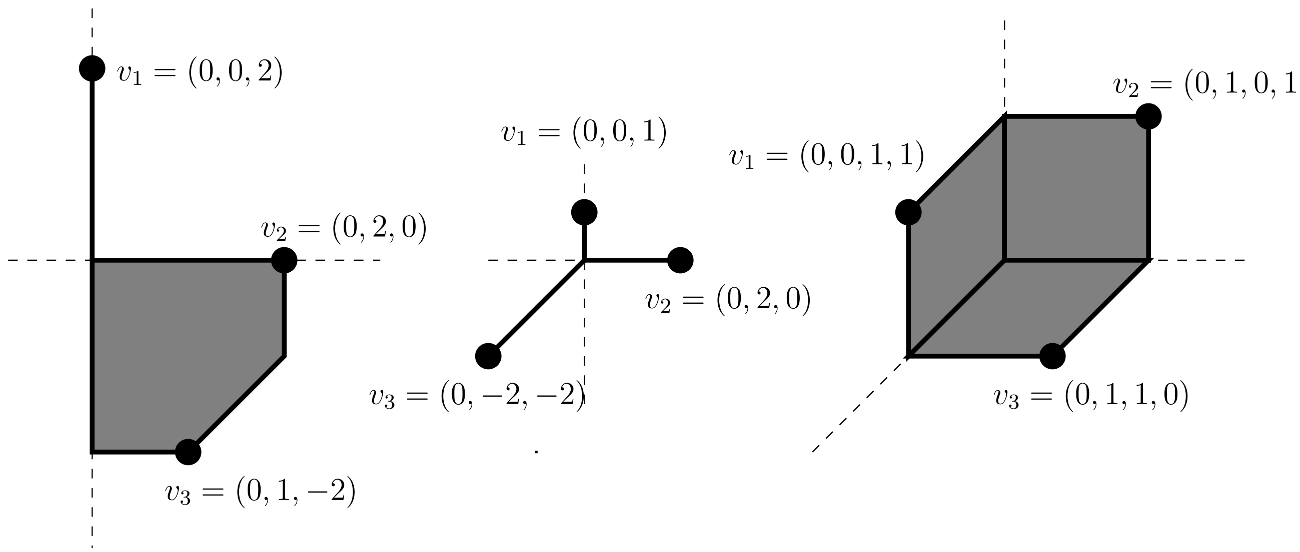

FIGURE 2. Three tropical polytopes in $\mathbb{TP}^2$

tropical linear combinations $x = \bigoplus_{i=1}^r c_i \odot v_i$ and $y = \bigoplus_{j=1}^r d_i \odot v_i$. By the distributive law, $a \odot x \oplus b \odot y$ is also a tropical linear combination of $v_1, \ldots, v_r \in V$. Hence the set of all tropical linear combinations of V is tropically convex, so it contains the tropical convex hull of V. ☐

If V is a finite subset of $\mathbb{TP}^{n-1}$ then $\mathrm{tconv}(V)$ is a *tropical polytope*. In Figure 2 we see three small examples of tropical polytopes. The first and second are tropical convex hulls of three points in $\mathbb{TP}^2$. The third tropical polytope lies in $\mathbb{TP}^3$ and is the union of three squares.

One of the basic results in the usual theory of convex polytopes is Carathéodory's theorem. This theorem holds in the tropical setting.

Proposition 5 (Tropical Carathéodory's Theorem). *If x is in the tropical convex hull of a set of r points v_i in $\mathbb{TP}^{n-1}$, then x is in the tropical convex hull of at most n of them.*

Proof. Let $x = \bigoplus_{i=1}^r a_i \odot v_i$ and suppose $r > n$. For each coordinate $j \in \{1, \ldots, n\}$, there exists an index $i \in \{1, \ldots, r\}$ such that $x_j = c_i + v_{ij}$. Take a subset I of $\{1, \ldots, r\}$ composed of one such i for each j. Then we also have $x = \bigoplus_{i \in I} a_i \odot v_i$, where $\#(I) \leq n$. ☐

The basic theory of tropical linear subspaces $\mathbb{TP}^{n-1}$ was developed in [17] and [18]. Recall that the *tropical hyperplane* defined by a tropical linear form $a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus \cdots \oplus a_n \odot x_n$ consists of all points $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{TP}^{n-1}$ such that the following holds (in ordinary arithmetic):

$$a_i + x_i \;=\; a_j + x_j \;=\; \min\{a_k + x_k \,:\, k = 1, \ldots, n\} \qquad \text{for some indices } i \neq j. \tag{3}$$

Just like in ordinary geometry, hyperplanes are convex sets:

Proposition 6. *Tropical hyperplanes in $\mathbb{TP}^{n-1}$ are tropically convex.*

Proof. Let H be the hyperplane defined by (3). Suppose that x and y lie in H and consider any tropical linear combination $z = c \odot x \oplus d \odot y$. Let i be an index which minimizes $a_i + z_i$. We need to show that this minimum is attained at least twice. By definition, z_i is equal to either $c + x_i$ or $d + y_i$, and, after permuting x and y, we may assume $z_i = c + x_i \leq d + y_i$. Since, for all k, $a_i + z_i \leq a_k + z_k$ and $z_k \leq c + x_k$, it follows that $a_i + x_i \leq a_k + x_k$ for all k, so that $a_i + x_i$ achieves the minimum of $\{a_1 + x_1, \ldots, a_n + x_n\}$. Since x is in H, there exists some index $j \neq i$ for which

$a_i+x_i = a_j+x_j$. But now $a_j+z_j \leq a_j+c+x_j = c+a_i+x_i = a_i+z_i$. Since a_i+z_i is the minimum of all a_j+z_j, the two must be equal, and this minimum is obtained at least twice as desired. $\square$

Proposition 6 implies that if V is a subset of $\mathbb{TP}^{n-1}$ which happens to lie in a tropical hyperplane H, then its tropical convex hull $\mathrm{tconv}(V)$ will lie in H as well. The same holds for tropical planes of higher codimension. Recall that every *tropical plane* is an intersection of tropical hyperplanes [18]. But the converse does not hold: not every intersection of tropical hyperplanes qualifies as a tropical plane (see [17, §5]). Proposition 6 and the first statement in Theorem 2 imply:

Corollary 7. *Tropical planes in $\mathbb{TP}^{n-1}$ are tropically convex.*

A theorem in classical geometry states that every point outside a closed convex set can be separated from the convex set by a hyperplane. The same statement holds in tropical geometry. This follows from the results in [4]. Some caution is needed, however, since the definition of hyperplane in [4] differs from our definition of hyperplane, as explained in [17]. In our definition, a tropical hyperplane is a fan which divides $\mathbb{TP}^{n-1}$ into n convex cones, each of which is also tropically convex. Rather than stating the most general separation theorem, we will now focus our attention on tropical polytopes, in which case the separation theorem is the Farkas Lemma stated below.

3. Tropical polytopes as cell complexes

Throughout this section we fix a finite subset $V = \{v_1, v_2, \ldots, v_r\}$ of tropical projective space $\mathbb{TP}^{n-1}$. Here $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$. Our goal is to study the tropical polytope $P = \mathrm{tconv}(V)$. We begin by describing the natural cell decomposition of $\mathbb{TP}^{n-1}$ induced by the fixed finite subset V.

Let x be any point in $\mathbb{TP}^{n-1}$. The *type* of x relative to V is the ordered n-tuple $(S_1, \ldots, S_n)$ of subsets $S_j \subseteq \{1, 2, \ldots, r\}$ which is defined as follows: An index i is in S_j if

$$v_{ij} - x_j \quad = \quad \min(v_{i1} - x_1, v_{i2} - x_2, \ldots, v_{in} - x_n).$$

Equivalently, if we set $\lambda_i = \min\{\, \lambda \in \mathbb{R} \,:\, \lambda \odot v_i \oplus x = x \,\}$ then S_j is the set of all indices i such that $\lambda_i \odot v_i$ and x have the same j-th coordinate. We say that an n-tuple of indices $S = (S_1, \ldots, S_n)$ is a *type* if it arises in this manner. Note that every i must be in some S_j.

Example 8. Let $r = n = 3$, $v_1 = (0, 0, 2)$, $v_2 = (0, 2, 0)$ and $v_3 = (0, 1, -2)$. There are 30 possible types as x ranges over the plane $\mathbb{TP}^2$. The corresponding cell decomposition has six convex regions (one bounded, five unbounded), 15 edges (6 bounded, 9 unbounded) and 6 vertices. For instance, the point $x = (0, 1, -1)$ has $\mathrm{type}(x) = \big(\{2\}, \{1\}, \{3\}\big)$ and its cell is a bounded pentagon. The point $x' = (0, 0, 0)$ has $\mathrm{type}(x') = \big(\{1, 2\}, \{1\}, \{2, 3\}\big)$ and its cell is one of the six vertices. The point $x'' = (0, 0, -3)$ has $\mathrm{type}(x'') = \big\{\{1, 2, 3\}, \{1\}, \emptyset\big)$ and its cell is an unbounded edge.

Our first application of types is the following separation theorem.

Proposition 9 (Tropical Farkas Lemma)**.** *For all $x \in \mathbb{TP}^{n-1}$, exactly one of the following is true.*
(i) the point x is in the tropical polytope $P = \mathrm{tconv}(V)$, or
(ii) there exists a tropical hyperplane which separates x from P.

The separation statement in part (ii) means the following: if the hyperplane is given by (3) and $a_k + x_k = \min(a_1 + x_1, \ldots, a_n + x_n)$ then $a_k + y_k > \min(a_1 + y_1, \ldots, a_n + y_n)$ for all $y \in P$.

Proof. Consider any point $x \in \mathbb{TP}^{n-1}$, with $\mathrm{type}(x) = (S_1, \ldots, S_n)$, and let $\lambda_i = \min\{\, \lambda \in \mathbb{R} \,:\, \lambda \odot v_i \oplus x = x \,\}$ as before. We define

$$\pi_V(x) \quad = \quad \lambda_1 \odot v_1 \,\oplus\, \lambda_2 \odot v_2 \,\oplus\, \cdots \,\oplus\, \lambda_r \odot v_r. \tag{4}$$

There are two cases: either $\pi_V(x) = x$ or $\pi_V(x) \neq x$. The first case implies (i). Since (i) and (ii) clearly cannot occur at the same time, it suffices to prove that the second case implies (ii).

Suppose that $\pi_V(x) \neq x$. Then S_k is empty for some index $k \in \{1,\ldots,n\}$. This means that $v_{ik} + \lambda_i - x_k > 0$ for $i = 1,2,\ldots,r$. Let $\varepsilon > 0$ be smaller than any of these r positive reals. We now choose our separating tropical hyperplane (3) as follows:

$$a_k \;:=\; -x_k - \varepsilon \qquad \text{and} \qquad a_j \;:=\; -x_j \;\text{ for } j \in \{1,\ldots,n\}\backslash\{k\}. \tag{5}$$

This certainly satisfies $a_k + x_k = \min(a_1+x_1 \ldots, a_n+x_n)$. Now, consider any point $y = \bigoplus_{i=1}^r c_i \odot v_i$ in $\operatorname{tconv}(V)$. Pick any m such that $y_k = c_m + v_{mk}$. By definition of the λ_i, we have $x_k \leq \lambda_m + v_{mk}$ for all k, and there exists some j with $x_j = \lambda_m + v_{mj}$. These equations and inequalities imply

$$\begin{aligned} a_k + y_k \;&=\; a_k + c_m + v_{mk} \;=\; c_m + v_{mk} - x_k - \varepsilon \;>\; c_m + v_{mk} - x_k \;\geq\; c_m - \lambda_m \\ &=\; c_m + v_{mj} - x_j \;\geq\; y_j - x_j \;=\; a_j + y_j \;\;\geq\;\; \min(a_1 + y_1, \ldots, a_n + y_n). \end{aligned}$$

Therefore, the hyperplane defined by (5) separates x from P as desired. □

The construction in (4) defines a map $\pi_V : \mathbb{TP}^{n-1} \to P$ whose restriction to P is the identity. This map is the tropical version of the *nearest point map* onto a closed convex set in ordinary geometry. Such maps were studied in [4] for convex subsets in arbitrary idempotent semimodules.

If $S = (S_1,\ldots,S_n)$ and $T = (T_1,\ldots,T_n)$ are n-tuples of subsets of $\{1,2,\ldots,r\}$, then we write $S \subseteq T$ if $S_j \subseteq T_j$ for $j = 1,\ldots,n$. We also consider the set of all points whose type contains S:

$$X_S \quad := \quad \{\, x \in \mathbb{TP}^{n-1} \,:\, S \subseteq \operatorname{type}(x) \,\}.$$

Lemma 10. *The set X_S is a closed convex polyhedron (in the usual sense). More precisely,*

$$X_S \quad = \quad \{\, x \in \mathbb{TP}^{n-1} \,:\, x_k - x_j \leq v_{ik} - v_{ij} \;\; \textit{for all } j,k \in \{1,\ldots,n\} \textit{ such that } i \in S_j \,\}. \tag{6}$$

Proof. Let $x \in \mathbb{TP}^{n-1}$ and $T = \operatorname{type}(x)$. First, suppose x is in X_S. Then $S \subseteq T$. For every i,j,k such that $i \in S_j$, we also have $i \in T_j$, and so by definition we have $v_{ij} - x_j \leq v_{ik} - x_k$, or $x_k - x_j \leq v_{ik} - v_{ij}$. Hence x lies in the set on the right hand side of (6). For the proof of the reverse inclusion, suppose that x lies in the right hand side of (6). Then, for all i,j with $i \in S_j$, and for all k, we have $v_{ij} - x_j \leq v_{ik} - x_k$. This means that $v_{ij} - x_j = \min(v_{i1} - x_1, \ldots, v_{in} - x_n)$ and hence $i \in T_j$. Consequently, for all j, we have $S_j \subset T_j$, and so $x \in X_S$. □

As an example for Lemma 10, we consider the region $X_{(2,1,3)}$ in the tropical convex hull of $v_1 = (0,0,2)$, $v_2 = (0,2,0)$, and $v_3 = (0,1,-2)$. This region is defined by six linear inequalities, one of which is redundant, as depicted in Figure 3. Lemma 10 has the following immediate corollaries.

Corollary 11. *The intersection $X_S \cap X_T$ is equal to the polyhedron $X_{S \cup T}$.*

Proof. The inequalities defining $X_{S\cup T}$ are precisely the union of the inequalities defining X_S and X_T, and points satisfying these inequalities are precisely those in $X_S \cap X_T$. □

Corollary 12. *The polyhedron X_S is bounded if and only if $S_j \neq \emptyset$ for all $j = 1,2,\ldots,n$.*

Proof. Suppose that $S_j \neq \emptyset$ for all $j = 1,2,\ldots,n$. Then for every j and k, we can find $i \in S_j$ and $m \in S_k$, which via Lemma 10 yield the inequalities $v_{mk} - v_{mj} \leq x_k - x_j \leq v_{ik} - v_{ij}$. This implies that each $x_k - x_j$ is bounded on X_S, which means that X_S is a bounded subset of $\mathbb{TP}^{n-1}$.

Conversely, suppose some S_j is empty. Then the only inequalities involving x_j are of the form $x_j - x_k \leq c_{jk}$. Consequently, if any point x is in S_j, so too is $x - ke_j$ for $k > 0$, where e_j is the j-th basis vector. Therefore, in this case, X_S is unbounded. □

Corollary 13. *Suppose we have $S = (S_1,\ldots,S_n)$, with $S_1 \cup \cdots \cup S_n = \{1,\ldots,r\}$. Then if $S \subseteq T$, X_T is a face of X_S, and furthermore all faces of X_S are of this form.*

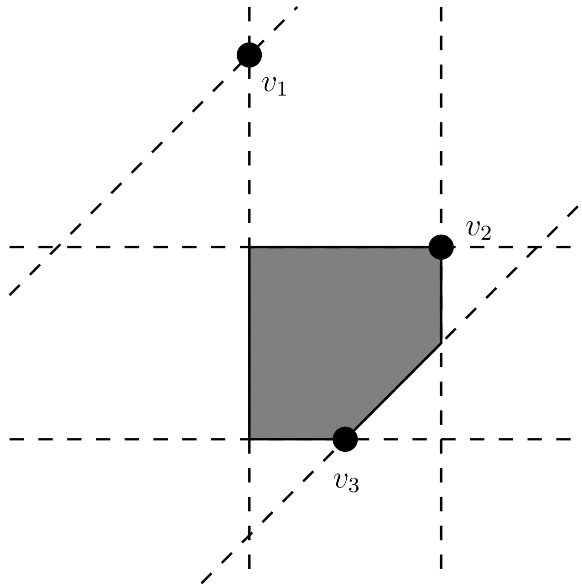

FIGURE 3. The region $X_{(2,1,3)}$ in the tropical convex hull of v_1, v_2 and v_3.

Proof. For the first part, it suffices to prove that the statement is true when T covers S in the poset of containment, i.e. when $T_j = S_j \cup \{i\}$ for some $j \in \{1, \ldots, n\}$ and $i \notin S_j$, and $T_k = S_k$ for $k \neq j$.

We have the inequality presentation of X_S given by Lemma 10. By the same lemma, the inequality presentation of X_T consists of the inequalities defining X_S together with the inequalities

$$\{x_k - x_j \leq v_{ik} - v_{ij} \mid k \in \{1, \ldots, n\}\}. \tag{7}$$

By assumption, i is in some S_m. We claim that X_T is the face of S defined by the equality

$$x_m - x_j = v_{im} - v_{ij}. \tag{8}$$

Since X_S satisfies the inequality $x_j - x_m \leq v_{ij} - v_{im}$, (8) defines a face F of S. The inequality $x_m - x_j \leq v_{im} - v_{ij}$ is in the set (7), so (8) is valid on X_T and $X_T \subseteq F$. However, any point in F, being in X_S, satisfies $x_k - x_m \leq v_{ik} - v_{im}$ for all $k \in \{1, \ldots, n\}$. Adding (8) to these inequalities proves that the inequalities (7) are valid on F, and hence $F \subseteq X_T$. So $X_T = F$ as desired.

By the discussion in the proof of the first part, prescribing equality in the facet-defining inequality $x_k - x_j \leq v_{ik} - v_{ij}$ yields X_T, where $T_k = S_k \cup \{i\}$ and $T_j = S_j$ for $j \neq k$. Therefore, all facets of X_S can be obtained as regions X_T, and it follows recursively that all faces of X_S are of this form. ☐

Corollary 14. *Suppose that $S = (S_1, \ldots, S_n)$ is an n-tuple of indices satisfying $S_1 \cup \cdots \cup S_n = \{1, \ldots, r\}$. Then X_S is equal to X_T for some type T.*

Proof. Let x be a point in the relative interior of X_S, and let $T = \operatorname{type}(x)$. Since $x \in X_S$, T contains S, and by Lemma 13, X_T is a face of X_S. However, since x is in the relative interior of X_S, the only face of X_S containing x is X_S itself, so we must have $X_S = X_T$ as desired. ☐

We are now prepared to state our main theorem in this section.

Theorem 15. *The collection of convex polyhedra X_S, where S ranges over all types, defines a cell decomposition $\mathcal{C}_V$ of $\mathbb{TP}^{n-1}$. The tropical polytope $P = \operatorname{tconv}(V)$ equals the union of all bounded cells X_S in this decomposition.*

Proof. Since each point has a type, it is clear that the union of the X_S is equal to $\mathbb{TP}^{n-1}$. By Corollary 13, the faces of X_S are equal to X_U for $S \subseteq U$, and by Corollary 14, $X_U = X_W$ for

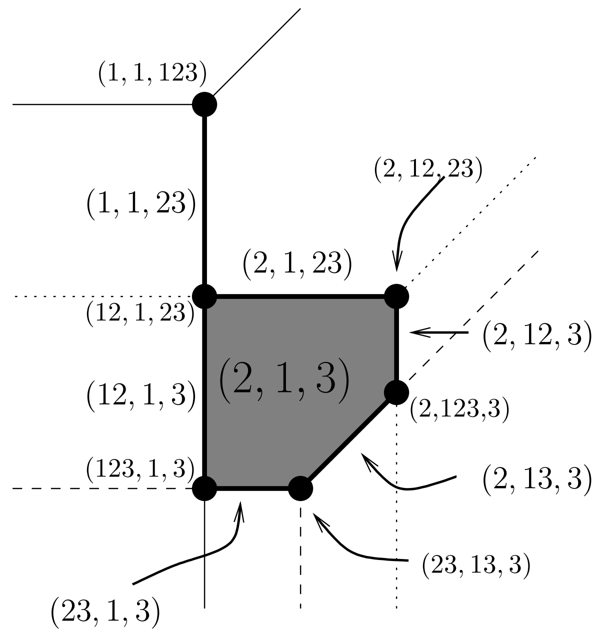

Figure 4. Tropical polytope expressed as the bounded cells in the common refinement of the fans $v_i - \mathcal{F}$, $v_2 - \mathcal{F}$ and $v_3 - \mathcal{F}$. Cells are labeled with their types.

some type W, and hence X_U is in our collection. The only thing remaining to check to show that this collection defines a cell decomposition is that $X_S \cap X_T$ is a face of both X_S and X_T, but $X_S \cap X_T = X_{S\cup T}$ by Corollary 11, and $X_{S\cup T}$ is a face of X_S and X_T by Corollary 13.

For the second assertion consider any point $x \in \mathbb{TP}^{n-1}$ and let $S = \mathrm{type}(x)$. We have seen in the proof of the Tropical Farkas Lemma (Proposition 9) that x lies in P if and only if no S_j is empty. By Corollary 12, this is equivalent to the polyhedron X_S being bounded. $\square$

Here is a nice geometric construction of the cell decomposition $\mathcal{C}_V$ of $\mathbb{TP}^{n-1}$ induced by $V = \{v_1, \ldots, v_r\}$. Let $\mathcal{F}$ be the fan in $\mathbb{TP}^{n-1}$ defined by the tropical hyperplane (3) with $a_1 = \cdots = a_n = 0$. Two vectors x and y lie in the same relatively open cone of the fan $\mathcal{F}$ if and only if

$$\{\, j \,:\, x_j \;=\; \min(x_1, \ldots, x_n)\,\} \quad = \quad \{\, j \,:\, x_j \;=\; \min(y_1, \ldots, y_n)\,\}.$$

If we translate the negative of $\mathcal{F}$ by the vector v_i then we get a new fan which we denote by $v_i - \mathcal{F}$. Two vectors x and y lie in the same relatively open cone of the fan $v_i - \mathcal{F}$ if and only

$$\{\, j \,:\, x_j - v_{ij} \;=\; \max(x_1 - v_{i1}, \ldots, x_n - v_{in})\,\} \quad = \quad \{\, j \,:\, y_j - v_{ij} \;=\; \max(y_1 - v_{i1}, \ldots, y_n - v_{in})\,\}.$$

Proposition 16. *The cell decomposition $\mathcal{C}_V$ is the common refinement of the r fans $v_i - \mathcal{F}$.*

Proof. We need to show that the cells of this common refinement are precisely the convex polyhedra X_S. Take a point x, with $T = \mathrm{type}(x)$ and define $S_x = (S_{x1}, \ldots, S_{xn})$ by letting $i \in S_{xj}$ whenever

$$x_j - v_{ij} \quad = \quad \max(x_1 - v_{i1}, \ldots, x_n - v_{in}). \tag{9}$$

Two points x and y are in the relative interior of the same cell of the common refinement if and only if they are in the same relatively open cone of each fan; this is tantamount to saying that $S_x = S_y$. However, we claim that $S_x = T$. Indeed, taking the negative of both sides of (9) yields exactly the condition for i being in T_j, by the definition of type. Consequently, the condition for two points having the same type is the same as the condition for the two points being in the relative interior of the same chamber of the common refinement of the fans $v_1 - \mathcal{F}$, $v_2 - \mathcal{F}$, $\ldots$, $v_r - \mathcal{F}$. $\square$

The next few results provide additional information about the polyhedron X_S. Let G_S denote the undirected graph with vertices $\{1, \ldots, n\}$, where $\{j, k\}$ is an edge if and only if $S_j \cap S_k \neq \emptyset$.

Proposition 17. *The dimension d of the polyhedron X_S is one less than the number of connected components of G_S, and X_S is affinely and tropically isomorphic to some polyhedron X_T in $\mathbb{TP}^d$.*

Proof. The proof is by induction on n. Suppose we have $i \in S_j \cap S_k$. Then X_S satisfies the linear equation $x_k - x_j = c$ where $c = v_{ik} - v_{ij}$. Eliminating the variable x_k (projecting onto $\mathbb{TP}^{n-2}$), X_S is affinely and tropically isomorphic to X_T where the type T is defined by $T_r = S_r$ for $r \neq j$ and $T_j = S_j \cup S_k$. The region X_T exists in the cell decomposition of $\mathbb{TP}^{n-2}$ induced by the vectors $w_1, \ldots, w_n$ with $w_{ir} = v_{ir}$ for $r \neq j$, and $w_{ij} = \max(v_{ij}, v_{ik} - c)$. The graph G_T is obtained from the graph G_S by contracting the edge $\{j, k\}$, and thus has the same number of connected components.

This induction on n reduces us to the case where all of the S_j are pairwise disjoint. We must show that X_S has dimension $n - 1$. Suppose not. Then X_S lies in $\mathbb{TP}^{n-1}$ but has dimension less than $n - 1$. One of the inequalities in (6) holds with equality, say $x_k - x_j = v_{ik} - v_{ij}$ for all $x \in X_S$. The inequality "$\leq$" implies $i \in S_j$ and the inequality "$\geq$" implies $i \in S_k$. Hence S_j and S_k are not disjoint, a contradiction. $\square$

The following proposition can be regarded as a converse to Lemma 10.

Proposition 18. *Let R be any polytope in $\mathbb{TP}^{n-1}$ defined by inequalities of the form $x_k - x_j \leq c_{jk}$. Then R arises as a cell X_S in the decomposition $\mathcal{C}_V$ of $\mathbb{TP}^{n-1}$ defined by some set $V = \{v_1, \ldots, v_n\}$.*

Proof. Define the vectors v_i to have coordinates $v_{ij} = c_{ij}$ for $i \neq j$, and $v_{ii} = 0$. (If c_{ij} did not appear in the given inequality presentation then simply take it to be a very large positive number.) Then by Lemma 10, the polytope in $\mathbb{TP}^{n-1}$ defined by the inequalities $x_k - x_j \leq c_{jk}$ is precisely the unique cell of type $(1, 2, \ldots, n)$ in the tropical conver hull of $\{v_1, \ldots, v_n\}$. $\square$

The region X_S is a polytope both in the ordinary sense and in the tropical sense.

Proposition 19. *Every bounded cell X_S in the decomposition $\mathcal{C}_V$ is itself a tropical polytope, equal to the tropical convex hull of its vertices. The number of vertices of the polytope X_S is at most $\binom{2n-2}{n-1}$, and this bound is tight for all positive integers n.*

Proof. By Proposition 17, if X_S has dimension d, it is affinely and tropically isomorphic to a region in the convex hull of a set of points in $\mathbb{TP}^d$, so it suffices to consider the full-dimensional case.

The inequality presentation of Lemma 10 demonstrates that X_S is tropically convex for all S, since if two points satisfy an inequality of that form, so does any linear combination thereof. Therefore, it suffices to show that X_S is contained in the tropical convex hull of its vertices.

The proof is by induction on the dimension of X_S. All proper faces of X_S are polytopes X_T of lower dimension, and by induction are contained in the tropical convex hull of their vertices. These vertices are a subset of the vertices of X_S, and so this face is in the tropical convex hull.

Take any point $x = (x_1, \ldots, x_n)$ in the interior of X_S. Since X_S has dimension n, we can travel in any direction from x while remaining in X_S. Let us travel in the $(1, 0, \ldots, 0)$ direction until we hit the boundary, to obtain points $y_1 = (x_1 + b, x_2, \ldots, x_n)$ and $y_2 = (x_1 - c, x_2, \ldots, x_n)$ in the boundary of X_S. These points are contained in the tropical convex hull by the inductive hypothesis, which means that $x = y_1 \oplus c \odot y_2$ is also, completing the proof of the first assertion.

For the second assertion, we consider the convex hull of all differences of unit vectors, $e_i - e_j$. This is a lattice polytope of dimension $n-1$ and normalized volume $\binom{2n-2}{n-1}$. To see this, we observe that this polytope is tiled by n copies of the convex hull of the origin and the $\binom{n}{2}$ vectors $e_i - e_j$ with $i < j$. The other $n-1$ copies are gotten by cyclic permutation of the coordinates. This latter polytope was studied by Gel'fand, Graev and Postnikov, who showed in [5, Theorem 2.3 (2)] that the normalized volume of this polytope equals the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$.

We conclude that every complete fan whose rays are among the vectors $e_i - e_j$ has at most $\binom{2n-2}{n-1}$ maximal cones. This applies in particular to the normal fan of X_S, hence X_S has at most $\binom{2n-2}{n-1}$ vertices. Since the configuration $\{e_i - e_j\}$ is unimodular, the bound is tight whenever the fan is simplicial and uses all the rays $e_i - e_j$. □

We close this section with two more results about arbitrary tropical polytopes in $\mathbb{TP}^{n-1}$.

Proposition 20. *If P and Q are tropical polytopes in $\mathbb{TP}^{n-1}$ then $P \cap Q$ is also a tropical polytope.*

Proof. Since P and Q are both tropically convex, $P \cap Q$ must also be. Consequently, if we can find a finite set of points in $P \cap Q$ whose convex hull contains all of $P \cap Q$, we will be done. By Theorem 15, P and Q are the finite unions of bounded cells $\{X_S\}$ and $\{X_T\}$ respectively, so $P \cap Q$ is the finite union of the cells $X_S \cap X_T$. Consider any $X_S \cap X_T$. Using Lemma 10 to obtain the inequality representations of X_S and X_T, we see that this region is of the form dictated by Proposition 18, and therefore obtainable as a cell X_W in some tropical polytope. By Proposition 19, it is itself a tropical polytope, and we can therefore find a finite set whose convex hull is equal to $X_S \cap X_T$. Taking the union of these sets over all choices of S and T then gives us the desired set of points whose convex hull contains all of $P \cap Q$. □

Proposition 21. *Let $P \subset \mathbb{TP}^{n-1}$ be a tropical polytope. Then there exists a unique minimal set V such that $P = \mathrm{tconv}(V)$.*

Proof. Suppose that P has two minimal generating sets, $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_r\}$. Write each element of W as $w_i = \oplus_{j=1}^m c_{ij} \odot v_j$. We claim that $V \subseteq W$. Consider $v_1 \in V$ and write

$$v_1 \;=\; \bigoplus_{i=1}^{r} d_i \odot w_i \;=\; \bigoplus_{j=1}^{m} f_j \odot v_j \qquad \text{where } f_j = \min_i(d_i + c_{ij}). \tag{10}$$

If the term $f_1 \odot v_1$ does not minimize any coordinate in the right-hand side of (10), then v_1 is a linear combination of $v_2, \ldots, v_m$, contradicting the minimality of V. However, if $f_1 \odot v_1$ minimizes any coordinate in this expression, it must minimize all of them, since $(v_1)_j - (v_1)_k = (f_1 \odot v_1)_j - (f_1 \odot v_1)_k$. In this case we get $v_1 = f_1 \odot v_1$, or $f_1 = 0$. Pick any i for which $f_1 = d_i + c_{i1}$; we claim that $w_i = c_{i1} \odot v_1$. Indeed, if any other term in $w_i = \oplus_{j=1}^m c_{ij} \odot v_j$ contributed nontrivially to w_i, that term would also contribute to the expression on the right-hand side of (10), which is a contradiction. Consequently, $V \subseteq W$, which means $V = W$ since both sets are minimal by hypothesis. □

Like many of the results presented in this section, Propositions 20 and 21 parallel results on ordinary polytopes. We have already mentioned the tropical analogues of the Farkas Lemma and of Carathéodory's Theorem (Propositions 5 and 9); Proposition 17 is analogous to the result that a polytope $P \subset \mathbb{R}^n$ of dimension d is affinely isomorphic to some $Q \subset \mathbb{R}^d$. Proposition 19 hints at a duality between an inequality representation and a vertex representation of a tropical polytope.

4. Subdividing products of simplices

Every set $V = \{v_1, \ldots, v_r\}$ of r points in $\mathbb{TP}^{n-1}$ begets a tropical polytope $P = \mathrm{tconv}(V)$. This polytope comes with a canonical cell decomposition $\mathcal{C}_P$ into ordinary polytopes defined by inequalities of the form $x_i - x_j \leq c_{ij}$. In particular, this is the subcomplex of bounded cells of the decomposition $\mathcal{C}_V$ of $\mathbb{TP}^{n-1}$, as seen in Theorem 15 and Proposition 16. Each cell of $\mathcal{C}_P$ is labelled by its type, which is an n-vector of finite subsets of $\{1, \ldots, r\}$. Two tropical polytopes P and Q have the same *combinatorial type* if the types occurring in C_P and in C_Q are identical; note that by Lemma 13, this implies that the face posets of these polyhedral complexes are isomorphic.

With the definition in the previous paragraph, the statement of Theorem 1 has now finally been made precise. We will prove this correspondence between tropical polytopes and subdivisions of products of simplices by constructing the polyhedral complex $\mathcal{C}_P$ in a higher-dimensional space.

Let W denote the $(r+n-1)$-dimensional real vector space $\mathbb{R}^{r+n}/(1,\ldots,1,-1,\ldots,-1)$. The natural coordinates on W are denoted $(y,z)=(y_1,\ldots,y_r,z_1,\ldots,z_n)$. As before, we fix an ordered subset $V=\{v_1,\ldots,v_r\}$ of $\mathbb{TP}^{n-1}$ where $v_i=(v_{i1},\ldots,v_{in})$. This defines the unbounded polyhedron

$$\mathcal{P}_V \quad = \quad \big\{ (y,z)\in W \,:\, y_i+z_j\leq v_{ij} \text{ for all } i\in\{1,\ldots,r\} \text{ and } j\in\{1,\ldots,n\} \big\}. \tag{11}$$

Lemma 22. *There is a piecewise-linear isomorphism between the tropical polytope $P=\mathrm{tconv}(V)$ and the complex of bounded faces of the $(r+n-1)$-dimensional polyhedron $\mathcal{P}_V$. The image of a cell X_S of $\mathcal{C}_P$ under this isomorphism is the bounded face $\{y_i+z_j=v_{ij} \,:\, i\in S_j\}$ of the polyhedron $\mathcal{P}_V$. That bounded face maps isomorphically to X_S via projection onto the z-coordinates.*

Proof. Let F be a bounded face of $\mathcal{P}_V$, and define S_j via $i\in S_j$ if $y_i+z_j=v_{ij}$ is valid on all of F. If some y_i or z_j appears in no equality, then we can subtract arbitrary positive multiples of that basis vector to obtain elements of F, contradicting the assumption that F is bounded. Therefore, each i must appear in some S_j, and each S_j must be nonempty.

Since every y_i appears in some equality, given a specific z in the projection of F onto the z-coordinates, there exists a unique y for which $(y,z)\in F$, so this projection is an affine isomorphism from F to its image. We need to show that this image is equal to X_S.

Let z be a point in the image of this projection, coming from a point (y,z) in the relative interior of F. We claim that $z\in X_S$. Indeed, looking at the jth coordinate of z, we find

$$-y_i+v_{ij} \;\geq\; z_j \quad \text{for all } i, \tag{12}$$

$$-y_i+v_{ij} \;=\; z_j \quad \text{for } i\in S_j. \tag{13}$$

The defining inequalities of X_S are $x_j-x_k\leq v_{ij}-v_{ik}$ with $i\in S_j$. Subtracting the inequality $-y_i+v_{ik}\geq z_k$ from the equality in (13) yields that this inequality is valid on z as well. Therefore, $z\in X_S$. Similar reasoning shows that $S=\mathrm{type}(z)$. We note that the relations (12) and (13) can be rewritten elegantly in terms of the tropical product of a row vector and a matrix:

$$z \quad = \quad (-y)\odot V \;\;=\;\; \bigoplus_{i=1}^{r}(-y_i)\odot v_i. \tag{14}$$

For the reverse inclusion, suppose that $z\in X_S$. We define $y=V\odot(-z)$. This means that

$$y_i \;=\; \min(v_{i1}-z_1, v_{i2}-z_2,\ldots,v_{in}-z_n). \tag{15}$$

We claim that $(y,z)\in F$. Indeed, we certainly have $y_i+z_j\leq v_{ij}$ for all i and j, so $(y,z)\in\mathcal{P}_V$. Furthermore, when $i\in S_j$, we know that $v_{ij}-z_j$ achieves the minimum in the right-hand side of (15), so that $v_{ij}-z_j=y_i$ and $y_i+z_j=v_{ij}$ is satisfied. Consequently, $(y,z)\in F$ as desired.

It follows immediately that the two complexes are isomorphic: if F is a face corresponding to X_S and G is a face corresponding to X_T, where S and T are both types, then X_S is a face of X_T if and only if $T\subseteq S$. However, by the discussion above, this is equivalent to saying that the equalities G satisfies (which correspond to T) are a subset of the equalities F satisfies (which correspond to S); this is true if and only if F is a face of G. So X_S is a face of X_T if and only if F is a face of G, which implies the isomorphism of complexes. □

The boundary complex of the polyhedron $\mathcal{P}_V$ is polar to the regular subdivision of the product of simplices $\Delta_{r-1}\times\Delta_{n-1}$ defined by the weights v_{ij}. We denote this regular polyhedral subdivision by $(\partial\mathcal{P}_V)^*$. Explicitly, a subset of vertices (e_i,e_j) of $\Delta_{r-1}\times\Delta_{n-1}$ forms a cell of $(\partial\mathcal{P}_V)^*$ if and only if the equations $y_i + z_j = v_{ij}$ indexed by these vertices specify a face of the polyhedron $\mathcal{P}_V$. We refer to the book of De Loera, Rambau and Santos [6] for basics on polyhedral subdivisions.

We now present the proof of the result stated in the introduction.

Proof of Theorem 1: The poset of bounded faces of $\mathcal{P}_V$ is antiisomorphic to the poset of interior cells of the subdivision $(\partial\mathcal{P}_V)^*$ of $\Delta_{r-1} \times \Delta_{n-1}$. Since every full-dimensional cell of $(\partial\mathcal{P}_V)^*$ is interior, the subdivision is uniquely determined by its interior cells. In other words, the combinatorial type of $\mathcal{P}_V$ is uniquely determined by the lists of facets containing each bounded face of $\mathcal{P}_V$. These lists are precisely the types of regions in $\mathcal{C}_P$ by Lemma 22. This completes the proof. □

Theorem 1, which establishes a bijection between the tropical polytopes spanned by r points in $\mathbb{TP}^{n-1}$ and regular subdivisions of a product of simplices $\Delta_{r-1} \times \Delta_{n-1}$, has many striking consequences. Perhaps the most astonishing concerns the row span and column span of a matrix.

Theorem 23. *Given any matrix $M \in \mathbb{R}^{r\times n}$, the tropical convex hull of its column vectors is isomorphic to the tropical convex hull of its row vectors. This isomorphism is gotten by restricting the piecewise linear maps $\mathbb{R}^n \to \mathbb{R}^r$, $z \mapsto M \odot (-z)$ and $\mathbb{R}^r \to \mathbb{R}^n$, $y \mapsto (-y) \odot M$.*

Proof. By Theorem 1, the matrix M corresponds via the polyhedron $\mathcal{P}_M$ to a regular subdivision of $\Delta_{r-1} \times \Delta_{n-1}$, and the complex of interior faces of this regular subdivision is combinatorially isomorphic to both the tropical convex hull of its row vectors, which are r points in $\mathbb{TP}^{n-1}$, and its column vectors, which are n points in $\mathbb{TP}^{r-1}$. Furthermore, Lemma 22 tells us that the cell in $\mathcal{P}_M$ is affinely isomorphic to its corresponding cell in both tropical convex hulls. Finally, in the proof of Lemma 22, we showed that the point (y, z) in a bounded face F of $\mathcal{P}_M$ satisfies $y = M \odot (-z)$ and $z = (-y) \odot M$. This point projects to y and z, and so the piecewise-linear isomorphism mapping these two convex hulls to each other is defined by the stated maps. □

The common cell complex of these two tropical polytopes is given by the complex of bounded faces of the common polyhedron $\mathcal{P}_M$, which lives in a space of dimension $r + n - 1$; the tropical polytopes are unfoldings of this complex into dimensions $r - 1$ and $n - 1$. Theorem 23 also gives a natural bijection between the combinatorial types of tropical convex hulls of r points in $\mathbb{TP}^{n-1}$ and the combinatorial types of tropical convex hulls of n points in $\mathbb{TP}^{r-1}$, incidentally proving that there are the same number of each. This duality statement extends a similar statement in [4].

We now discuss the generic case when the subdivision $(\partial\mathcal{P}_V)^*$ is a regular triangulation of $\Delta_{r-1} \times \Delta_{n-1}$. We refer to [17, §5] for the geometric interpretation of the *tropical determinant*.

Proposition 24. *For a configuration V of r points in $\mathbb{TP}^{n-1}$ with $r \geq n$ the following are equivalent:*

1. *The regular subdivision $(\partial\mathcal{P}_V)^*$ is a triangulation of $\Delta_{r-1} \times \Delta_{n-1}$.*
2. *No k of the points in V have projections onto a k-dimensional coordinate subspace which lie in a tropical hyperplane, for any $2 \leq k \leq n$.*
3. *No $k\times k$-submatrix of the $r\times n$-matrix (v_{ij}) is tropically singular, i.e. has vanishing tropical determinant, for any $2 \leq k \leq n$.*

Proof. The last equivalence is proven in [17, Lemma 5.1]. We will prove that (1) and (3) are equivalent. The tropical determinant of a k by k matrix M is the tropical polynomial $\oplus_{\sigma\in S_k}(\odot_{i=1}^k M_{i\sigma(i)})$. The matrix M is tropically singular if the minimum $\min_{\sigma\in S_k}(\sum_{i=1}^k M_{i\sigma(i)})$ is achieved twice.

The regular subdivision $(\partial\mathcal{P}_V)^*$ is a triangulation if and only if the polyhedron $\mathcal{P}_V$ is simple, which is to say if and only if no $r + n$ of the facets $y_i + z_j \leq v_{ij}$ meet at a single vertex. For each vertex v, consider the bipartite graph G_v consisting of vertices $y_1, \ldots, y_n$ and $z_1, \ldots, z_j$ with an edge connecting y_i and z_j if v lies on the corresponding facet. This graph is connected, since each y_i and z_j appears in some such inequality, and thus it will have a cycle if and only if it has at least $r + n$ edges. Consequently, $\mathcal{P}_V$ is not simple if and only there exists some G_v with a cycle.

If there is a cycle, without loss of generality it reads $y_1, z_1, y_2, z_2, \dots, y_k, z_k$. Consider the submatrix M of (v_{ij}) given by $1 \le i \le k$ and $1 \le j \le k$. We have $y_1 + z_1 = M_{11}$, $y_2 + z_2 = M_{22}$, and so on, and also $z_1 + y_2 = M_{12}, \dots, z_k + y_1 = M_{k1}$. Adding up all of these equalities yields $y_1 + \cdots + y_s + z_1 + \cdots + z_s = M_{11} + \cdots + M_{kk} = M_{12} + \cdots + M_{k1}$. But consider any permutation $\sigma \in S_k$. Since we have $M_{i\sigma(i)} = v_{i\sigma(i)} \ge y_i + z_{\sigma(i)}$, we have $\sum M_{i\sigma(i)} \ge x_1 + \cdots + x_s + y_1 + \cdots + y_s$. Consequently, the permutations equal to the identity and to $(12\cdots s)$ simultaneously minimize the determinant of the minor M. This logic is reversible, proving the equivalence of (1) and (3). □

If a tropical polytope is the span of r points in general position, we call it a *generic tropical polytope*. These polyhedral complexes are then polar to the complexes of interior faces of regular triangulations of $\Delta_{r-1} \times \Delta_{n-1}$.

Corollary 25. *All tropical polytopes spanned by r points in general position in $\mathbb{TP}^{n-1}$ have the same f-vector. Specifically, the number of faces of dimension k is equal to the multinomial coefficient*

$$\binom{r+n-k-2}{r-k-1,\, n-k-1,\, k} = \frac{(r+n-k-2)!}{(r-k-1)!\cdot(n-k-1)!\cdot k!}.$$

Proof. By Proposition 24, these objects are in bijection with regular triangulations of $P = \Delta_{r-1} \times \Delta_{n-1}$. The polytope P is equidecomposable [1], meaning that all of its triangulations have the same f-vector. The number of faces of dimension k of the tropical convex hull of the given r points is equal to the number of interior faces of codimension k in the corresponding triangulation. Since all triangulations of all products of simplices have the same f-vector, they must also have the same interior f-vector, which can be computed by taking the f-vector and subtracting off the f-vectors of the induced triangulations on the proper faces of P. These proper faces are all products of simplices and hence equidecomposable, so all of these induced triangulations have f-vectors independent of the original triangulation as well.

To compute this number, we therefore need only compute it for one tropical convex hull. Let the vectors v_i, $1 \le i \le r$, be given by $v_i = (i, 2i, \cdots, ni)$. By Theorem 10, to count the faces of dimension k in this tropical convex hull, we enumerate the existing types with k degrees of freedom. Consider any index i. We claim that for any x in the tropical convex hull of $\{v_i\}$, the set $\{j \mid i \in S_j\}$ is an interval I_i, and that if $i < m$, the intervals I_m and I_i meet in at most one point, which in that case is the largest element of I_m and the smallest element of I_i.

Suppose we have $i \in S_j$ and $m \in S_l$ with $i < m$. Then we have by definition $v_{ij} - x_j \le v_{il} - x_l$ and $v_{ml} - x_l \le v_{mj} - x_j$. Adding these inequalities yields $v_{ij} + v_{ml} \le v_{il} + v_{mj}$, or $ij + ml \le il + mj$. Since $i < m$, it follows that we must have $l \le j$. Therefore, we can never have $i \in S_j$ and $m \in S_l$ with $i < m$ and $j < l$. The claim follows immediately, since the I_i cover $[1, n]$.

The number of degrees of freedom of an interval set $(I_1, \dots, I_r)$ is easily seen to be the number of i for which I_i and I_{i+1} are disjoint. Given this, it follows from a simple combinatorial counting argument that the number of interval sets with k degrees of freedom is the multinomial coefficient given above. Finally, a representative for every interval set is given by $x_j = x_{j+1} - c_j$, where if S_j and S_{j+1} have an element i in common (they can have at most one), $c_j = i$, and if not then $c_j = (\min(S_j) + \max(S_{j+1}))/2$. Therefore, each interval set is in fact a valid type, and our enumeration is complete. □

Corollary 26. *The number of combinatorially distinct generic tropical polytopes spanned by r points in $\mathbb{TP}^{n-1}$ equals the number of distinct regular triangulations of $\Delta_{r-1} \times \Delta_{n-1}$. The number of respective symmetry classes under the natural action of $S_r \times S_n$ on both spaces is also the same.*

The symmetries in $S_r \times S_n$ correspond to a natural action on $\Delta_{r-1} \times \Delta_{n-1}$ given by permuting the vertices of the two component simplices; the symmetries in S_r correspond to permuting the

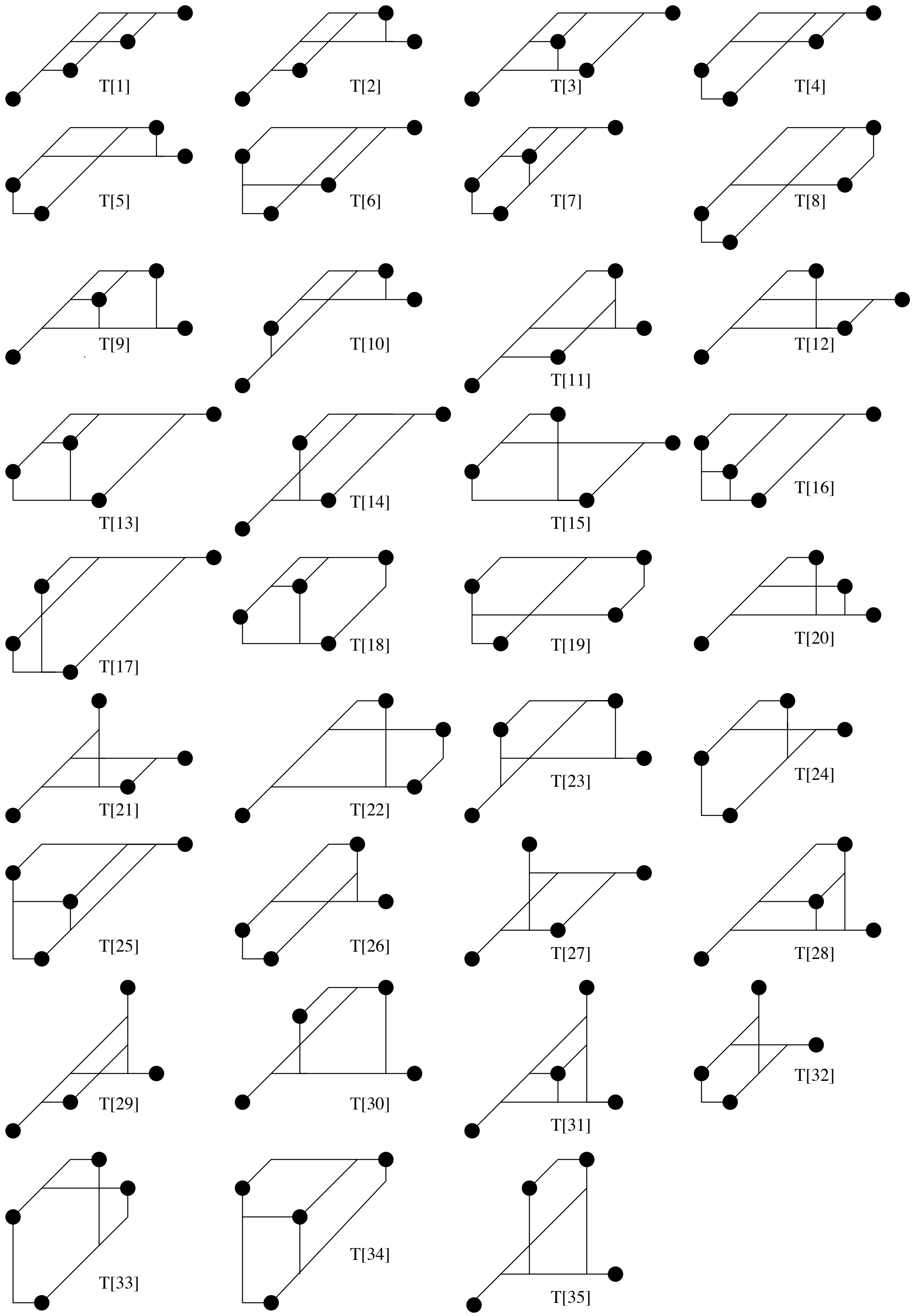

FIGURE 5. The 35 symmetry classes of tropical quadrangles in $\mathbb{TP}^2$.

points in a tropical polytope, while those in S_n correspond to permuting the coordinates. (These are dual, as per Corollary 23.) The number of symmetry classes of regular triangulations of the polytope $\Delta_{r-1} \times \Delta_{n-1}$ is computable via Jörg Rambau's TOPCOM [16] for small r and n:

	2	3
2	5	35
3	35	7,955
4	530	
5	13,631	

For example, the $(2, 3)$ entry of the table divulges that there are 35 symmetry classes regular triangulations of $\Delta_2 \times \Delta_3$. These correspond to the 35 combinatorial types of quadrangles in $\mathbb{TP}^2$, or to the 35 combinatorial types of triangles in $\mathbb{TP}^3$. These 35 quadrangles are shown in Figure 5; the labeling corresponds to Rambau's labeling (see [16]) of the regular triangulations of $\Delta_3 \times \Delta_2$.

5. Phylogenetic analysis using tropical polytopes

A fundamental problem in bioinformatics is the reconstruction of phylogenetic trees from approximate distance data. In this section we show how tropical convexity might help provide new algorithmic tools for this problem. Our approach augments the results in [18, §4] and it provides a tropical interpretation of the work on T-theory by Andreas Dress and his collaborators [7], [8], [9].

Consider a symmetric $n \times n$-matrix $D = (d_{ij})$ whose entries d_{ij} are non-negative real numbers and whose diagonal entries d_{ii} are all zero. We say that D is a *metric space* if the triangle inequality $d_{ij} \leq d_{ik} + d_{jk}$ holds for all indices i, j, k. Our starting point is the following easy observation:

Proposition 27. *The symmetric matrix D is a metric space if and only if all principal 3×3-minors of the negated symmetric matrix $-D = (-d_{ij})$ are tropically singular.*

Proof. Both properties involve only three points, so we may assume $n = 3$, in which case

$$-D \quad = \quad \begin{pmatrix} 0 & -d_{12} & -d_{13} \\ -d_{12} & 0 & -d_{23} \\ -d_{13} & -d_{23} & 0 \end{pmatrix}.$$

The tropical determinant of this matrix is the minimum of the six expressions

$$0,\ -2d_{12},\ -2d_{13},\ -2d_{23},\ -d_{12} - d_{13} - d_{23} \text{ and } -d_{12} - d_{13} - d_{23}.$$

This minimum is attained twice if and only if it is attained by the last two (identical) expressions, which occurs if and only if the three triangle inequalities are satisfied. □

In what follows we assume that $D = (d_{ij})$ is a metric space. Let P_D denote the tropical convex hull in $\mathbb{TP}^{n-1}$ of the n row vectors (or column vectors) of the negated matrix $-D = (-d_{ij})$. Proposition 27 tells us that the tropical polytope P_D is always one-dimensional for $n = 3$.

The finite metric space $D = (d_{ij})$ is said to be a *tree metric* if there exists a tree T with n leaves, embedded in some Euclidean space, such that d_{ij} denotes the distance between the i-th leaf and the j-th leaf along the unique path between these leaves in T. The next theorem characterizes tree metrics among all metrics by the dimension of the tropical polytope P_D. It is the tropical interpretation of results that are quite classical and well-known in the phylogenetics literature.

Theorem 28. *For a given finite metric space $D = (d_{ij})$ the following conditions are equivalent:*

(1) *D is a tree metric,*
(2) *the tropical polytope P_D has dimension one,*
(3) *all 4×4-minors of the matrix $-D$ are tropically singular,*
(4) *all principal 4×4-minors of the matrix $-D$ are tropically singular,*

(5) *For any choice of four indices $i, j, k, l \in \{1, 2, \ldots, n\}$, the maximum of the three numbers $d_{ij} + d_{kl}$, $d_{ik} + d_{jl}$ and $d_{il} + d_{ik}$ is attained at least twice.*

Proof. The condition (5) is the familiar *Four Point Condition* for tree metrics. The equivalence of (1) and (5) is a classical result due to Buneman [3]. See equation (B3) on page 57 in [7].

Suppose that the condition (5) holds. By the discussion in [18, §4], this means that $-D$ is a point in the tropical Grassmannian of lines, in symbols $-D \in \mathrm{Gr}(2, n) \subset \mathbb{TP}^{\binom{n}{2}}$. By [18, Theorem 3.8], the point $-D$ corresponds to a tropical line L_D in $\mathbb{TP}^{n-1}$. The n distinguished points whose coordinates are the rows of $-D$ lie on the line L_D. By Corollary 7, it follows that their tropical convex hull P_D is contained in L_D. This means that P_D has dimension one, that is, (2) holds.

The equivalence of (2) and (3) holds because the dimension of any tropical polytope $\mathrm{trop}(V)$ is one less than the tropical rank of its defining matrix $V = (v_{ij})$. This can be derived easily from the results on tropical linear spaces in [17] and [18].

Obviously, the condition (3) implies the condition (4). What remains is to prove the implication from (4) to (5). For this we note that the tropical determinant of the 4×4-matrix

$$\begin{pmatrix} 0 & -d_{12} & -d_{13} & -d_{14} \\ -d_{12} & 0 & -d_{23} & -d_{24} \\ -d_{13} & -d_{23} & 0 & -d_{34} \\ -d_{14} & -d_{24} & -d_{34} & 0 \end{pmatrix}$$

equals twice the minimum of $-d_{12} - d_{34}$, $-d_{13} - d_{24}$ and $-d_{14} - d_{23}$. (It's the tropicalization of a 4×4-Pfaffian). The matrix is tropically singular if and only if the minimum is attained twice. □

If the five equivalent conditions of Theorem 28 are satisfied then the metric tree T coincides with the one-dimensional tropical polytope P_D. To make sense of this statement, we regard tropical projective space $\mathbb{TP}^{n-1}$ as a metric space with respect to the infinity norm induced from $\mathbb{R}^n$,

$$||x - y|| \quad = \quad \max\{\, |x_i + y_j - x_j - y_i| \;:\; 1 \le i < j \le n \,\},$$

and we note that the finite metric space D embeds isometrically into P_D via the rows of $-\frac{1}{2}D$:

$$i \quad \mapsto \quad \frac{1}{2} \cdot (-d_{i1}, -d_{i2}, -d_{i3}, \ldots, -d_{in}) \qquad \text{for } i = 1, 2, \ldots, n$$

We learned from [8] that the tropical polytope P_D first appeared in the 1964 paper [10] by John Isbell. For the proof of the following result we assume familiarity with results from [7] and [8].

Theorem 29. *The tropical polytope P_D equals Isbell's injective hull of the metric space D.*

Proof. According to Lemma 22, the tropical polytope P_D is the bounded complex of the following unbounded polyhedron in the $(2n-1)$-dimensional space $W = \mathbb{R}^{2n}/\mathbb{R}(1, \ldots, 1, -1, \ldots, -1)$:

$$\mathcal{P}_{-D} \quad = \quad \{\, (y, z) \in W \;:\; y_i + z_j \le -d_{ij} \text{ for all } 1 \le i, j \le n \,\}.$$

Dress et.al. [7] showed that the injective hull $T(D)$ of the finite metric space D coincides with the complex of bounded faces of the following n-dimensional unbounded polyhedron:

$$\mathcal{Q}_{-D} \quad = \quad \{\, x \in \mathbb{R}^n \;:\; x_i + x_j \ge d_{ij} \text{ for all } 1 \le i, j \le n \,\}.$$

What we need to show is that the two polyhedra have the same bounded complex.

The metric space D satisfies the tropical matrix identity $-D = D \odot (-D)$, because $-d_{ij} = \min_k(d_{ik} - d_{kj})$. This implies that any column vector y of $-D$ satisfies $y = (-y) \odot (-D)$.

Consider any vertex (y, z) of $\mathcal{P}_{-D}$. Then y is a column vector of $-D$. Equation (14) implies $z = (-y) \odot (-D) = y$. Hence every vertex of $\mathcal{P}_{-D}$ lies in the subspace defined by $y = z$, and so does the complex of bounded faces of $\mathcal{P}_{-D}$. Therefore the linear map $(y, z) \mapsto -y$ induces an isomorphism between the bounded complex of $\mathcal{P}_{-D}$ and the bounded complex of $\mathcal{Q}_{-D}$. □

Theorem 23 specifies an involution on the set of all tropical polytopes. We are interested in the fixed points of this canonical involution. A necessary condition is that $r = n$ and V is a symmetric matrix. The previous result and its proof can be reinterpreted as follows:

Corollary 30. *A tropical polytope P is pointwise fixed under the canonical involution (on the set of all tropical polytopes) if and only if P is the injective hull of a metric space on $\{1, 2, \ldots, n\}$.*

Proof. The identity $-D = D \odot (-D)$ is equivalent to D being a metric space. □

Dress, Huber and Moulton [7] emphasize that the tropical polytope P_D records many important invariants of a given finite metric space D. For instance, the dimension of P_D gives information about how far the metric is from being a tree metric. In practical biological applications of phylogenetic reconstruction, the distances d_{ij} are not known exactly, and P_D appears to contain all the various trees which are found by existing software for phylogenetic reconstruction.

It would be a worthwhile project to classify all combinatorial types of tropical polytopes $\mathrm{trop}(-D)$ of small dimension, say $d \leq 3$, where D is metric space on few points, say $n \leq 8$. The dimension d is always computed as the tropical rank of the matrix $-D$. It can be characterized combinatorially by tropicalizing the sub-Pfaffians of a skew-symmetric $n \times n$-matrix. The tropical Pfaffians of format 4×4 specify the four point condition (5) in Theorem 28, while the tropical sub-Pfaffians of format 6×6 specify the six-point condition which is discussed in [7, page 25]. The combinatorial study of *k-compatible split systems* can be interpreted in the setting of tropical algebraic geometry (cf. [14], [17], [18]) as the study of the k-th *secant variety* in Grassmannian of lines $\mathrm{Gr}(2, n) \subset \mathbb{TP}^{\binom{n}{2}}$.

Tropical convexity provides a convenient language to study numerous extensions of the classical problem of tree reconstruction. As an example, imagine the following scenario, which would correspond to the Grassmannians of lines in $\mathbb{TP}^{n-1}$, denoted $\mathrm{Gr}(3, n)$.

Suppose there are n taxa, labeled $1, 2, \ldots, n$, and rather than having a distance for any pair i, j, we are now given a similarity measure d_{ijk} for any triple $i, j, k \in \{1, 2, \ldots, n\}$. We can then construct a tropical polytope by taking the tropical convex hull of $\binom{n}{2}$ points as follows:

$$P \quad = \quad \mathrm{tconv}\left\{ \left(-d_{ij1}, -d_{ij2}, -d_{ij3}, \ldots - d_{ijn}\right) \in \mathbb{TP}^{n-1} \;:\; 1 \leq i < j \leq n \right\}.$$

Under the certain hypotheses, the tropical polytope P can be realized as the complex of bounded faces of the polyhedron in $\mathbb{R}^n$ defined by the inequalities $x_i + x_j + x_k \geq d_{ijk}$. It provides a polyhedral model for the tree-like nature of the similarity data (d_{ijk}). The case of most interest is when P is two-dimensional in which case it plays the role of a *two-dimensional phylogenetic tree*.

The construction of this particular tropical polytope P was pioneered by Dress and Terhalle in the important paper [9]. There they discuss *valuated matroids*, which are essentially the points on the tropical Grassmannian of [18], and they call P the *tight span of a valuated matroid*. We share their view that these tropical polytopes constitute a promising tool for phylogenetic analysis.

Acknowledgements

This work was conducted while Mike Develin held the AIM Postdoctoral Fellowship 2003-2008 and Bernd Sturmfels held the MSRI-Hewlett Packard Professorship 2003/2004. Sturmfels also acknowledges partial support from the National Science Foundation (DMS-0200729).

References

[1] L.J. Billera, R. Cushman, and J.A. Sanders: The Stanley decomposition of the harmonic oscillator, *Nederl. Akad. Wetensch. Indag. Math.* **50** (1988), 375–393.

[2] L.J. Billera, S. Holmes, and K. Vogtmann: Geometry of the space of phylogenetic trees, *Advances in Applied Mathematics* **27** (2001), 733–767.

[3] P. Buneman: A note on metric properties of trees, *Journal of Combinatorial Theory, Ser. B* **17** (1974) 48–50.

[4] G. Cohen, S. Gaubert, and J.-P. Quadrat: Duality and separation theorems in idempotent semimodules, `arXiv: math.FA/0212294`.

[5] I.M. Gel'fand, M.I. Graev and A. Postnikov: Combinatorics of hypergeometric functions associated with positive roots, in *Arnold-Gel'fand Mathematical Seminars, Geometry and Singularity Theory*, (Eds. V.I. Arnold, I.M. Gel'fand, M. Smirnov and V.S. Retakh), Birkhäuser, Boston, 1997, pp. 205–221.

[6] J. De Loera, J. Rambau and F. Santos: *Triangulations of Point Configurations*, Algorithms and Computation in Mathematics, Springer Verlag, Heidelberg, to appear.

[7] A. Dress, K.T. Huber and V. Moulton: An explicit computation of the injective hull of certain finite metric spaces in terms of their associated Buneman complex, *Advances in Mathematics* **168** (2002) 1–28.

[8] A. Dress, V. Moulton and W. Terhalle: T-Theory – an overview, *European Journal of Combinatorics* **17** (1996) 161–175.

[9] A. Dress and W. Terhalle: The tree of life and other affine buildings. Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998), *Documenta Mathematica*, Extra Vol. III, 1998, 565–574.

[10] J. Isbell: Six theorems about metric spaces, *Comment. Math. Helv.* **39** (1964) 65–74.

[11] A.N. Kirillov: Introduction to tropical combinatorics, in *Physics and Combinatorics 2000*, Proceedings of the Nagoya 2000 International Workshop, (Eds. A.N. Kirillov and N. Liskova), pp. 82–150, World Scientific, 2001.

[12] G. Litvinov, V. Maslov, and G. Shpiz: Idempotent functional analysis: an algebraic approach, *Mathematical Notes* **69** (2001) 696–729, `arXiv:math.FA/0009128`.

[13] J. Matoušek, On directional convexity, *Discrete and Computational Geometry* **25** (2001) 389–403.

[14] G. Mikhalkin: Counting curves via lattice paths in polygons, to appear in *C. R. Acad. Sci. Paris*, `arXiv:math.AG/0209253`.

[15] M. Noumi and Y. Yamada: Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions, `arXiv: math-ph/0203030`.

[16] J. Rambau: TOPCOM (Triangulations of Point Configurations and Oriented Matroids), software available at `http://www.zib.de/rambau/TOPCOM/`. The numbers cited for regular triangulations of products of simplices can be found at `http://www.zib.de/rambau/TOPCOM/TOPCOM-examples/`.

[17] J. Richter-Gebert, B. Sturmfels, and T. Theobald: First steps in tropical geometry, `arXiv:math.AG/0306366`.

[18] D. Speyer and B. Sturmfels: The tropical Grassmannian, `arXiv:math.AG/0304218`.

Mike Develin, American Institute of Mathematics, 360 Portage Ave., Palo Alto, CA 94306-2244, USA

E-mail address: `develin@post.harvard.edu`

Bernd Sturmfels, Department of Mathematics, University of California, Berkeley, CA 94720, USA

E-mail address: `bernd@math.berkeley.edu`