Some Recent Results

1. Agol - H - Thurston
Knot Genus in a 3-Manifold is NP-Complete.

2. H - Lagarias - Thurston
Isoperimetric Inequalities for embedded disks spanning unknots

K

What is the area of a smallest embedded disk spanning K?

$$C_0^{\left(\frac{L}{R}\right)} L^2 \leq A \leq C_1^{\left(\frac{L}{R}\right)^2} L^2 \qquad R = \text{thickness}$$

These inequalities of area and length are derived from a detour to normal surfaces.

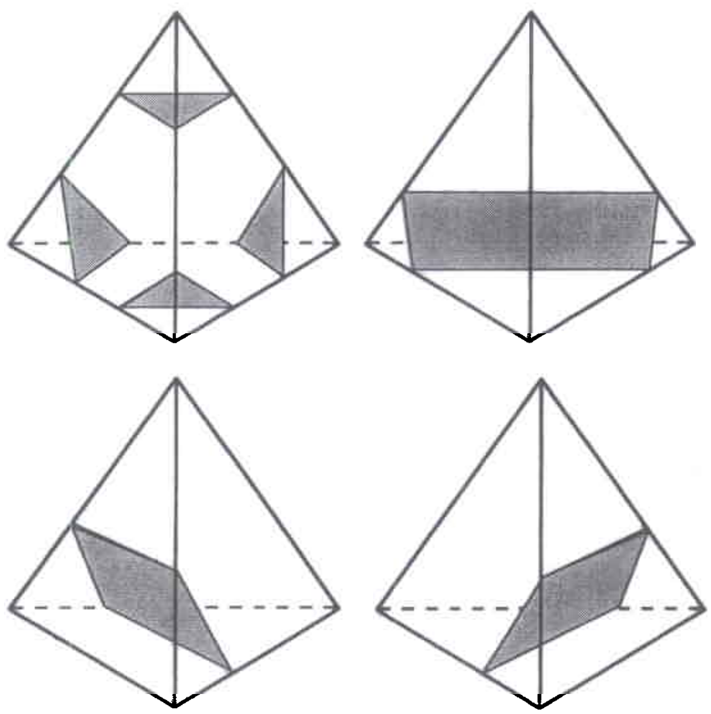
3. S. King, A. Mijatovic
How far is a triangulation of a 3-sphere from polytopal?

a. Number of moves needed is $\leq C_1^{n^2}$
(n = number of three simplices)

b. Examples with number needed $\geq C_2^{n}$

Enumerating normal surfaces

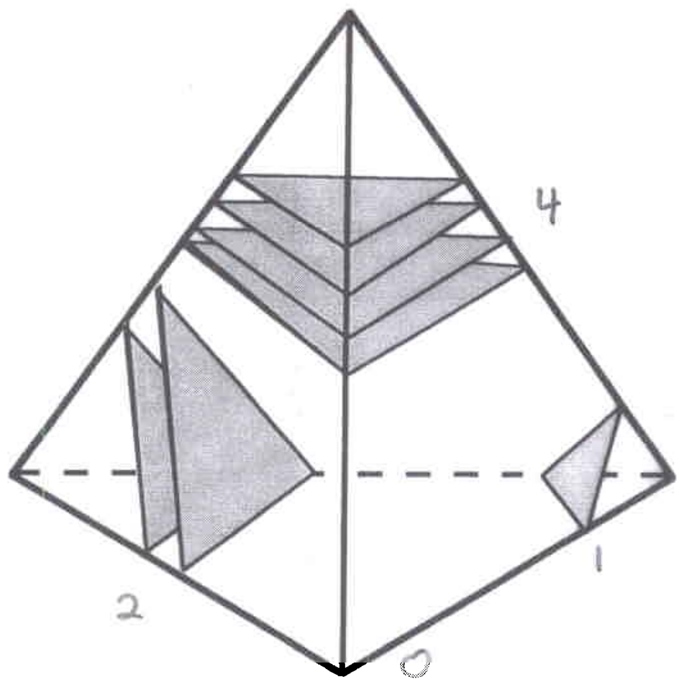
Haken noticed that normal surfaces can be used for solving algorithmic problems in 3-manifolds.



There are four kinds of triangle and three kinds of quadrilateral in each tetrahedron.
There are four kinds of triangle and three kinds of quadrilateral in each tetrahedron. A normal surface is completely determined by specifying how many of each type there are in each tetrahedron.

A normal surface is determined by a vector of 7t non-negative integers

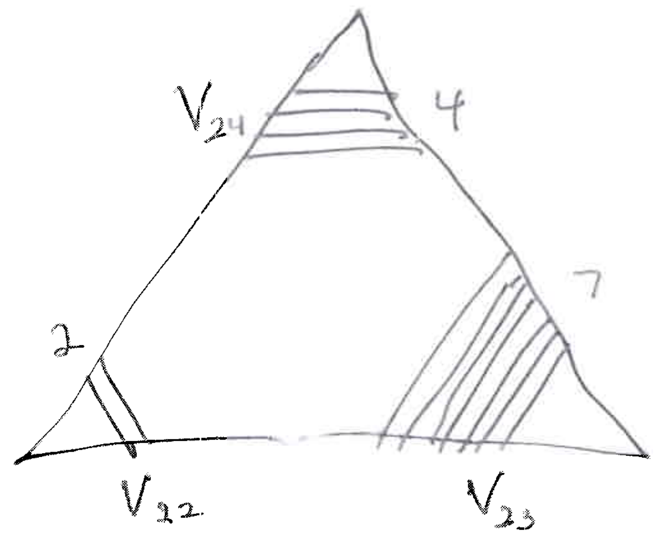
$$(v_1, v_2, v_3, \ldots\ v_{7t})$$



0 quadrilaterals

Of the seven types of triangle and quadrilateral in this tetrahedron, three appear. The vector corresponding to this normal surface looks like

$$(v_1, v_2, v_3, \ldots\ v_{7t}) = (\ldots, 4, 2, 0, 1, 0, 0, 0\ \ldots)$$



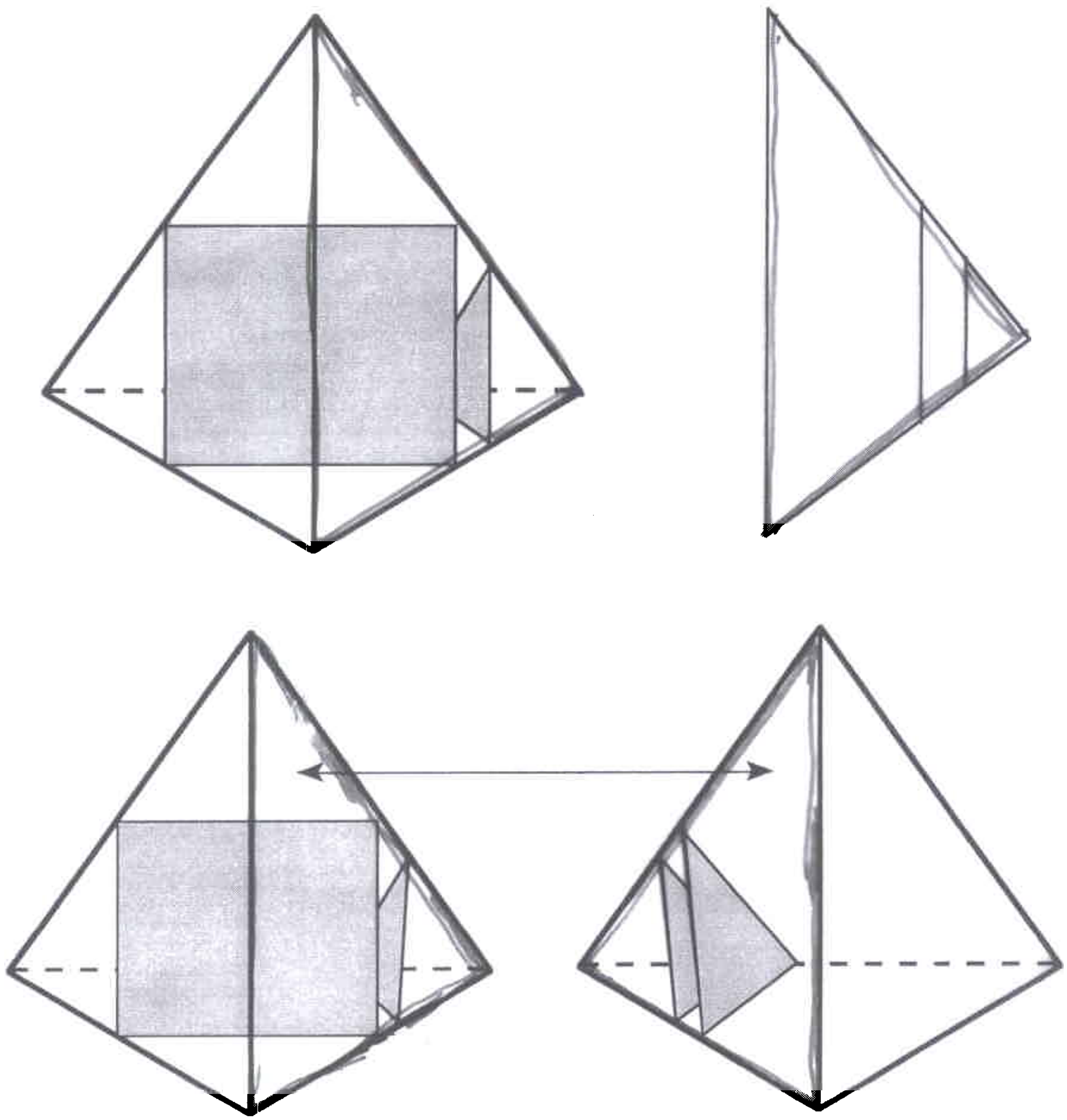
$V = (\ldots, 2, 7, 4, \ldots)$

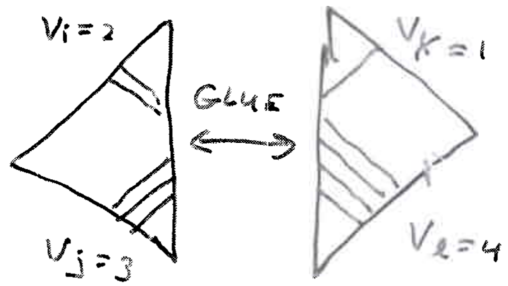
For a surface with t triangles in a triangulation, a normal curve is determined by $3t$ non-negative integers

$(V_1, V_2, \ldots\ldots V_{3t})$

Not all vectors give normal surfaces. We now ask which non-negative integer vectors in $\mathbf{Z}_+^{7t}$ give rise to a normal surface. A single condition must be met.

The pieces must match up across tetrahedra with common faces.



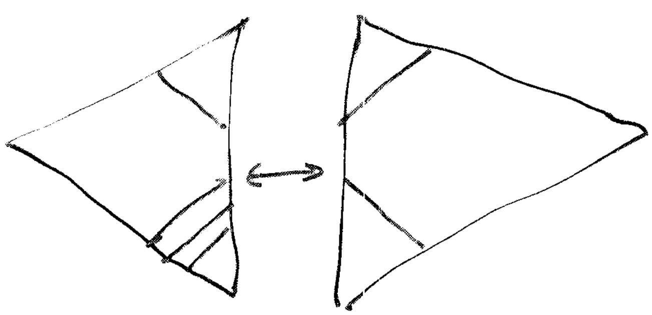


WANT NO LOOSE BOUNDARY

$$2+3 = 1+4$$

$$\rightarrow \quad V_i + V_j = V_k + V_l$$

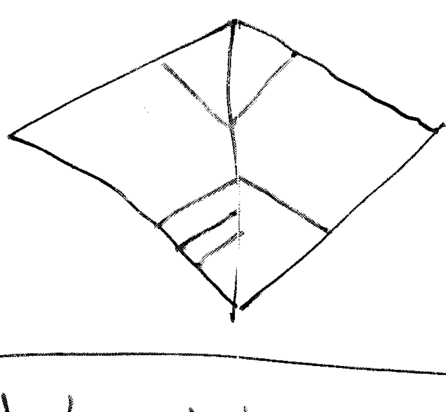
Get one such equation for each edge



Here

$$V_i + V_j \neq V_k + V_l$$

$$1+3 \neq 1+1$$



Loose Ends of Edges.

NOT A CLOSED CURVE

Need $\boxed{V_i + V_j = V_k + V_l}$ NORMAL EQUATIONS

This leads to linear equations for the coordinates of the vector $(v_1, v_2, v_3, \ldots\ v_{7t})$ of the form

$$v_i + v_j = v_k + v_l$$

v_i here counts the number of one elementary triangle in a tetrahedron, while v_j counts the number of one type of quadrilateral. These have parallel edges on a triangle of the tetrahedron.

Also have : $v_i > 0$

(AND: A QUADRILATERAL CONDITION)

Normal surfaces give rise to integer vectors subject to linear equations and inequalities.

Finding normal surfaces can now be formulated algebraically as problem in integer linear programming.

Normal surfaces correspond to integer vectors $(v_1, v_2, v_3, \ldots\ v_{7t})$ satisfying linear equations.

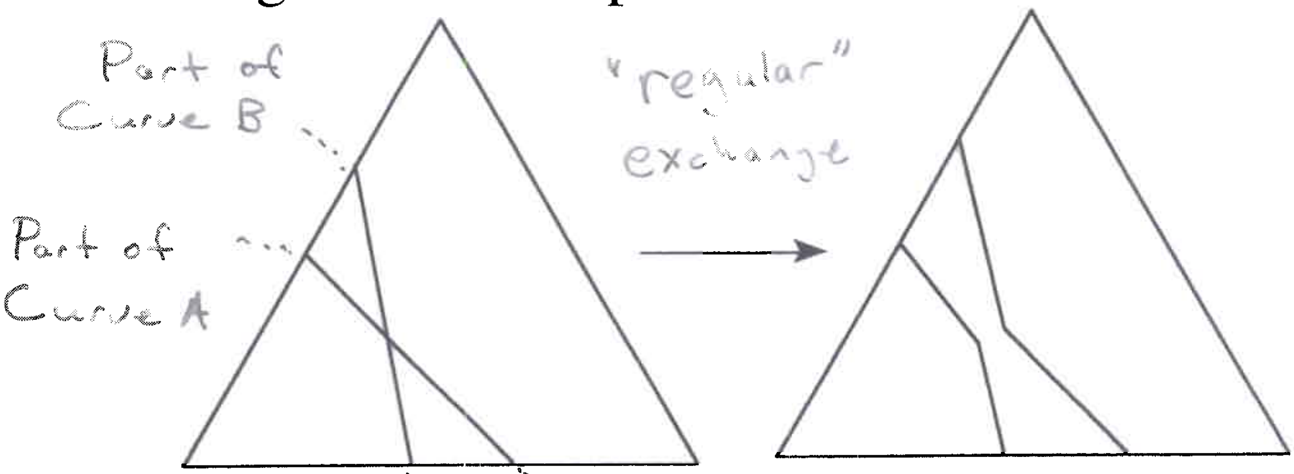
$V_i + V_j = V_k + V_l$ $\qquad V_i \geq 0$

Two normal vectors can be added to get a new normal vector, still satisfying the normal surface equations.

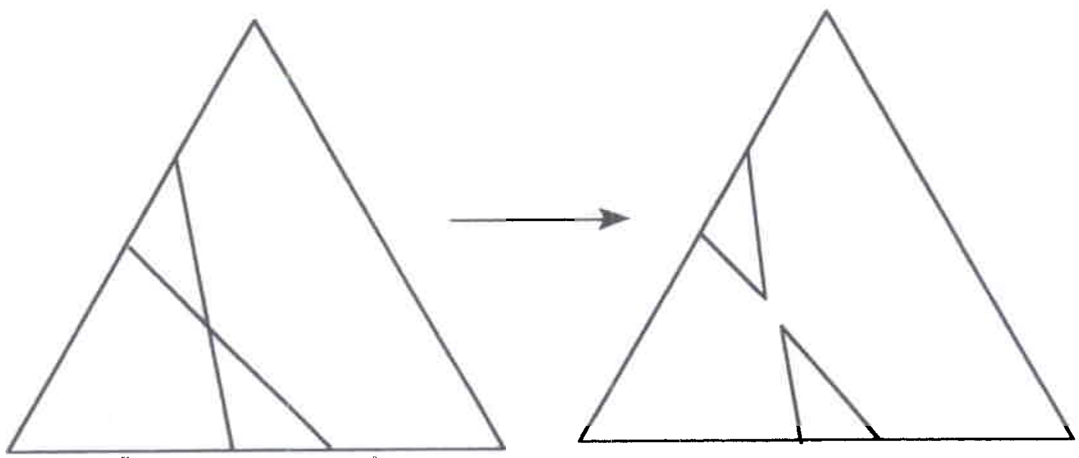
This would not be very useful if there were not an amazing occurrence. Normal vector addition has a natural geometric interpretation.

ALGEBRA ⇔ GEOMETRY

Sums of normal vectors corresponds to "regular cut and paste" of normal surfaces.

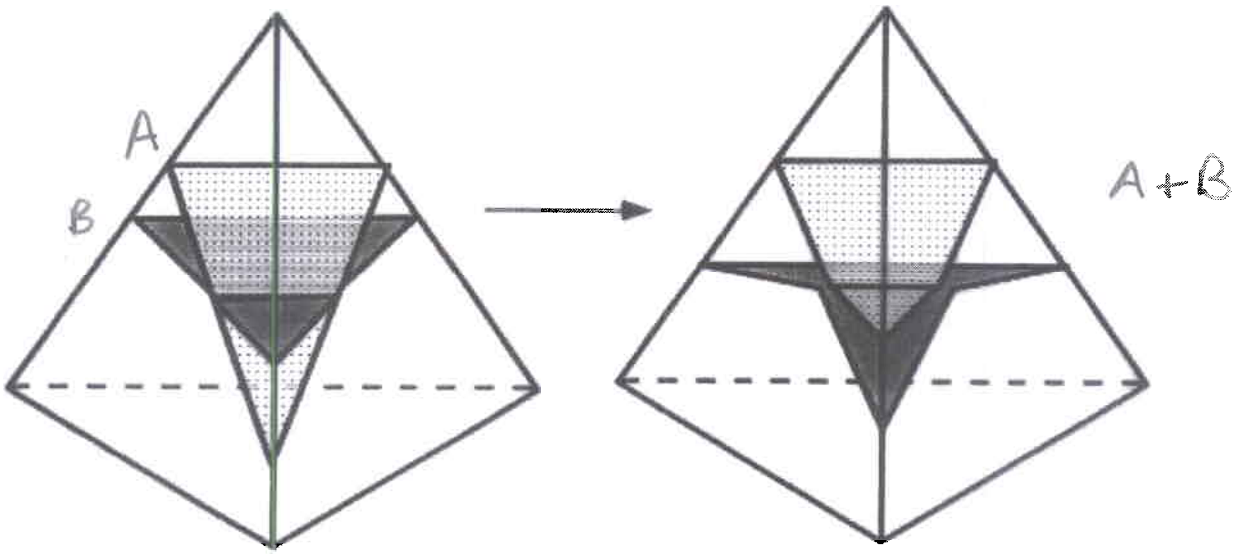


Two intersecting normal curves give a new embedded normal curve.



This "irregular" cut and paste leads to a non-normal surface.

There is exactly one way to cut and paste to keep the "sum" normal.



Regular exchange of two intersecting normal surfaces, A and B.

Note ~~both~~ all surfaces remain normal after the exchange. The normal vector representing A+B is the sum of those of A and B.

$(a_1+b_1, a_2+b_2, a_3+b_3, \ldots\ a_{7t}+b_{7t}) =$
$(a_1,a_2,a_3, \ldots\ a_{7t})+ (b_1,b_2,b_3, \ldots\ b_{7t})$

Each of A, B have a 1 in the ith coordinate

A = (., ..., 1, ...)   B = (..., 1, ...)

A+B has a 2

A+B = (...., 2, ...)

Furthermore, the Euler characteristic is linear under this sum.

$$\chi(V) + \chi(W) = \chi(V+W)$$

This follows because exchange preserves the number of vertices, edges, faces.

This allows us to control the genus when we work with normal surfaces.

Finally:

Weight Also Adds.

$$\text{weight}(V) + \text{weight}(W) = \text{weight}(V+W)$$

Weight is like Area, in a metric concentrated along edges of a triangulation.

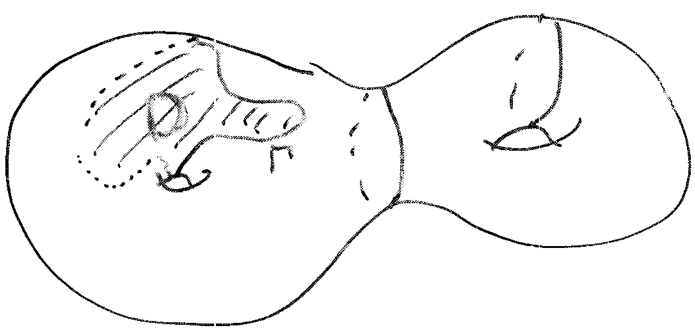
29A. There are many theorems in Differential Geometry about surfaces that minimize area.

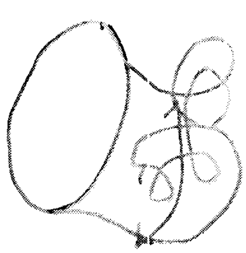
e.g. <u>Minimal Dehn's Lemma</u> (Meeks-Yau)

Suppose M is a Riemannian 3-manifold with convex boundary and Γ is a simple null-homotopic curve in ∂M. Then

1) Γ bounds a least area disk D in M

2) This disk is embedded





<u>NOTE</u>: NOT OBVIOUS that any disk with boundary Γ is embedded

LEAST WEIGHT NORMAL DISKS SHOULD BE SIMILAR.

A finite Hilbert basis:

Fundamental Normal Surfaces

A normal surface F is *fundamental* if its associated vector is not the sum of two normal curve vectors.

$$F \neq A + B,$$

whereA and B satisfy normal equations

Since the entries of normal vectors are non-negative integers, it is immediate that any normal vector is a sum of fundamental vectors.

Less obvious, but well known in linear programming, is that there are only finitely many fundamental vectors. (Hilbert basis.)

Finiteness is the key to constructing algorithms.

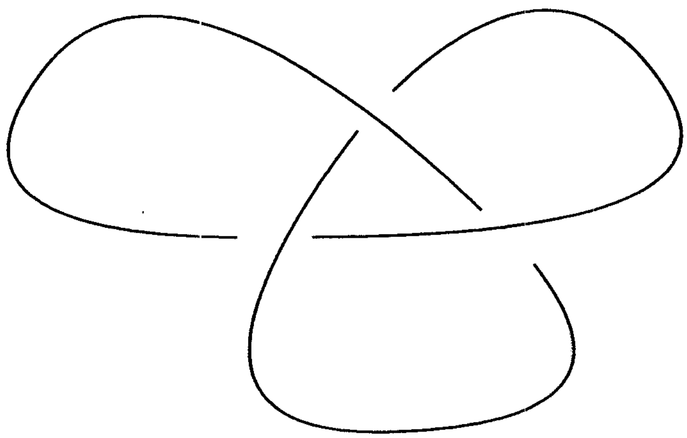
CONCLUSION:

We have found a class of surfaces that are sufficiently constrained that there are only finitely many, yet rich enough to contain representatives of many interesting classes of surfaces, giving useful algorithms.

Some classes of surfaces that have representatives among the fundamental ~~curves.~~ Vectors.

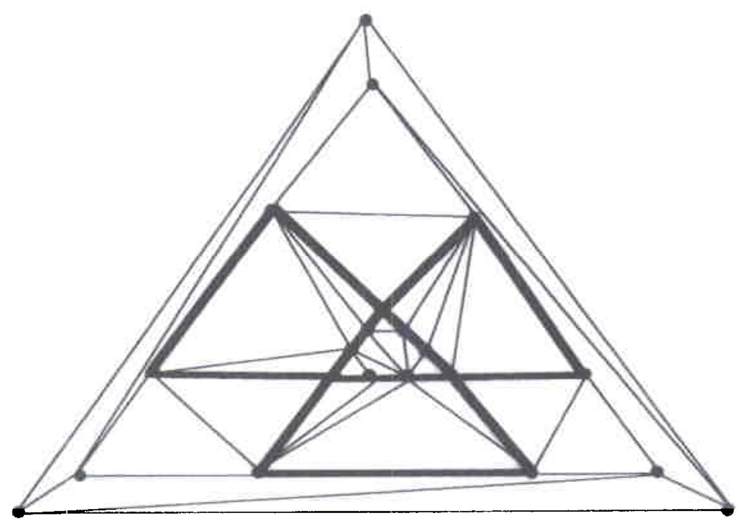
Example 1: Unknotting disks → Unknotting Algorithms

Start with an n-crossing knot diagram.



(n=3)

Make everything PL: triangulate a ball so that the knot lies on its 1-skeleton.



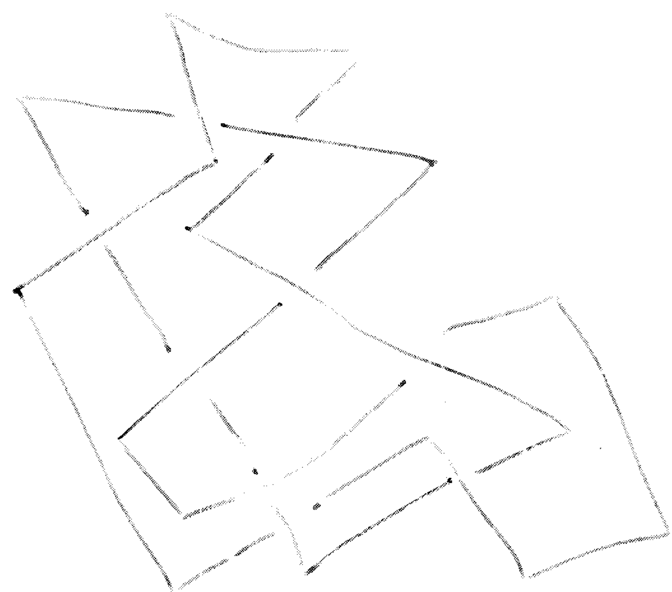
In the complexity analysis, a typical issue that arises is

Question. How many tetrahedra are needed?

Open Problem:
If K is a polygonal knot in R^3 with n edges,

how many tetrahedra are needed to triangulate B^3 so that K lies on the 1-skeleton? (Allow non-linear tetrahedra).

Best at present is $O(n^2)$.
Linear?

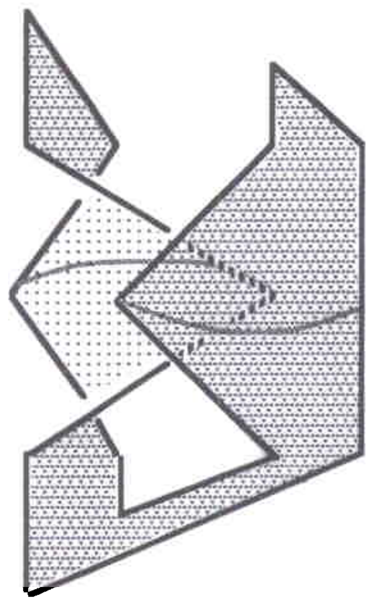


n edges

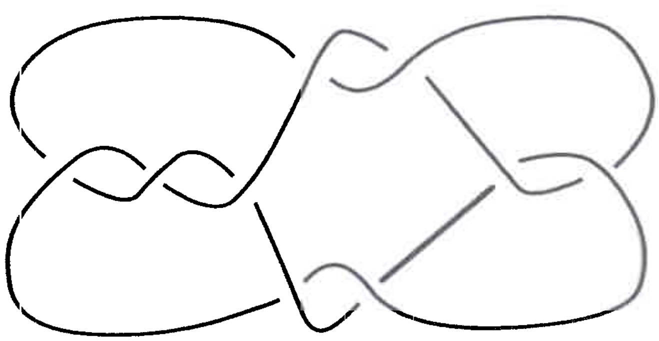
How many tetrahedra required?

Outline of Haken's Unknotting Algorithm

Step 1: K is unknotted $\Leftrightarrow$ K is the boundary of an embedded disk



A disk spanning an unknot.



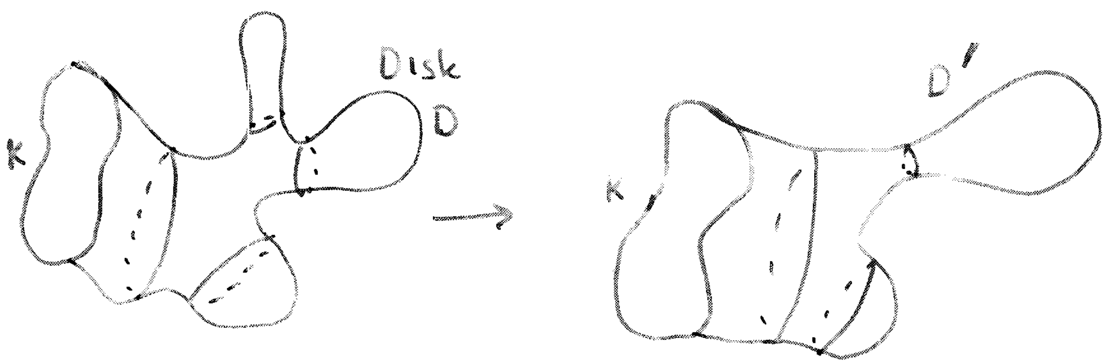
The disk is less obvious for this unknot.

Step 2: If M is triangulated and K is unknotted then K bounds a normal disk:

K is unknotted ⇔ K is the boundary of a normal disk D

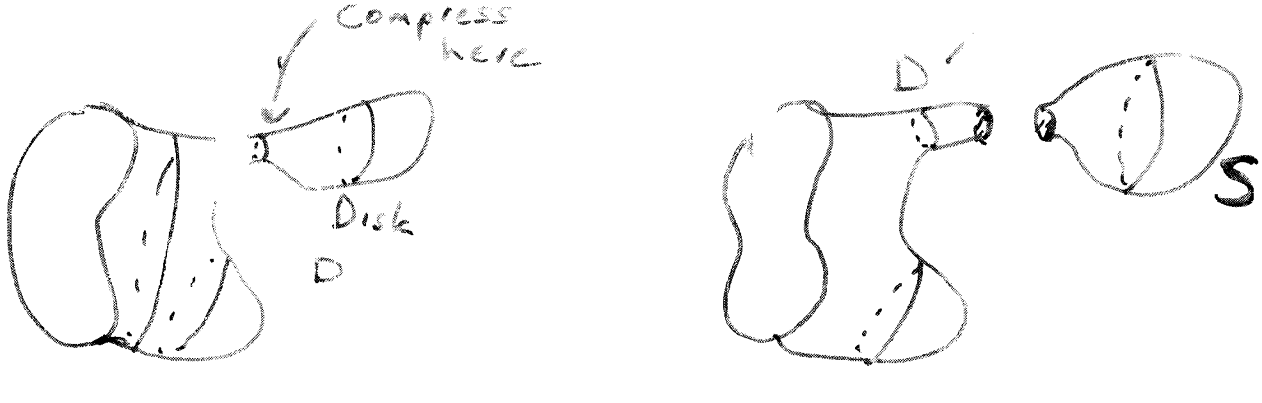
Recall: Starting with any surface, we get a normal surface by Isotopy and compression.

1.



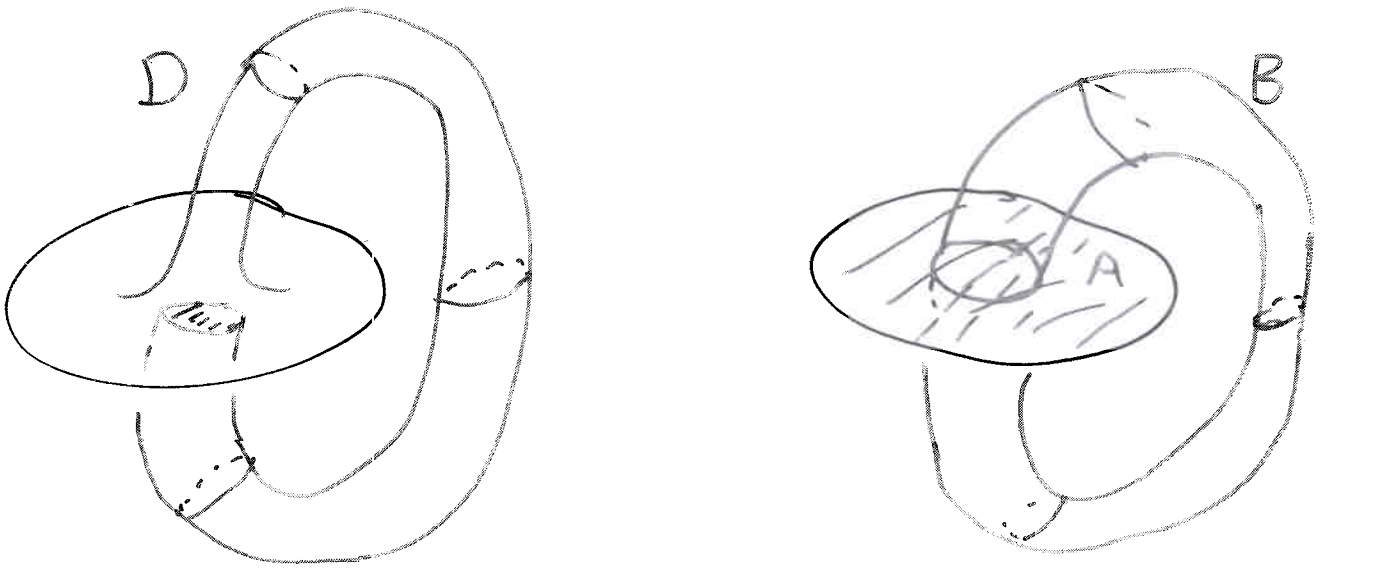
Isotopy moves D to a new unknotting disk. No problem.

2. <u>Compression</u> D → D' ∪ 2-Sphere S



D' is a New, Simpler Unknotting Disk

Step 3: Analyze how surfaces sum:
If D = A+B, then one of A and B is a disk of smaller weight that spans the curve K. We can continue until we arrive at a fundamental disk with boundary K.

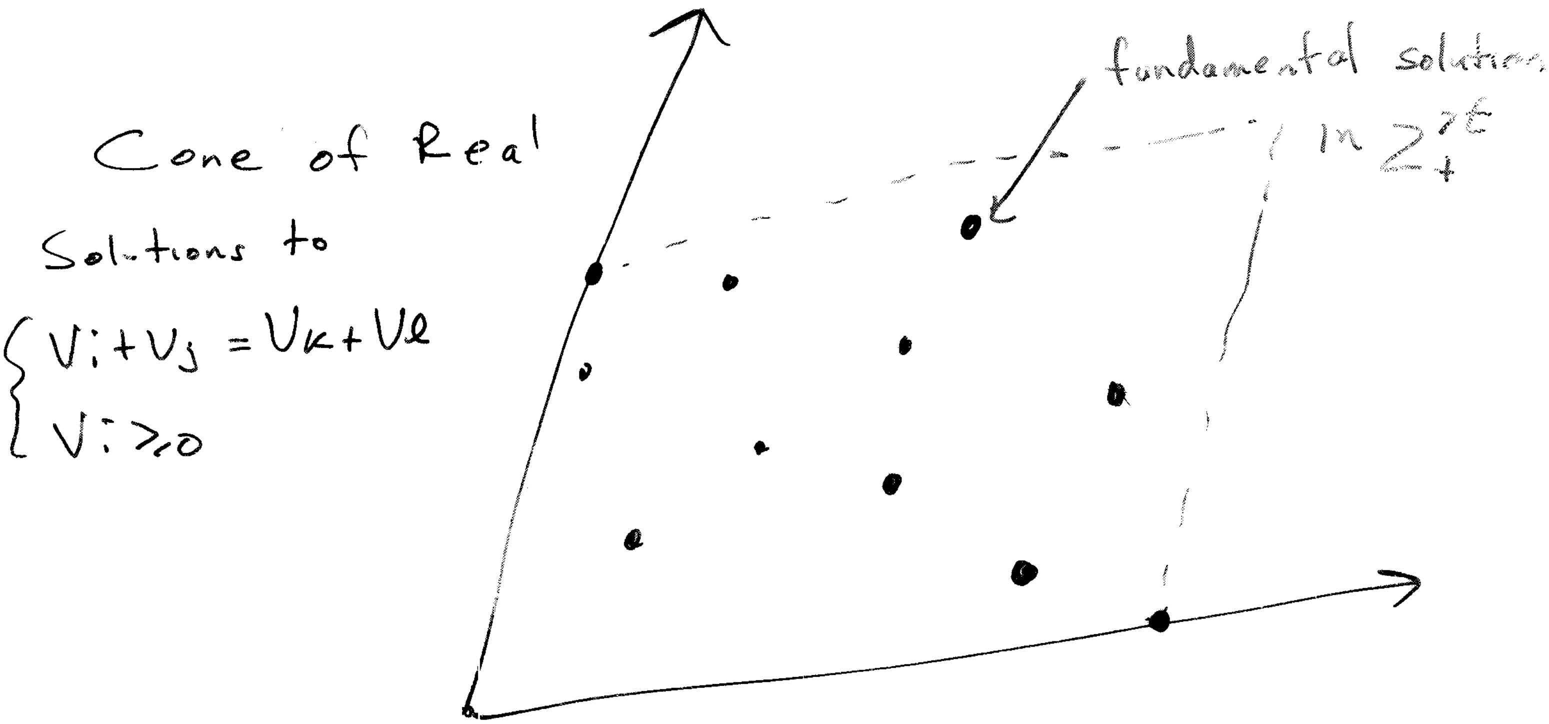


Here A is a disk, B is a torus. Their sum is a disk D.

K is unknotted ⇔ K is the boundary of a fundamental normal disk

Replace D with A.
Reduce weight each time,
so eventually stop.

Step 4: To see if K is unknotted, check the (finitely many) fundamental normal surfaces one by one, to see if any is a disk with boundary K. (If F is a fundamental surface and $\chi(F) = 1$, then F is a disk)



Finitely many fundamental vectors in $\mathbb{Z}_+^{7t}$

Check Euler characteristic of each one to see if it's a disk.

$\chi = 1 \Rightarrow$ Disk

(also check only one quadrilateral type per tetrahedron)

To obtain complexity bounds, (bounds on the running time of this algorithm), we need to explicitly bound the size of a fundamental normal disk in terms of the number of Knot crossings n and the number of tetrahedra t.

Lemma (H-Lagarias-Pippenger)
Any fundamental surface has normal coordinates $(v_1, v_2, v_3, \ldots\ v_{7t})$ with

$$v_i < t \cdot 2^{7t+2}$$

Corollary
There are at most $t^{7t}\, 2^{49t^2+14t}$ fundamental surfaces.

This is how many surfaces need to be checked to see if any is an unknotting disk.
(At most)

Corollary An algorithm for the UNKNOTTING PROBLEM runs in time

$$O(c^{t^2})$$

$t =$ crossing number
?
number of tetrahedra

Can improve to $O(c_1^t)$ or $O(c_2^n)$

To show that Unknotting is in NP, we need to give a certificate of the triviality of a knot that can be checked in time polynomial in the number of crossings of the knot.

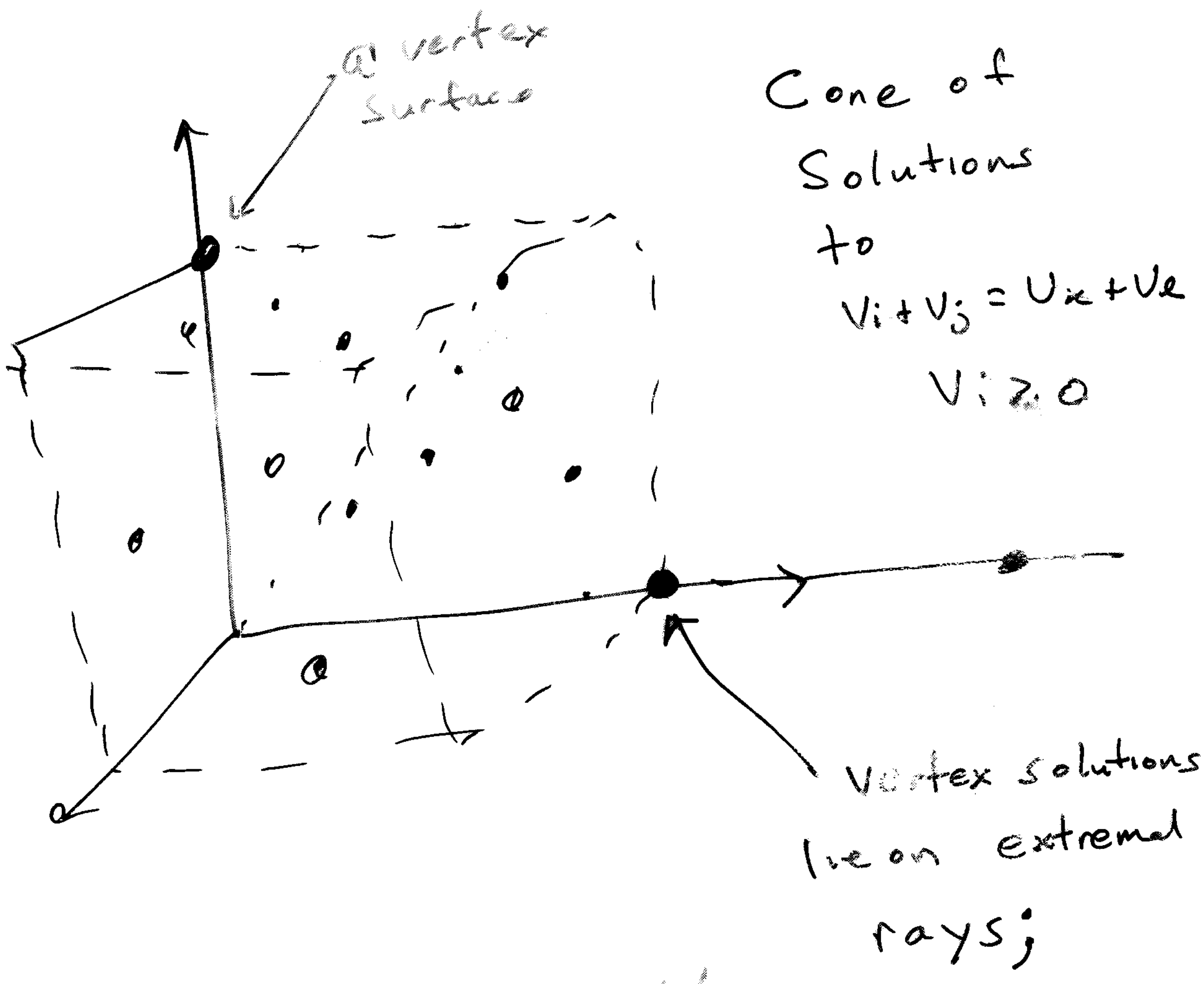
This can be done by giving the normal coordinates of a fundamental normal disk that spans the knot. All properties demonstrating that this is a disk spanning the knot are verifiable in polynomial time.

The hardest property to verify is the following:

Given a normal vector $(v_1, v_2, v_3, \ldots\ v_{7t})$ satisfying the normal surface equations, is the corresponding surface connected?

If yes, then it is easy to compute the Euler characteristic and thus the genus.

For the Unknotting problem, Jaco and Tollefson showed that there is a vertex solution to the normal surface equations, which is a disk. This vertex solution must be connected, and this is easily verified. This allowed the proof that Unknotting is NP.



Vertex Surfaces V are fundamental ∴ connected.

If $V = A \cup B$ then V is a sum of two normal vectors.

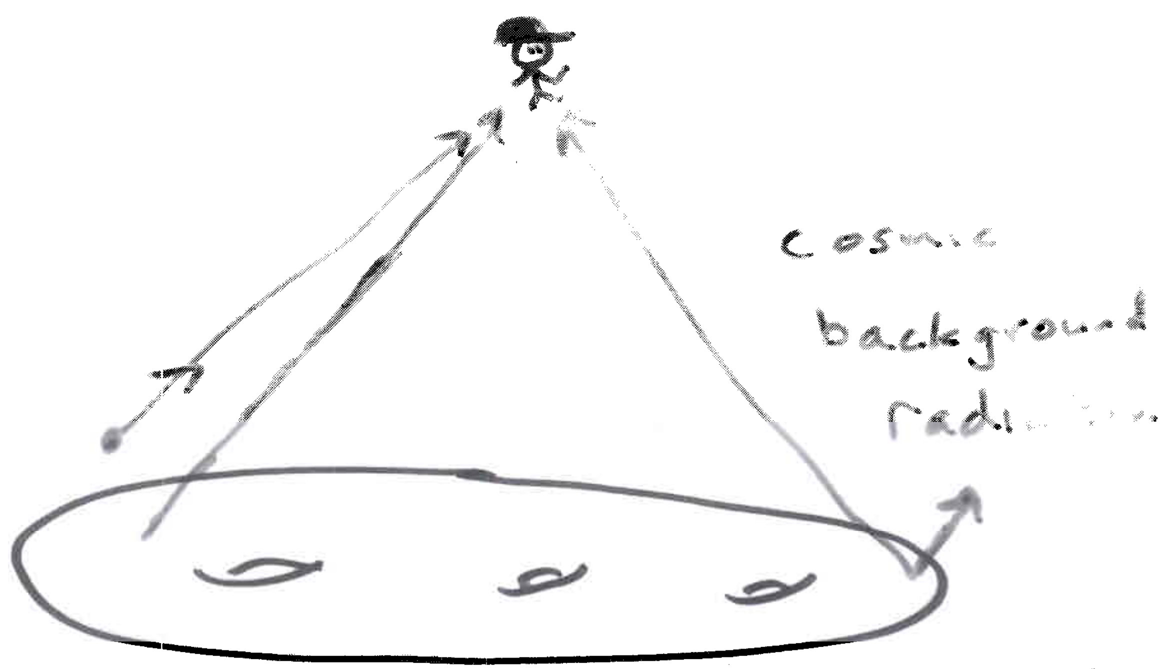
It's easy to check that V is a vertex surface

Recognition problems old and new

RECOGNITION PROBLEMS

OLD

NEW



RECOGNIZE THE UNIVERSE
(POINCARE HOMOLOGY SPHERE)
?

Recognizing the 3-sphere

Problem: given a description of a 3-manifold as a union of tetrahedra, how can we decide whether it is the 3-sphere?

Novikov: Don't bother trying for the 5-sphere – no algorithm exists to recognize it.

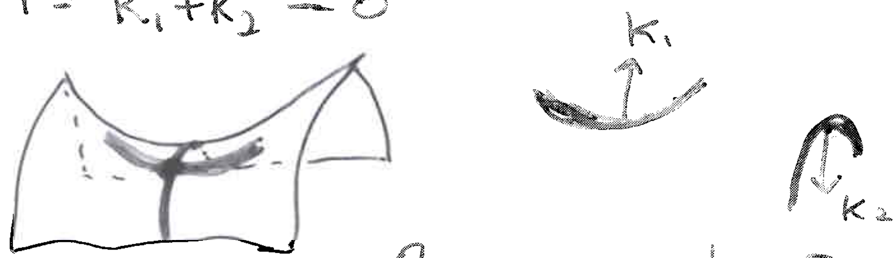
Solution: There is an algorithm to recognize the 3-sphere. (Rubinstein-Thompson, 1993)

The key idea in this algorithm lies in the connection between minimal and normal surfaces. It was inspired by a study of minimal 2-spheres in 3-manifolds.

Minimal surfaces

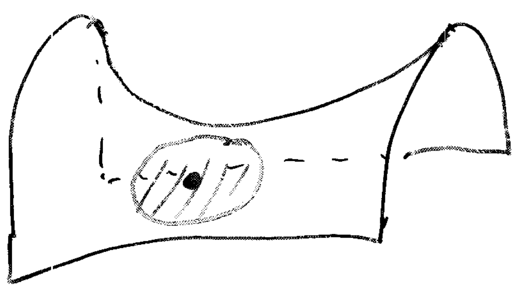
1. A surface with mean curvature zero

$H = k_1 + k_2 = 0$



Average curvature = 0

2. A surface that locally minimizes area:
Each point lies in a small disk on the surface that has the least area among all surfaces with the same boundary as that disk.

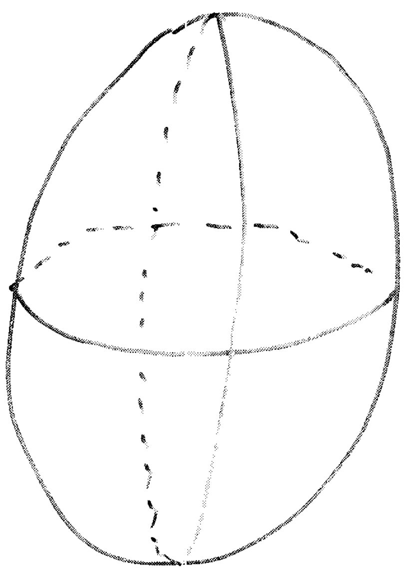


3. Conformal, harmonic mappings

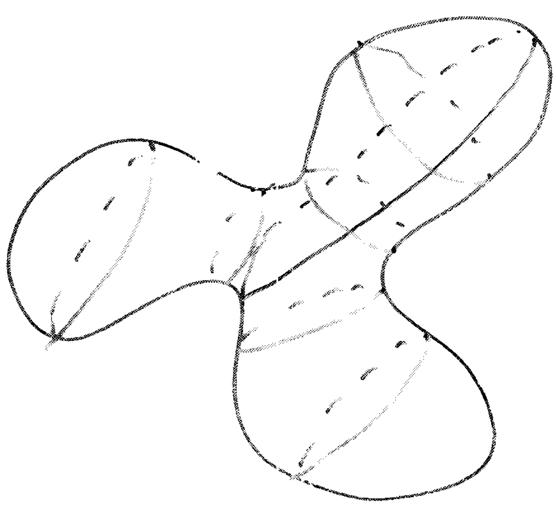
etc

A diversion to some differential geometry.

Lusternik-Schnirelmann studied geodesics on the 2-sphere. They showed
Theorem: (LS, 1929) There are always at least three simple closed geodesics on a 2-sphere, no matter what shape (Riemannian metric) it has.

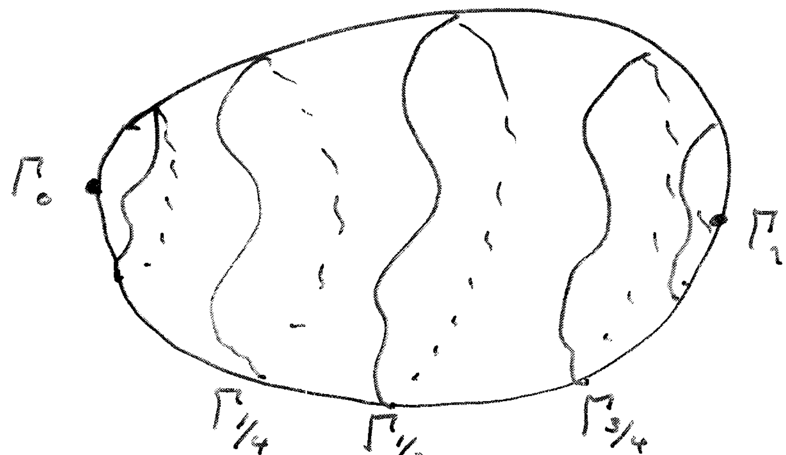


Some Ellipsoids have exactly 3 simple geodesics.

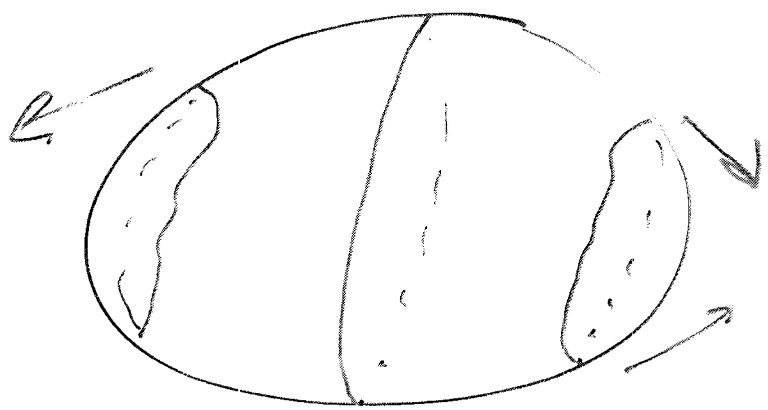


May have lots.

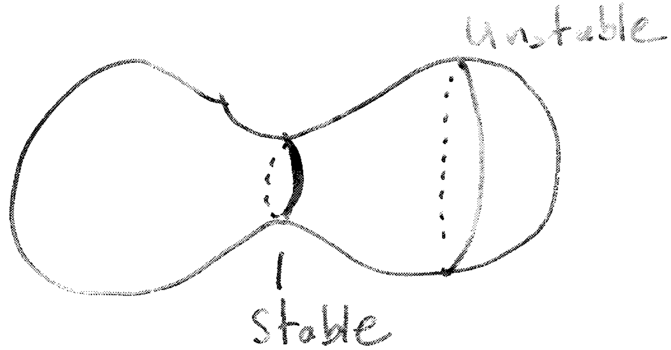
WHY A 2-SPHERE HAS AN UNSTABLE GEODESIC



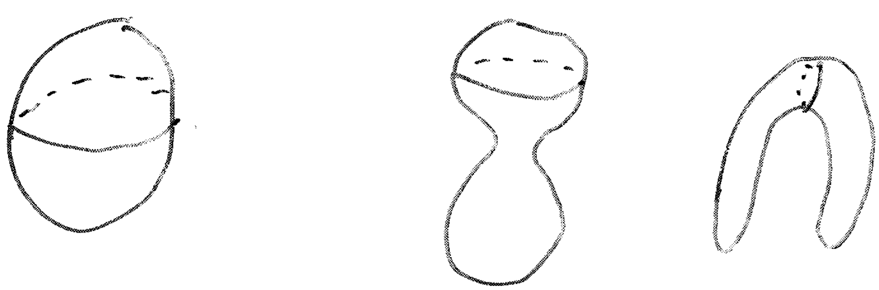
Shorten a family of curves Γ_t, $0 \le t \le 1$. Some curve gets stuck in the middle.



We distinguish between stable and unstable geodesics.



With ANY metric, a 2-sphere always has an unstable geodesic.

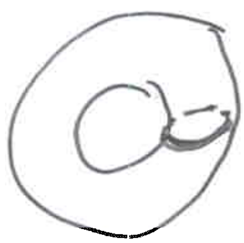


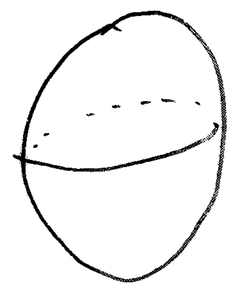
Other surfaces may or may not have unstable geodesics.

We can use these observations to make a (rather useless) recognition algorithm for the 2-sphere:

Take a mystery surface F. Is it the 2-sphere?

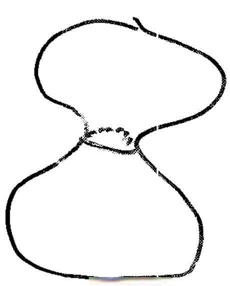
1. Find a maximal family of stable disjoint geodesics in F. If none, then we have a 2-sphere.



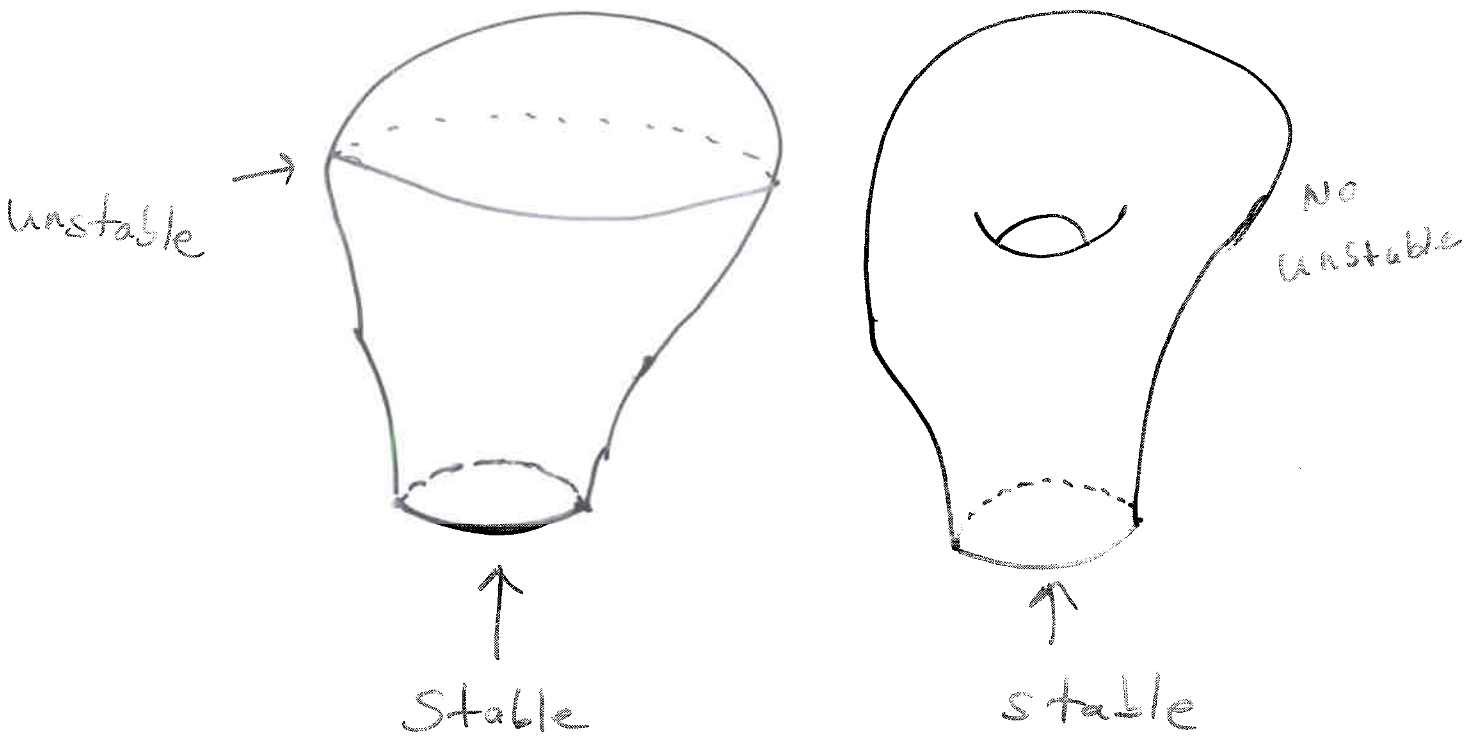


None

If there is one, could be S^2, or not.



2. If there is a complementary region X bounded by exactly one stable geodesic, then X is a disk if and only if it contains an unstable geodesic.



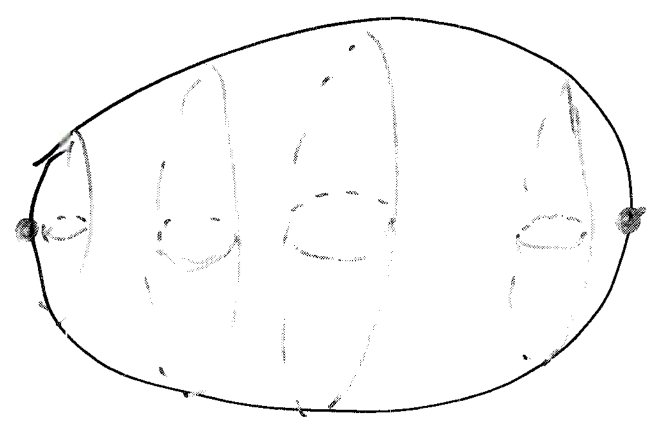
This tells disks from non-disks

~~3. If there is a complementary region bounded by a subcollection of more than one stable geodesics, then it is a punctured disk (disk with holes) if and only if it contains no stable geodesics.~~

Minimal 2-spheres in the 3-sphere

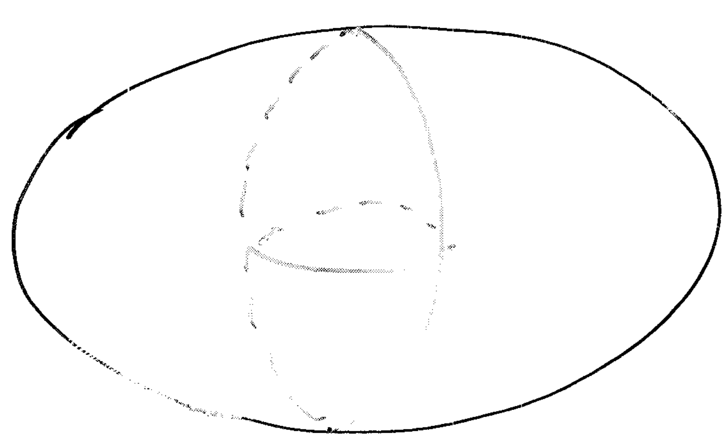
The Lusternik-Schnirelmann theorem has an analog in dimension three.

Theorem: In any Riemannian metric on the 3-sphere, there is always an embedded unstable minimal 2-sphere (in fact four). (Pitts, Rubinstein, Smith, Simon, Jost).



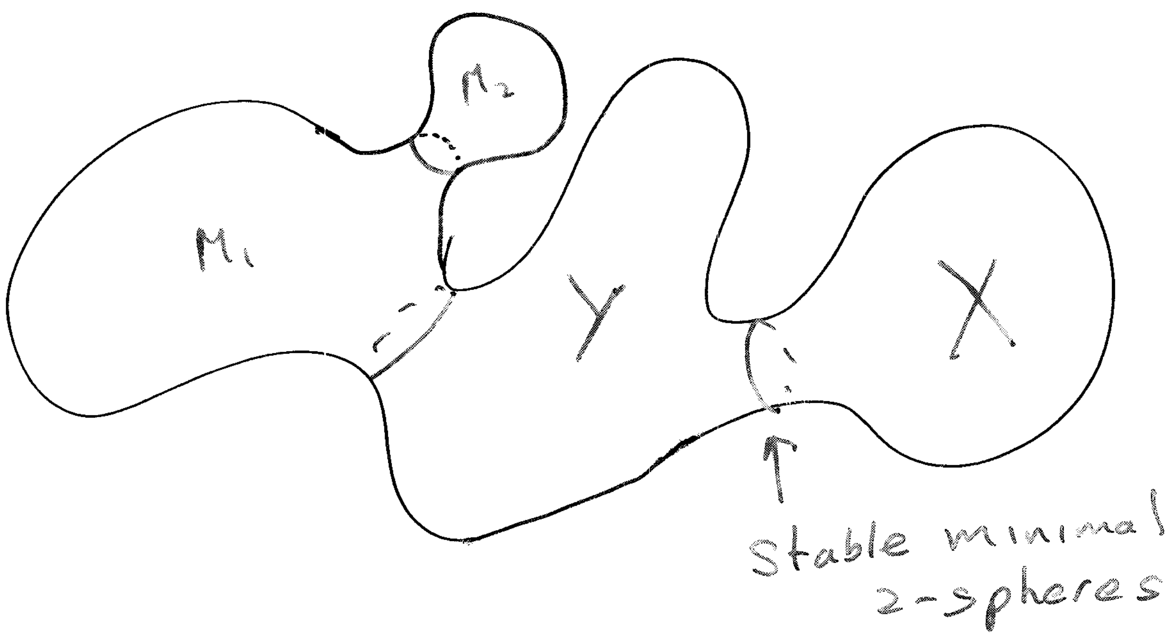
S_t = family of 2-spheres

Such unstable minimal 2-spheres are found by pulling down the area of a whole family of spheres, and showing that at least one gets stuck on a bulge.



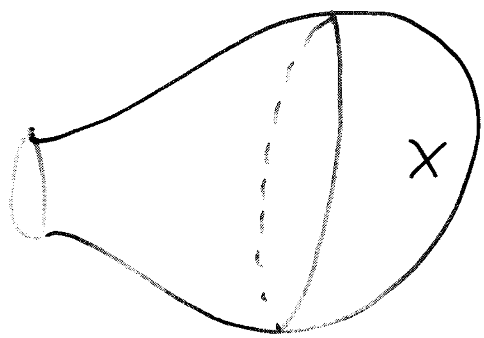
This gives us a seemingly unuseful way of recognizing the 3-sphere.
Suppose we have a mystery 3-manifold M.

1. Find a maximal collection of disjoint, stable minimal 2-spheres in M.
(How? We'll see shortly).

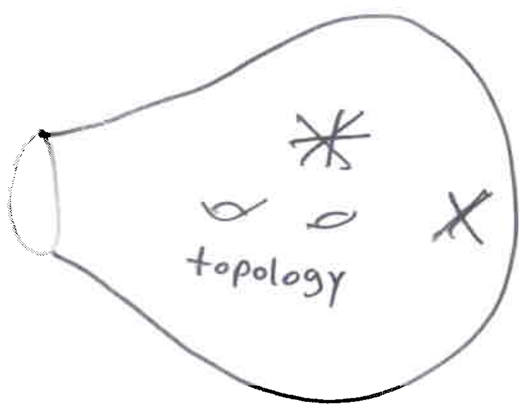


M is a 3-sphere if and only if each piece is a punctured ball (ball with holes).

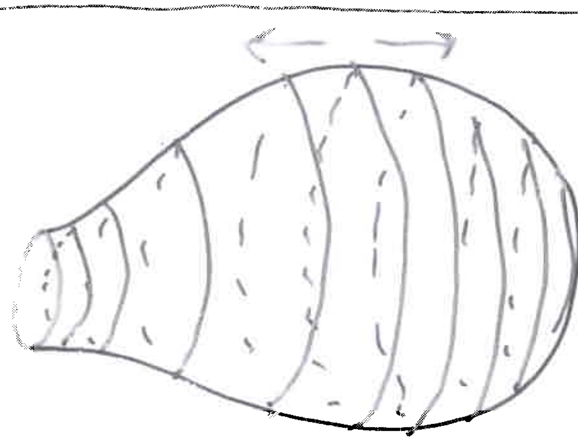
3. Pieces like X, with a single boundary component, may or may not be balls. They are balls if and only if they contain an unstable 2-sphere.



Unstable Minimal 2-sphere
$\Rightarrow B^3$



NO UNSTABLE Minimal 2-spheres
$\Rightarrow$ NOT B^3

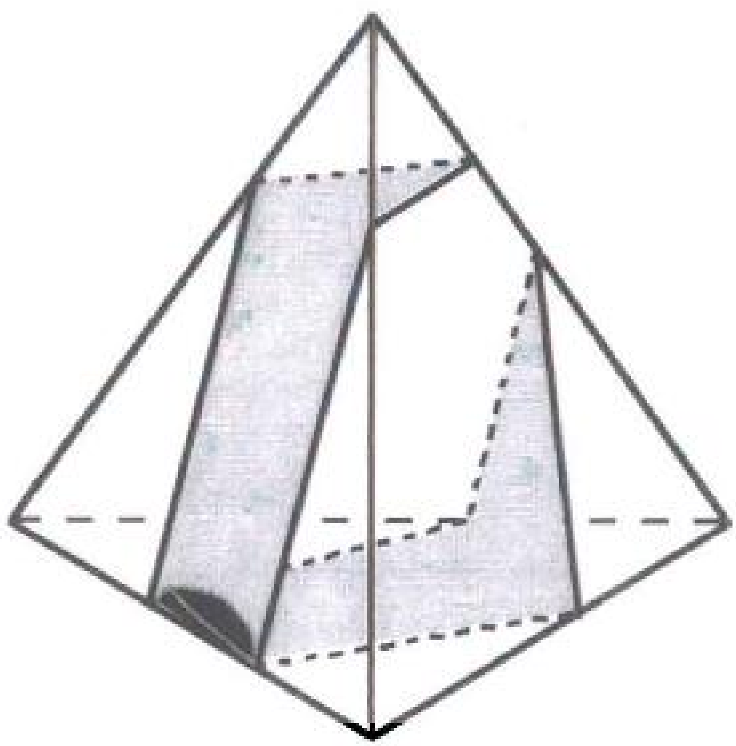


WHY B^3?

Push off
Sweeps out
all of X

This idea was pursued by Rubinstein, who defined Almost Normal Surfaces

An almost normal surface intersects each tetrahedron in triangles and quadrilaterals except for one. In one tetrahedron it intersects in an octogon.



This is part of an "Unstable" minimal 2-sphere in the PL setting.

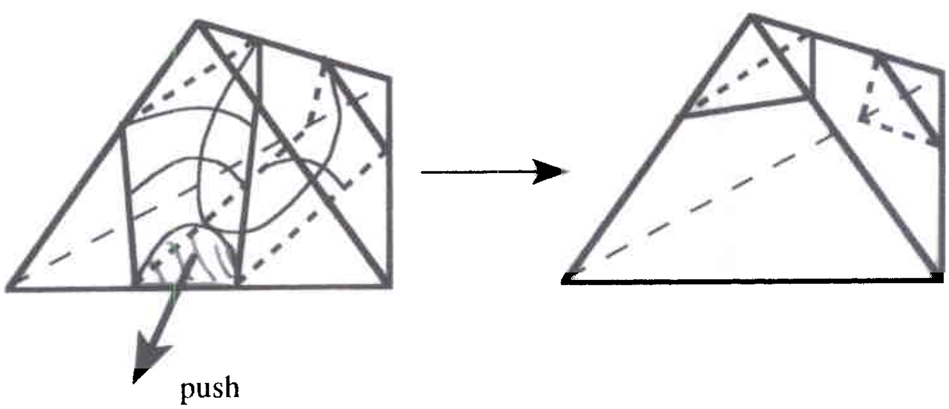
If we could somehow get hold of stable and unstable minimal 2-spheres, we could decide whether each piece is a punctured ball, and whether the whole manifold is the 3-sphere.

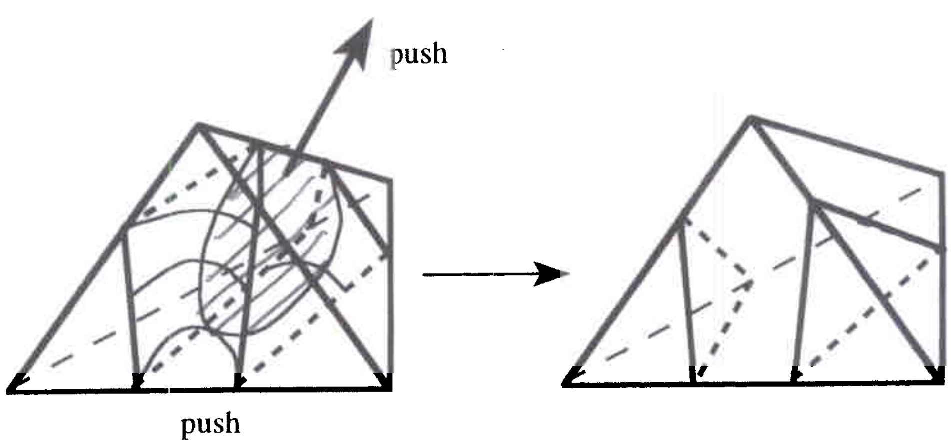
At first this seems harder than the recognition problem. Finding minimal surfaces is very difficult. Doesn't lend itself to an algorithm.

BUT, we have seen that minimal surfaces have discrete analogs, normal surfaces, with triangulations replacing metrics,. Normal surfaces can be found by a finite procedure. What remains is to find the analog in the discrete setting of the idea of stable and unstable minimal surfaces.

Like an equatorial 2-sphere of a 3-sphere, these can be pushed slightly in two different directions to decrease their area (or weight).

(Technically, an unstable minimal 2-sphere has a single Jacobi field up to scaling. This means that in each direction there is one way to push the 2-sphere off itself to decrease area, to first order.)


push

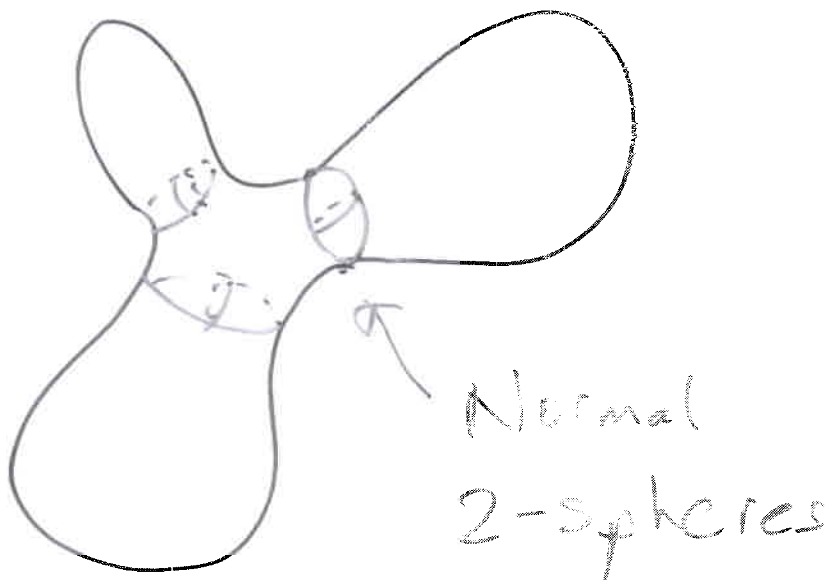

push
push

Recognizing the 3-Sphere (Rubinstein - Thompson)

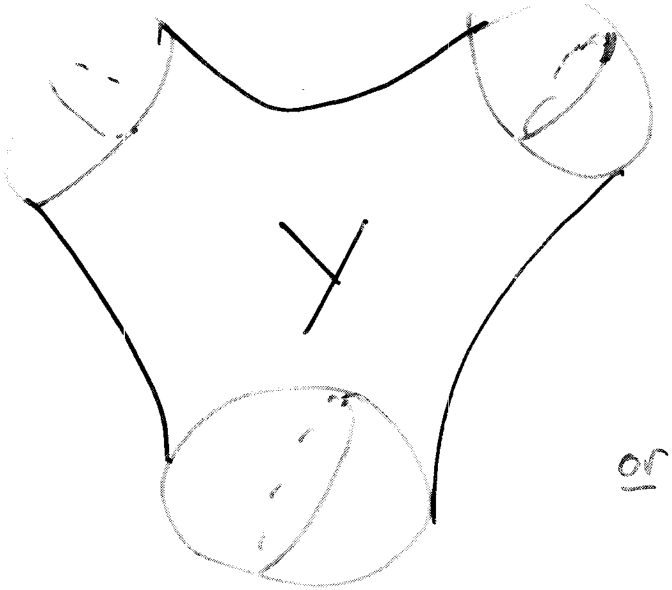
Some highlights of the algorithm:

1. Take a triangulated mystery 3-manifold M.

2. Find a maximal family of disjoint, non-parallel normal 2-spheres.
(Look among the fundamental surfaces)

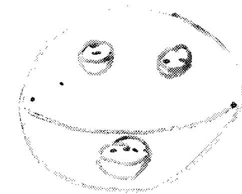


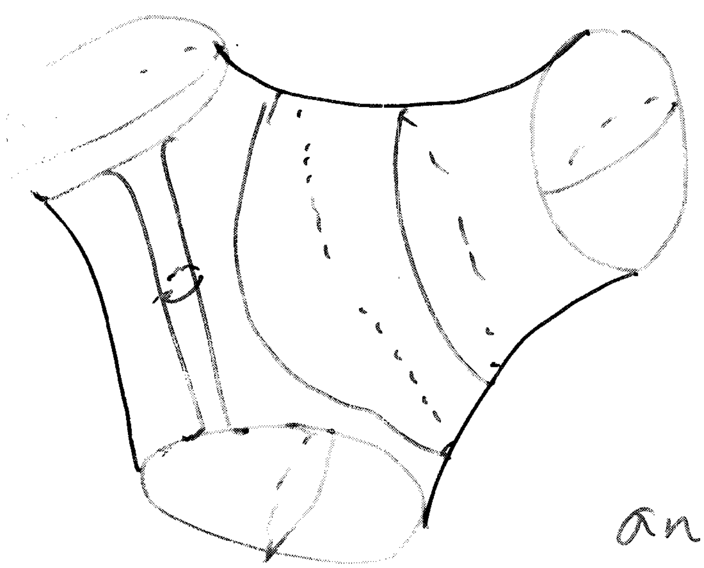
3. Check complementary pieces.
a. Those with more than one boundary component are punctured balls.
(Nothing to do. ALWAYS Punctured Balls)



Y is a ball with holes

or





Tube together two spheres. They sweep out Y, ending at another sphere.

b. Those with exactly one boundary component are balls if and only if they contain an almost normal 2-sphere.

X is a ball

$\iff$

X contains an almost normal surface

This can be checked algorithmically.

WHY?

SWEEP OUT IN EACH DIRECTION. NOTHING TO GET STUCK ON.

The running time of this algorithm was analyzed by Casson (2000).

He showed that if the manifold M has t tetrahedra, we can decide it is the 3-sphere in time $O(3^t)$.