# Folding & Unfolding: Origami

**Erik Demaine**
**M.I.T.**

edemaine@mit.edu
http://theory.lcs.mit.edu/~edemaine/folding

# Folding and Unfolding Talks

| | | |
|---|---|---|
| Linkage folding | Yesterday | Erik Demaine |
| **Paper folding** | **Today** | **Erik Demaine** |
| Folding polygons into convex polyhedra | *Saturday morning* | Joe O'Rourke |
| Unfolding polyhedra | Saturday afternoon | Joe O'Rourke |

# Outline

- History and Definitions
- Foldability
  - Crease patterns
  - Map folding
- Origami design
  - Silhouettes and gift wrapping
  - Tree method
  - One complete straight cut
  - Flattening polyhedra

# History of Paper in Asia

- **Origami** believed to have followed shortly after making of paper (not papyrus)
- **Paper**
  - Believed to have been invented by Ts'ai Lun, Chinese court official, 105 AD, following the 250 BC invention of the camel hair brush
  - Spread by Buddhist monks through Korea to Japan from 538 AD to 610 AD
  - Spread by Arabs occupying Samarkand, Uzbekistan from 751 AD to Egypt in 900's and continued west

# History of Paper in Europe

- Moors brought paper (and mathematics) to Spain during their invasion in 700's
  - Established paper making in 1100's in Jativa, Spain
- Arab occupation of Sicily brought paper to Italy
- Paper mills built in Fabriano, Italy in 1276, in Troyes, France in 1348, and in Hertford, England in 1400's
- By ~1350, paper was widespread for literary work in Europe
- First paper mill in North America built in 1690 in Roxboro, Pennsylvania

# Modern History of Origami

- Origami popular throughout the world
  - North America: mainly U.S.
  - Europe: particularly England, Spain, Italy
  - Asia: particularly Japan, China, Korea
- Until recently, most origami models were relatively simple—e.g., most animals had just 4 “limbs” (head and three legs, etc.)
- In the last ~25 years, **complex origami** has evolved to attain incredible feats

# Modern Artistic Origami

Black Forest Cuckoo Clock by Robert Lang

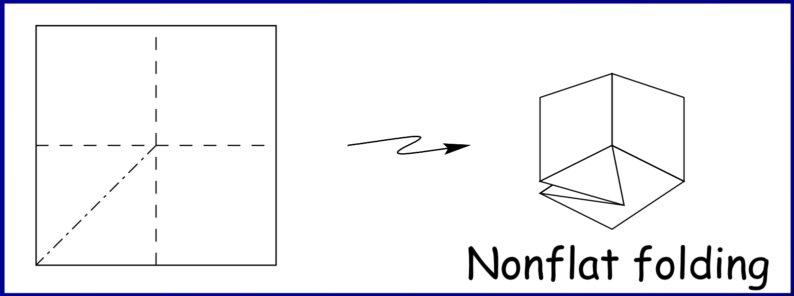
Mask by Eric Joisel

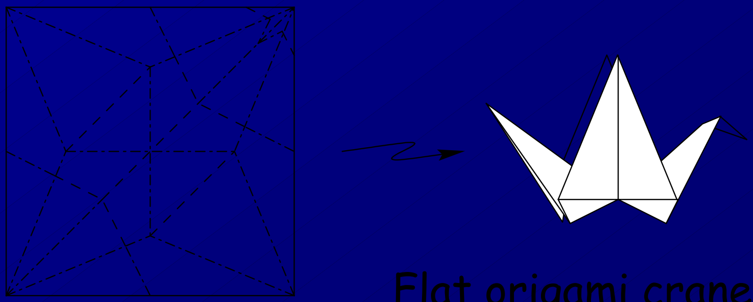
Bat by Michael LaFosse

Pangolin by Eric Joisel

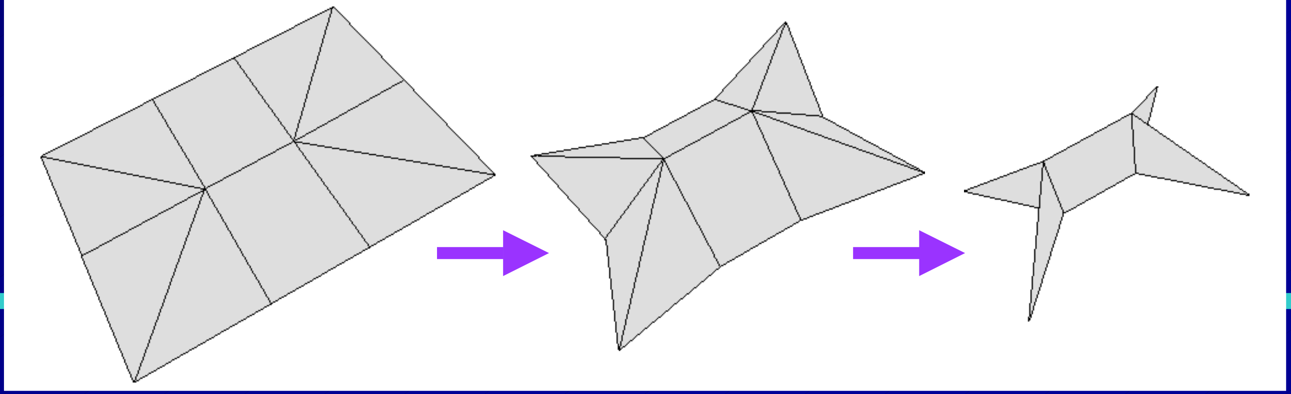
# Foldings

- **Piece of paper** = 2D surface
  - Square, or polygon, or polyhedral surface
- **Folded state** = isometric "embedding"
  - **Isometric** = preserve intrinsic distances (measured along paper surface)
  - **"Embedding"** = no self-intersections except that multiple surfaces can "touch" with infinitesimal separation



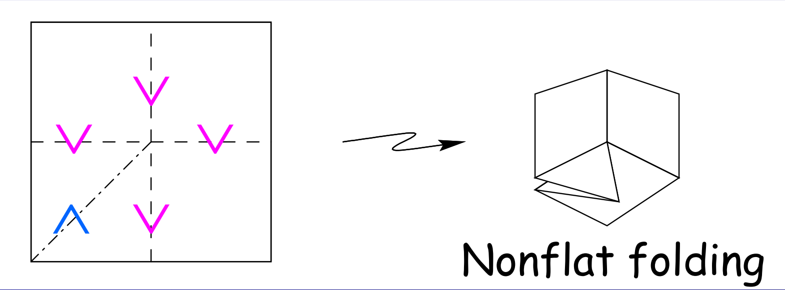
# Foldings



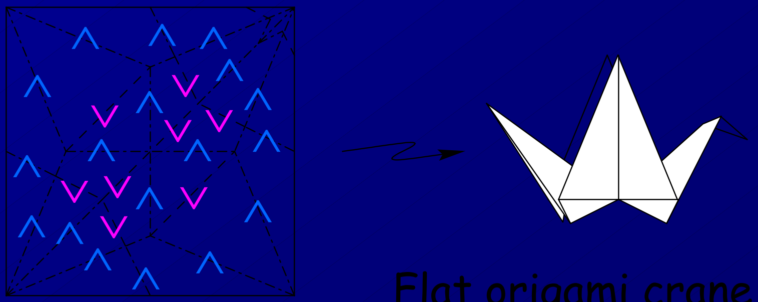
- **Configuration space** of piece of paper = uncountable-dim. space of all folded states
- **Folding motion** = path in this space = continuum of folded states
- Fortunately, configuration space of a rectangular piece of paper is path-connected [Demaine & Mitchell 2001]
  - $\Rightarrow$ Focus on finding interesting folded states
- **Open:** Nonrectangular paper?

# Structure of Foldings

- **Creases** in folded state = discontinuities in the derivative
- **Crease pattern** = planar graph drawn with straight edges (creases) on the paper, corresponding to unfolded creases




- **Mountain-valley assignment** = specify crease directions as ∧ or ∨



# What Can You Fold?

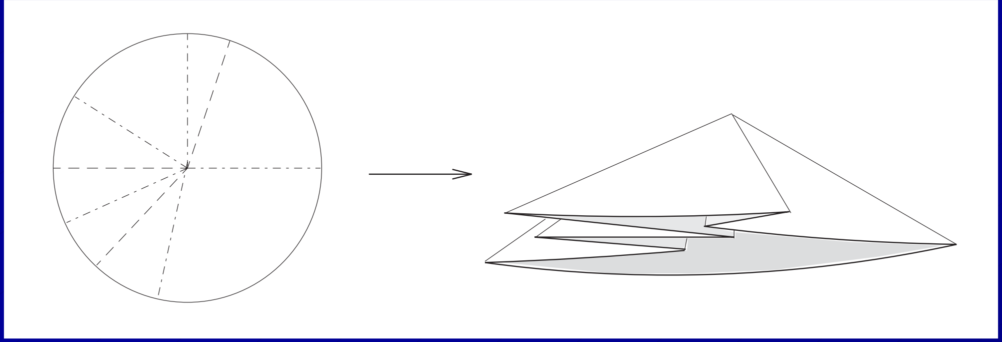
- **Universality result:** Everything is foldable, and there is an efficient algorithm to find the foldings
- **Efficient decision result:** Efficient algorithms for deciding whether something is foldable, and when it is, exhibiting a folding
- **Hardness result:** Deciding foldability is computationally intractable

# Outline

- History and Definitions
- Foldability
  - Crease patterns
  - Map folding
- Origami design
  - Silhouettes and gift wrapping
  - Tree method
  - One complete straight cut
  - Flattening polyhedra

# Single-Vertex Origami

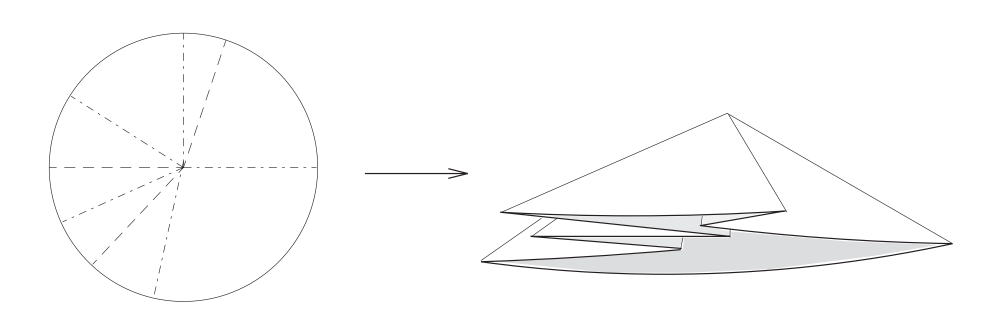
- Consider a disk surrounding a lone vertex in a crease pattern (**local foldability**)
- When can it be folded flat?



- Depends on
  - Circular sequence of angles between creases: $\theta_1 + \theta_2 + \ldots + \theta_n = 360°$
  - Mountain-valley assignment

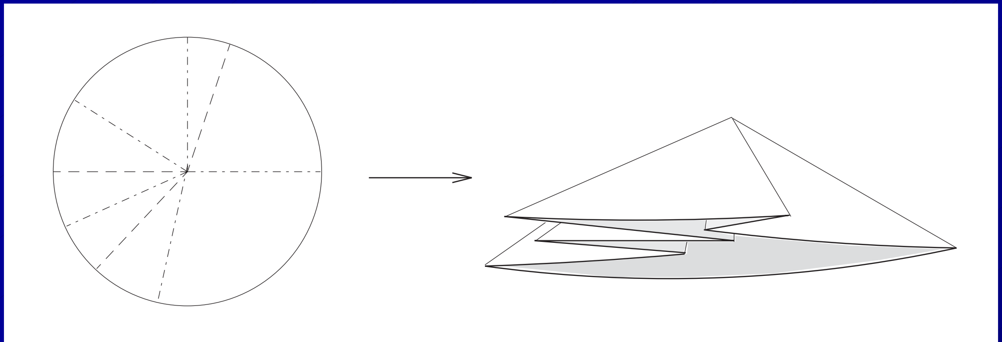
# Single-Vertex Origami without Mountain-Valley Assignment

- **Kawasaki's Theorem:**
  Without a mountain-valley assignment, a vertex is flat-foldable precisely if sum of alternate angles is 180°
  ($\Theta_1 + \Theta_3 + \ldots + \Theta_{n-1} = \Theta_2 + \Theta_4 + \ldots + \Theta_n$)
  - Tracing disk's boundary along folded arc moves $\Theta_1 - \Theta_2 + \Theta_3 - \Theta_4 + \ldots + \Theta_{n-1} - \Theta_n$
  - Should return to starting point $\Rightarrow$ equals 0



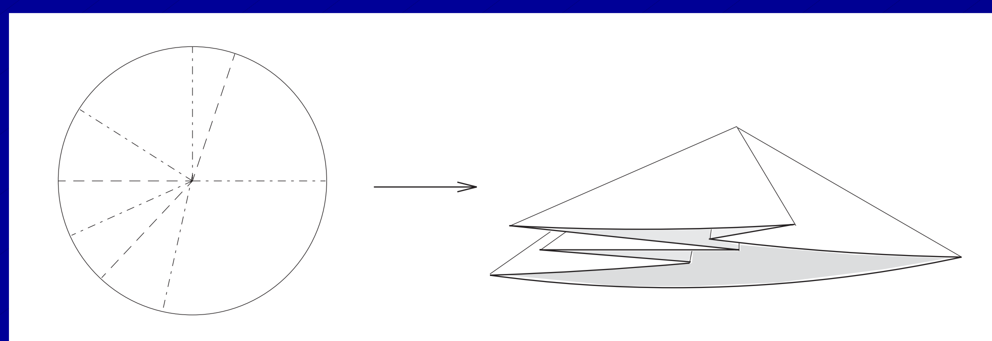
# Single-Vertex Origami with Mountain-Valley Assignment

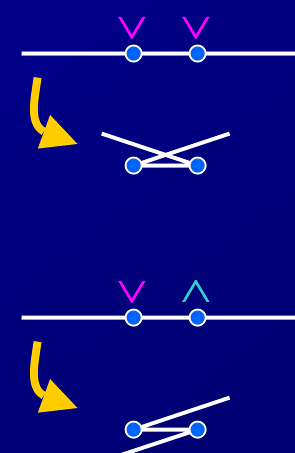
- **Maekawa's Theorem:**
  For a vertex to be flat-foldable, need
  |# mountains - # valleys| = 2
  - Total turn angle = ±360°
    = 180° × # mountains - 180° × # valleys



# Single-Vertex Origami with Mountain-Valley Assignment

- **Another Kawasaki Theorem:** If one angle is smaller than its two neighbors, the two surrounding creases must have opposite direction
  - Otherwise, the two large angles would collide

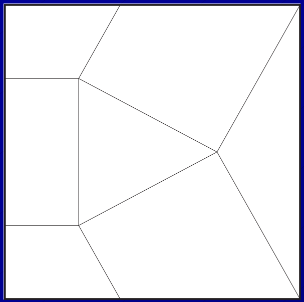


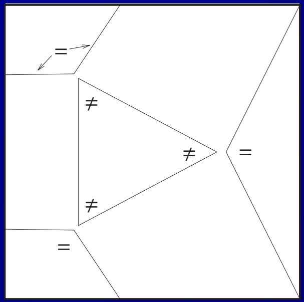


- These theorems essentially characterize all flat foldings

# Local Flat Foldability

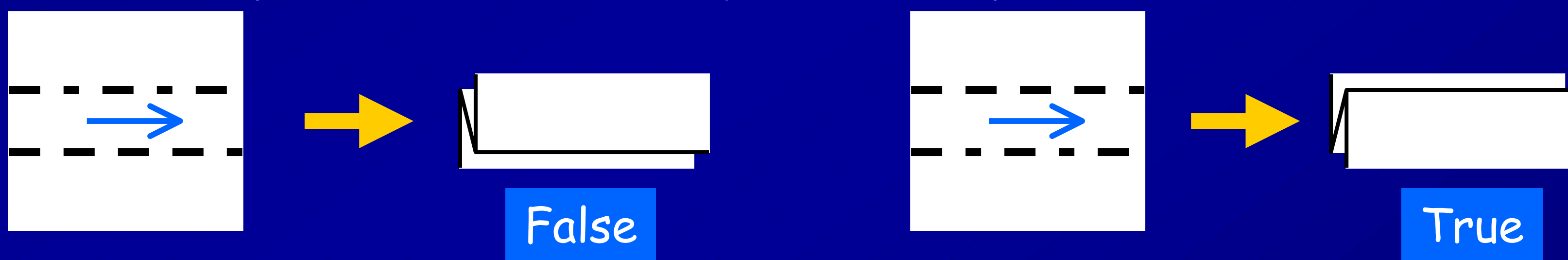
- **Locally flat-foldable** crease pattern
  = each vertex is flat-foldable if cut out
  = flat-foldable except possibly for nonlocal self-intersection
- Testable in linear time [Bern & Hayes 1996]
  - Check Kawasaki's Theorem
  - Solve a kind of matching problem to find a valid mountain-valley assignment, if one exists
  - Barrier:

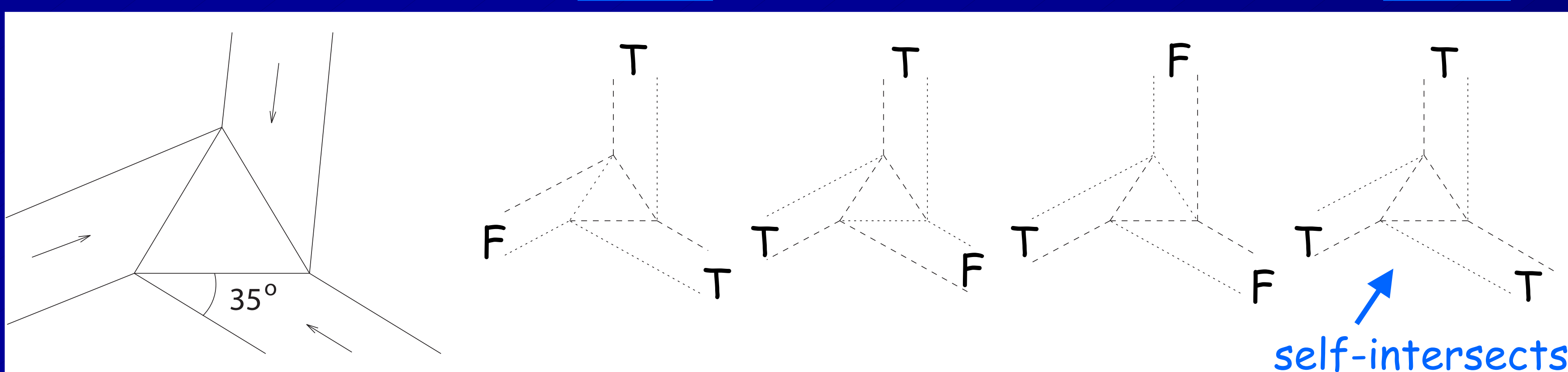




# Global Flat Foldability

- Testing (global) flat foldability is strongly NP-hard [Bern & Hayes 1996]
- Wire represented by "crimp" direction:



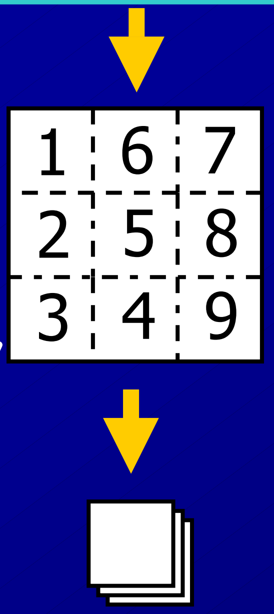

Not-all-equal 3-SAT clause

# Outline

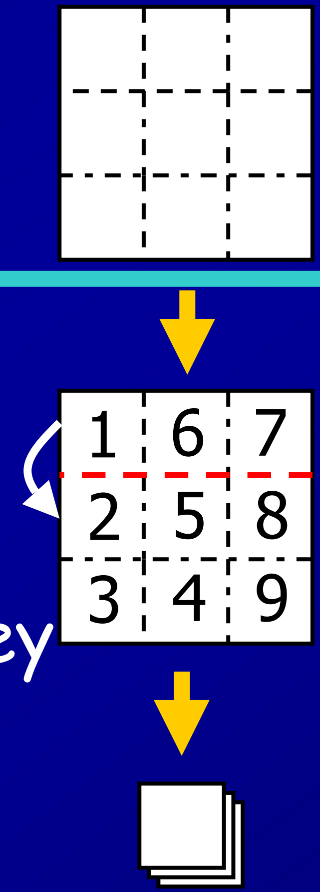
- History and Definitions
- Foldability
  - Crease patterns
  - **Map folding**
- Origami design
  - Silhouettes and gift wrapping
  - Tree method
  - One complete straight cut
  - Flattening polyhedra

# Map Folding

- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line



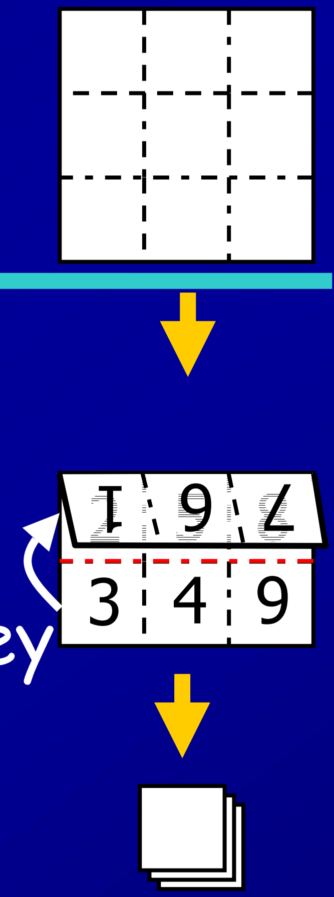
# Map Folding



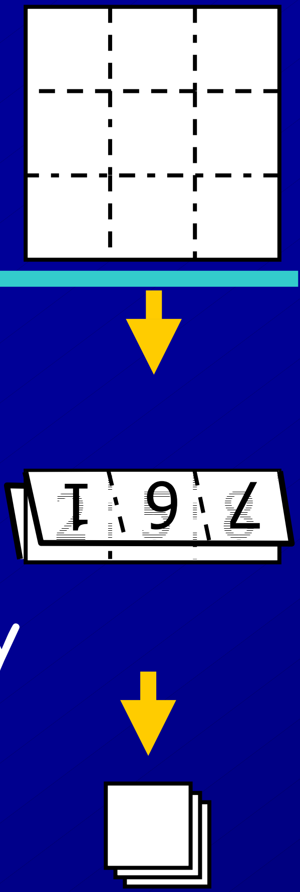

- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line

# Map Folding

- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line

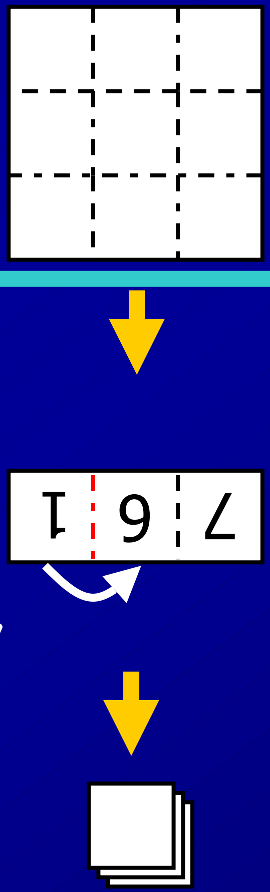


# Map Folding



- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line

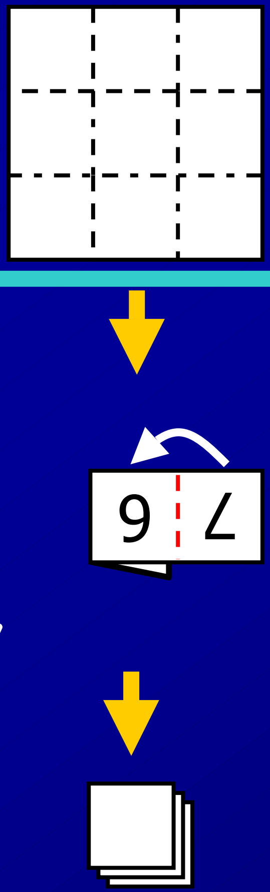
# Map Folding



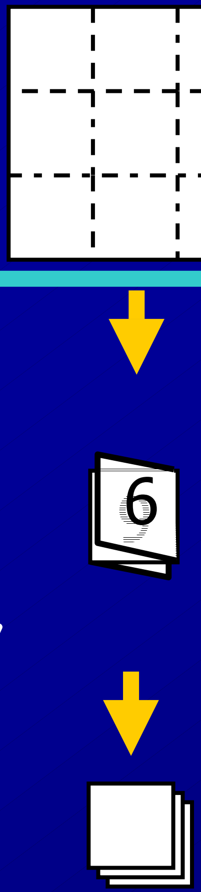
- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line

# Map Folding

- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line



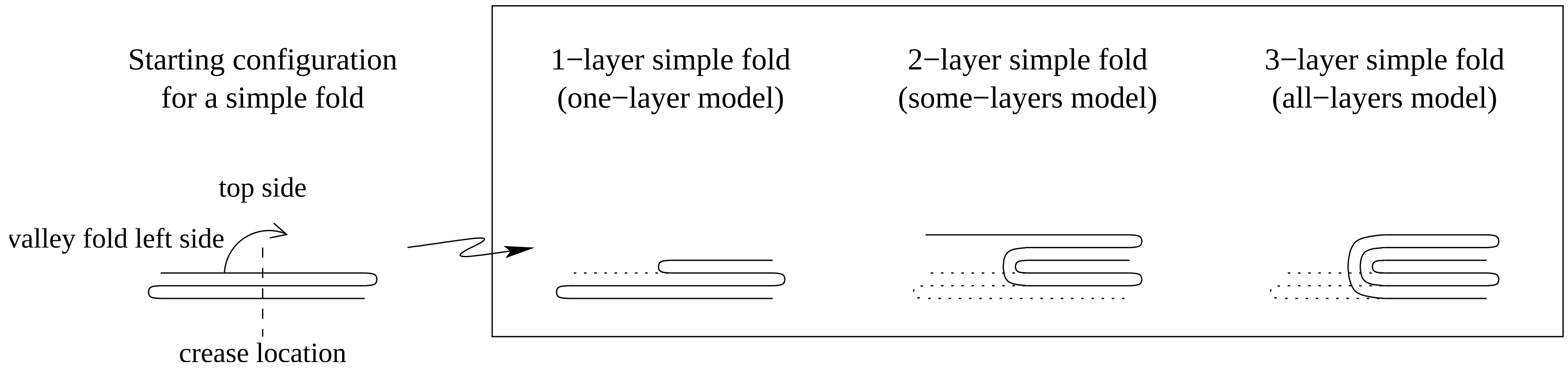
# Map Folding



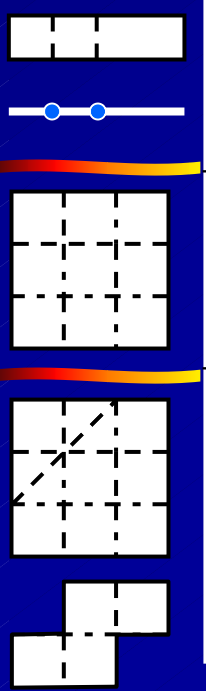

- Motivating problem:
  - Given a **map** (grid of unit squares), each crease marked mountain or valley
  - Can it be folded into a **packet** (whose silhouette is a unit square) via a sequence of simple folds?
  - Simple fold = fold along a line
- More generally: Given an arbitrary crease pattern, is it flat-foldable by simple folds?

# Models of Simple Folds

- A single line can admit several different simple folds, depending on # layers folded
  - Extremes: one-layer or all-layers simple fold
  - In general: some-layers simple fold
- Example in 1D:



# Simple Foldability [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2001]



| | All layers | Some layers | One layer | Folded state |
|---|---|---|---|---|
| **1-D** | O(n) exp.<br>O(n lg m) | O(n) | = O(n) | = O(n) |
| **2-D Map** | O(n) exp.<br>O(n lg m) | O(n) | = O(n) | Open [Edmonds] |
| **2-D** | Weakly NP-hard | Weakly NP-hard | Weakly NP-hard | Strongly NP-hard [B&H '96] |

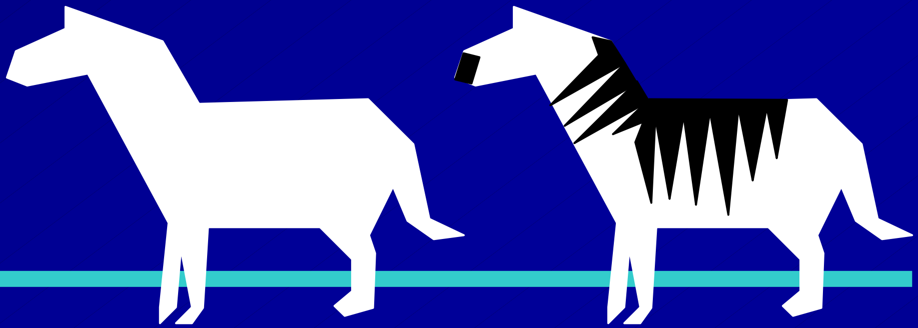
# Open Problems

- **Open:** Pseudopolynomial-time algorithms?
- **Open:** Orthogonal creases on non-axis-aligned rectangular piece of paper?
- **Open** (Edmonds): Complexity of deciding whether an $m \times n$ grid can be folded flat (has a flat folded state) with specified mountain-valley assignment
  - Would strengthen Bern & Hayes result
- **Open:** What about orthogonal polygons with orthogonal creases, etc.?

# Outline

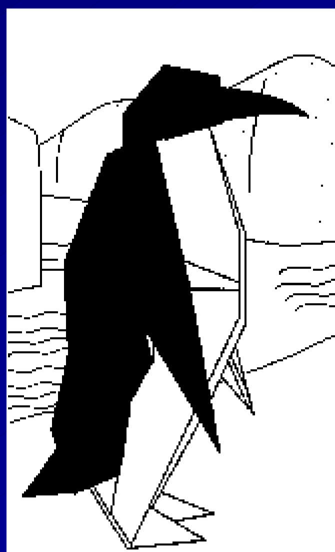
- History and Definitions
- Foldability
  - Crease patterns
  - Map folding
- **Origami design**
  - **Silhouettes and gift wrapping**
  - Tree method
  - One complete straight cut
  - Flattening polyhedra

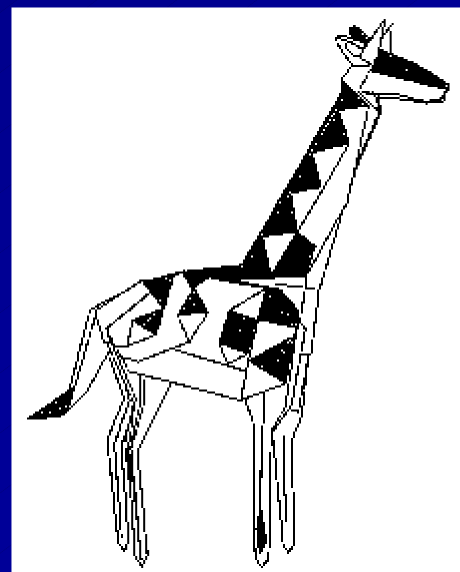
# The Problems

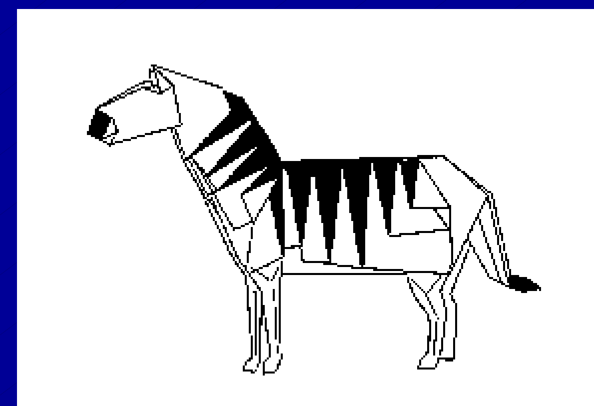


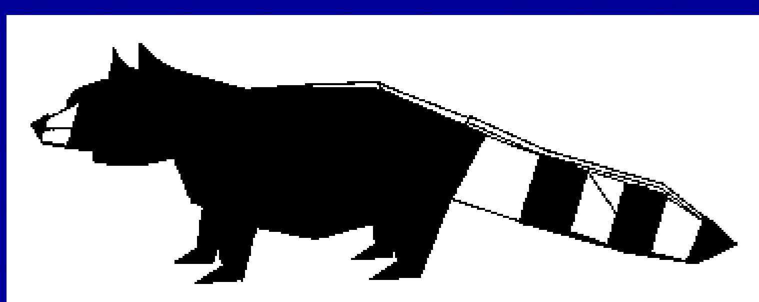
- **Silhouette question** (Bern & Hayes 1996): Is every polygon the silhouette of a flat origami?
- **2-color origami problem:** Construct a given 2-color pattern with bicolor paper
  - **2-color pattern** = polygonal region partitioned into subregions, each assigned one of 2 colors
  - **Bicolor paper** has different color on each side

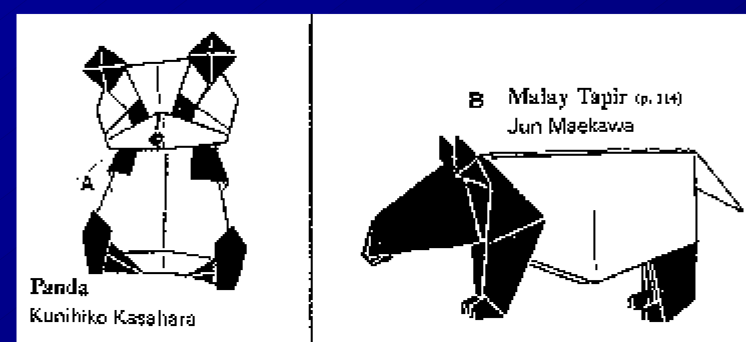
# Flat Foldings of Single Sheets of Paper



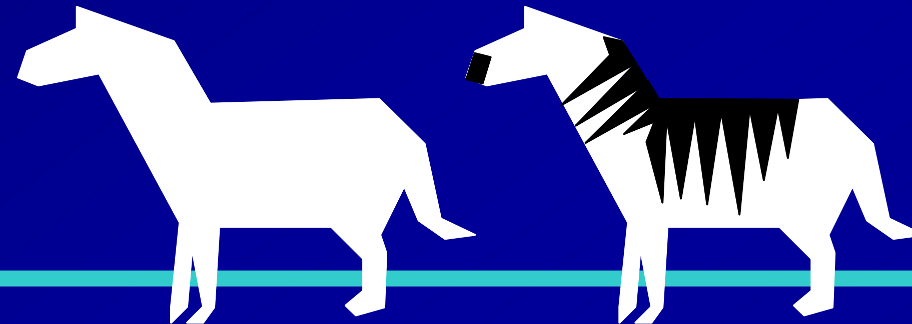




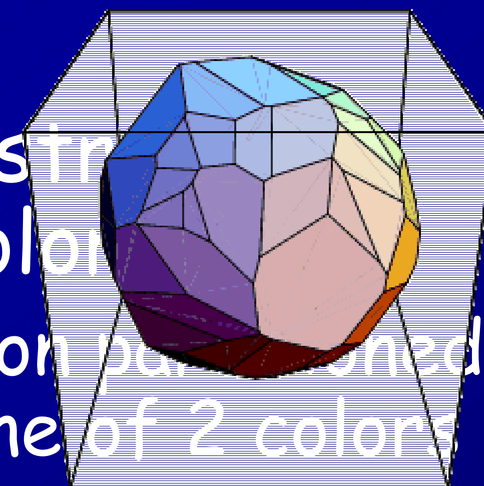




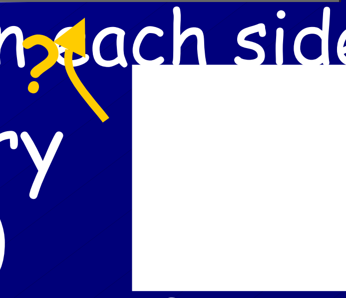
# The Problems



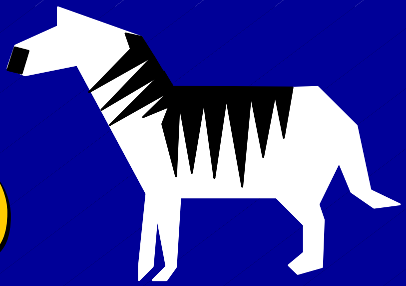
- **Silhouette question** (Bern & Hayes 1996): Is every polygon the silhouette of a flat origami?
- **2-color origami problem:** Cons given 2-color pattern with bicolo
  - **2-color pattern** = polygonal region partitioned into subregions, each assigned one of 2 colors
  - **Bicolor paper** has different color on each side



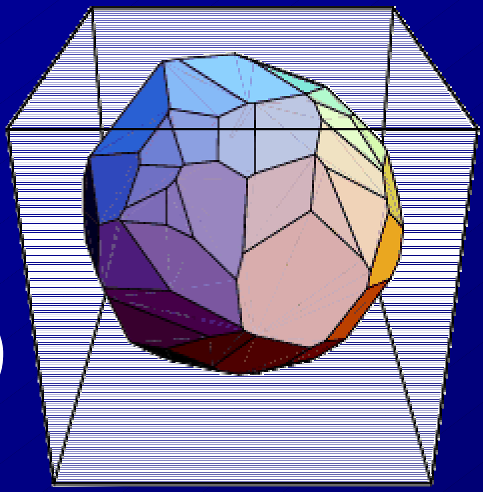
- **Gift wrapping question:** Can every polyhedron be "wrapped" (folded) by a sufficiently large piece of paper?



# General Theorem (Demaine, Demaine, Mitchell 1999)

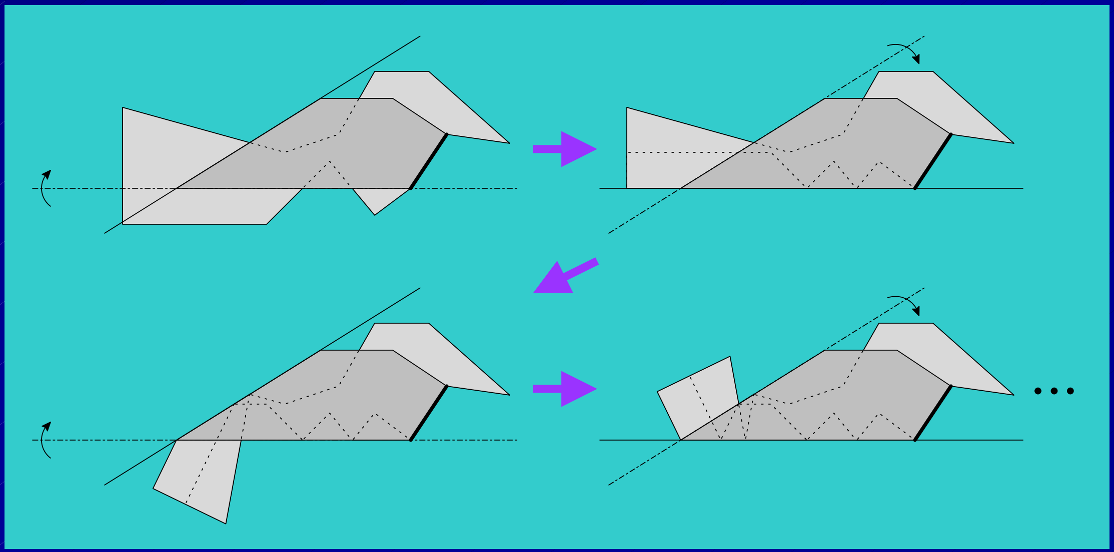


- Given a polyhedron, each face assigned one of two colors, there is a folding of a sufficiently large piece of bicolor paper into the colored surface
- Can optimize:
  - Paper usage (area of paper = ε + surface area)
  - "Strip width"
  - Visible "seams" (creases/paper edges)

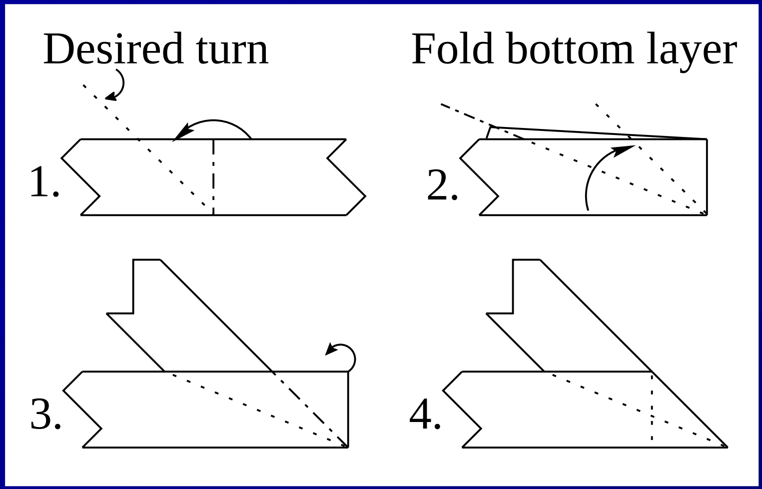


# Strips

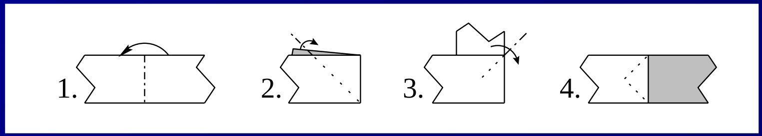
- Basic idea: Use a **strip** = a long rectangle
- Several gadgets for "navigating" strips:



Hiding excess paper under a convex polygon



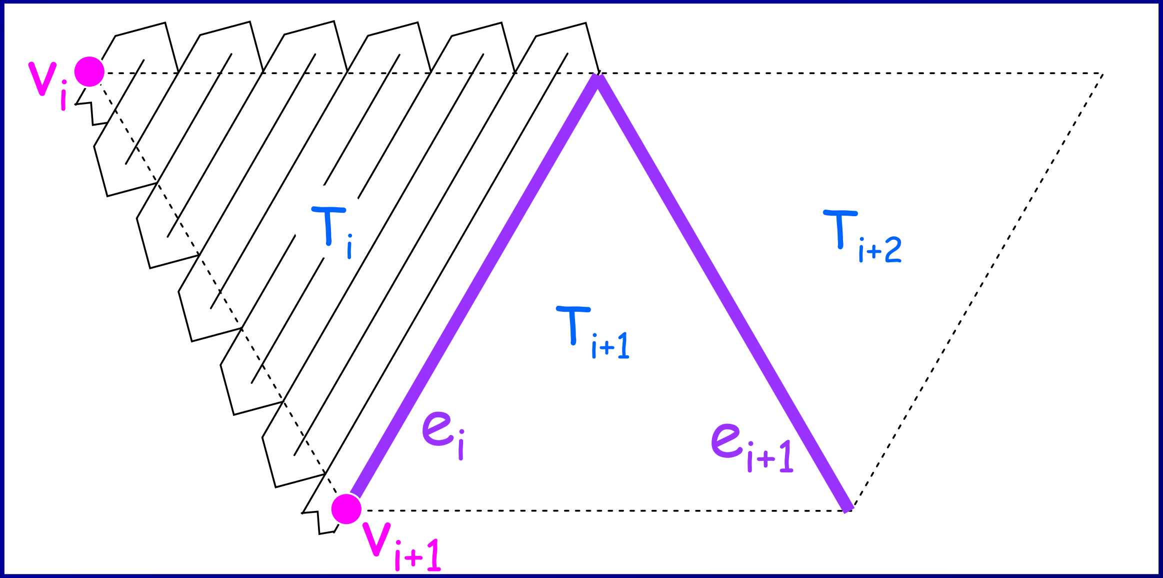

Turn gadget




Color-reversal gadget

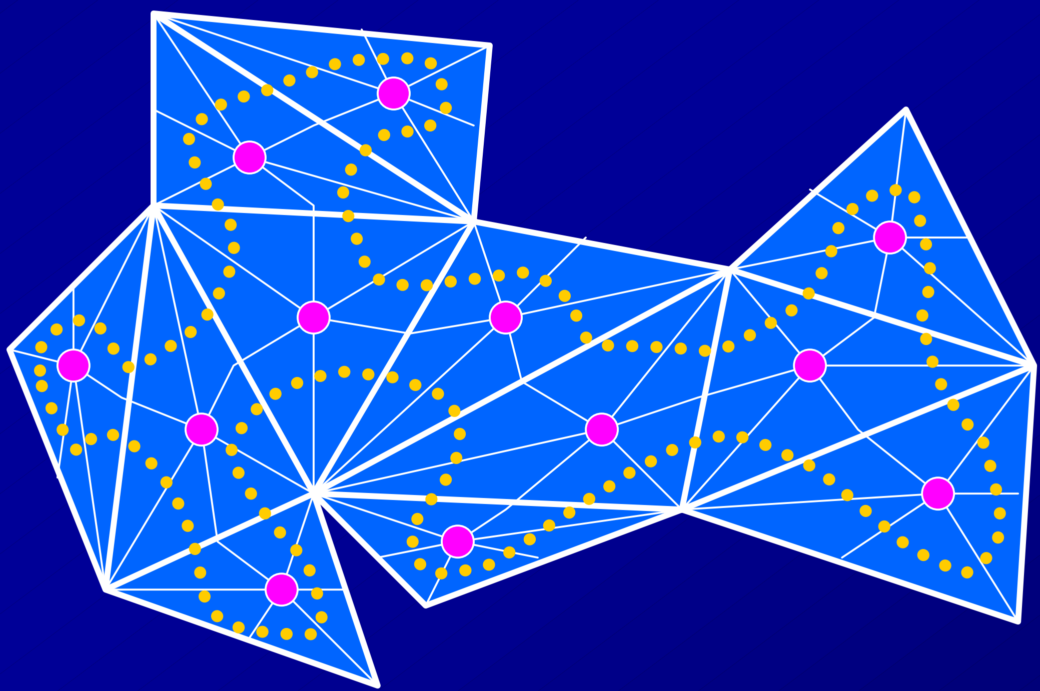
# Navigating a Triangulation

- Zig-zag to cover each triangle $T_i$
- Parallel to edge $e_i$ adjacent to next triangle $T_{i+1}$
- Choose initial direction to end at vertex $v_{i+1}$ opposite next edge $e_{i+1}$



# Minimizing Paper Usage

- Triangulate polyhedron so that dual graph has Hamiltonian cycle



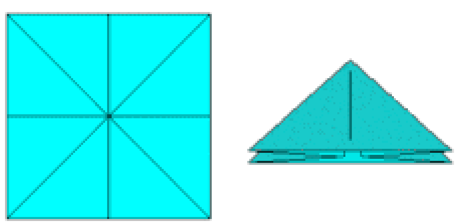
- Paper wastage $\rightarrow$ 0 with strip width

# What If We Start from a Square?

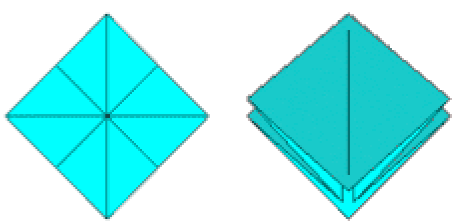
- Strip folding extremely inefficient; used paper $\to 0$ with strip width
- **Open:** What is the largest $k \times k$ checkerboard foldable from a unit square?
  - **Conjecture:** $\sim 2/k \times 2/k$
- **Open:** What is the largest regular tetrahedron/octahedron/dodecahedron/icosahedron foldable from a unit square?
  - Only the cube has been solved

# Origami Bases

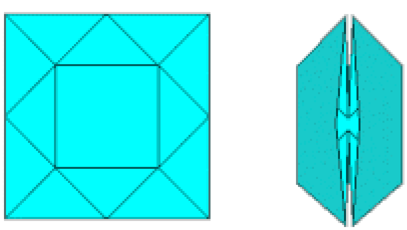
- Concentrate on one type of polyhedron: **origami base**
- 6 standard origami bases, with limited numbers of flaps for shaping into limbs, ...



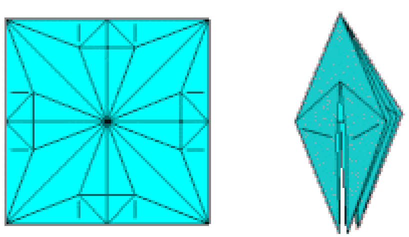
Waterbomb base



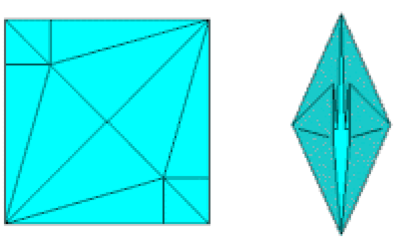
Preliminary base



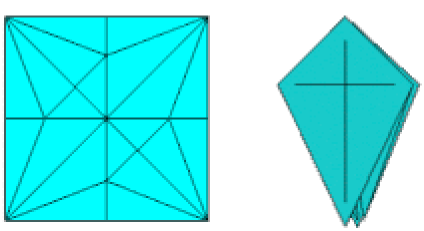
Windmill base



Frog base



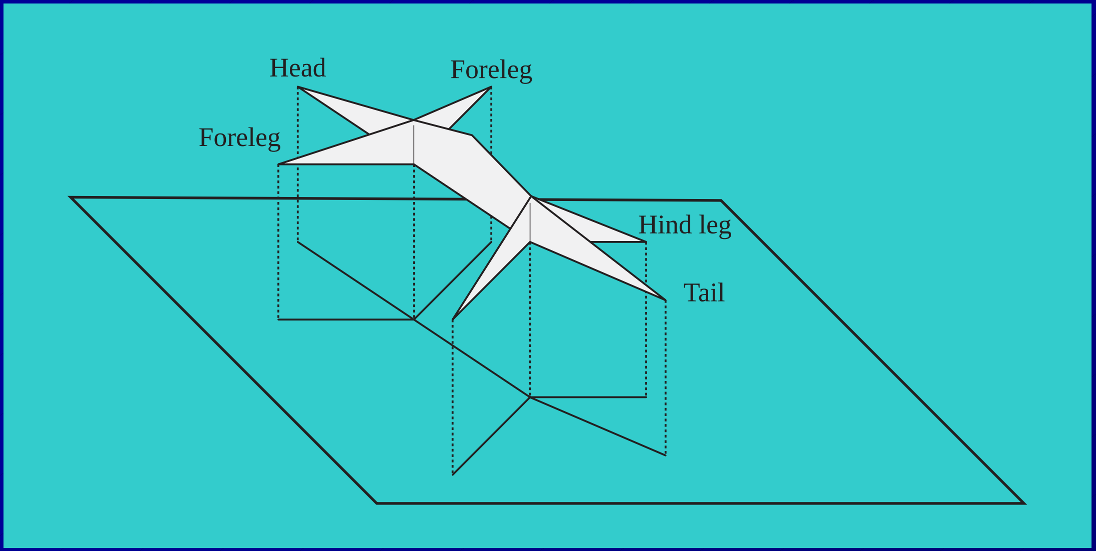
Fish base



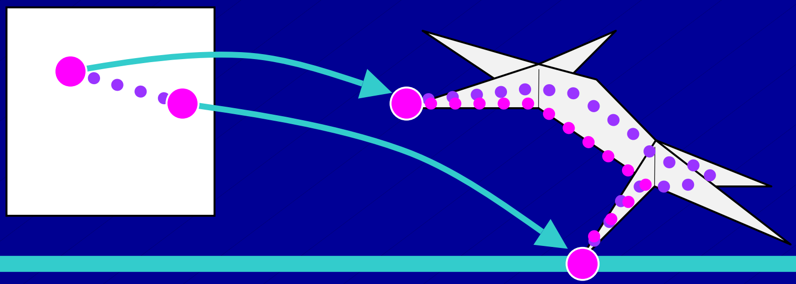
Bird base

# Tree Method [Lang]

- What if we want more limbs?
- **Uniaxial origami base:** Projection = intersection with xy plane = tree
- Can represent any "stick figure" with such a **shadow tree**

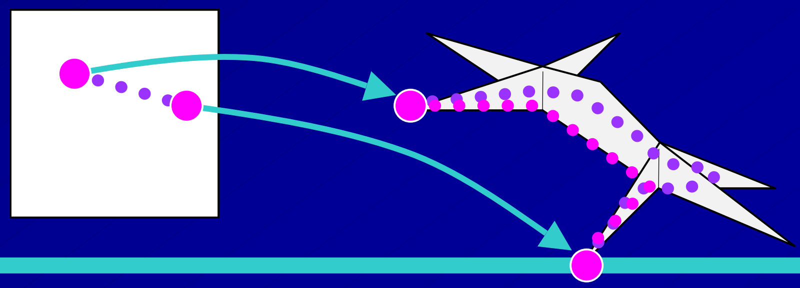


# Tree Lemma



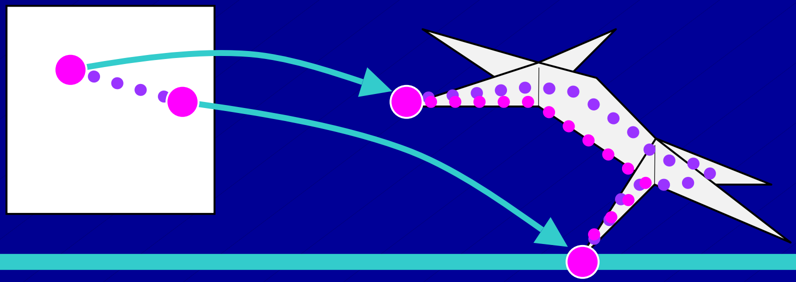
- Consider two points of paper that fold to two points on the shadow tree
- Draw line segment on unfolded piece of paper (assuming convex polygon)
- Line segment folds to a continuous path
- Path at least as long as direct path in tree
- Distance between two points on the shadow tree is a lower bound on the distance between corresponding points on the unfolded piece of paper

# Tree Conditions



- Consider an assignment of points on paper to leaves of shadow tree
- Tree lemma says when paper is too small:
  - unfolded-distance (p, q) = tree-distance (p, q)
- **Conjecture:** Tree conditions are sufficient
- **Theorem:** If tree conditions are satisfied, "slight modifications" make it feasible
- **Goal:** Find paper size and point assignment satisfying tree conditions

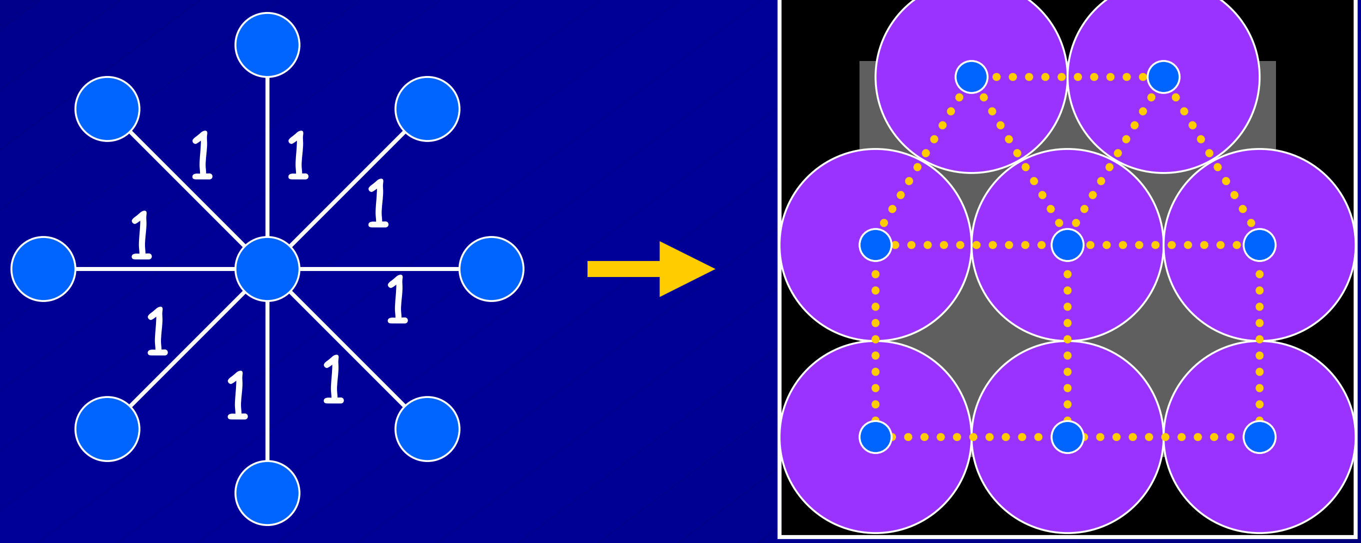
# Scale Optimization



- Allow tree to scale by factor ? > 0
- Tree condition becomes
  - unfolded-dist. (p, q) = ? × tree-dist. (p, q)
- Now almost all point assignments are valid:
  - ? = min {unfolded-dist. (p, q) / tree-dist. (p, q)}
- **Goal:** Maximize ? among point assignments for leaves of shadow tree
  - Difficult nonlinear optimization
  - Approximate/heuristic solutions OK

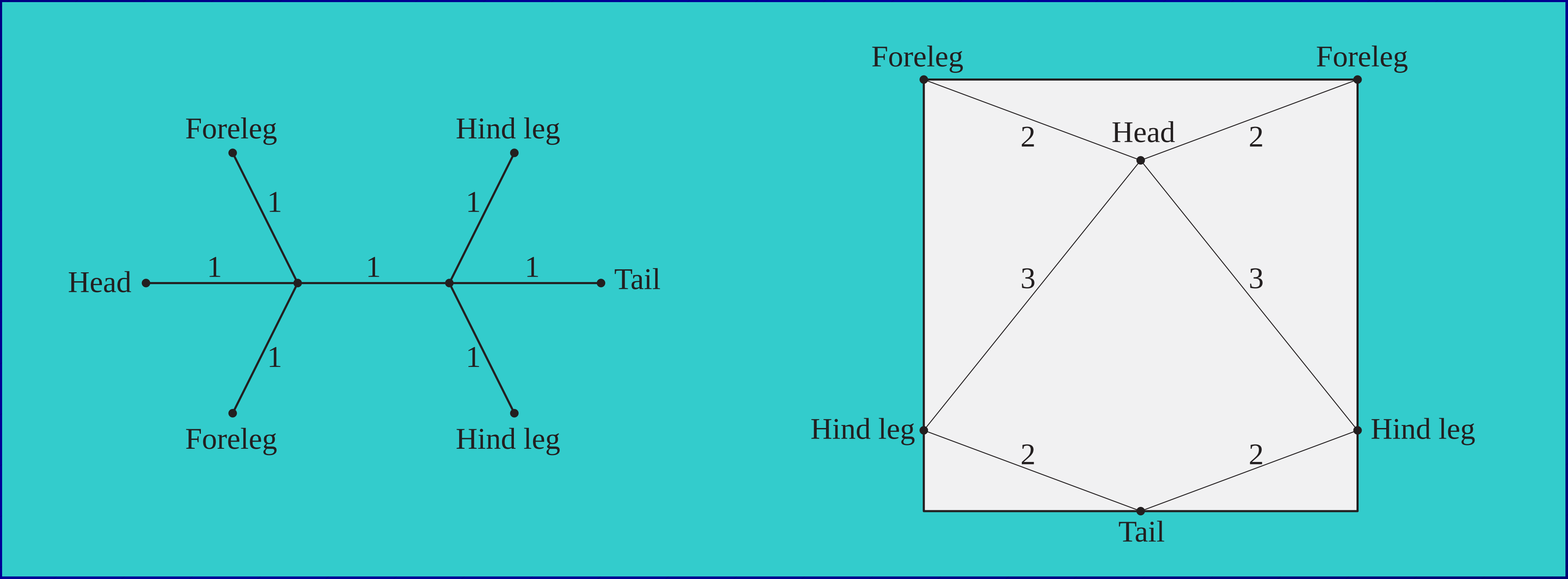
# Scale Optimization is as Hard as Disk Packing

- Consider unit star tree:



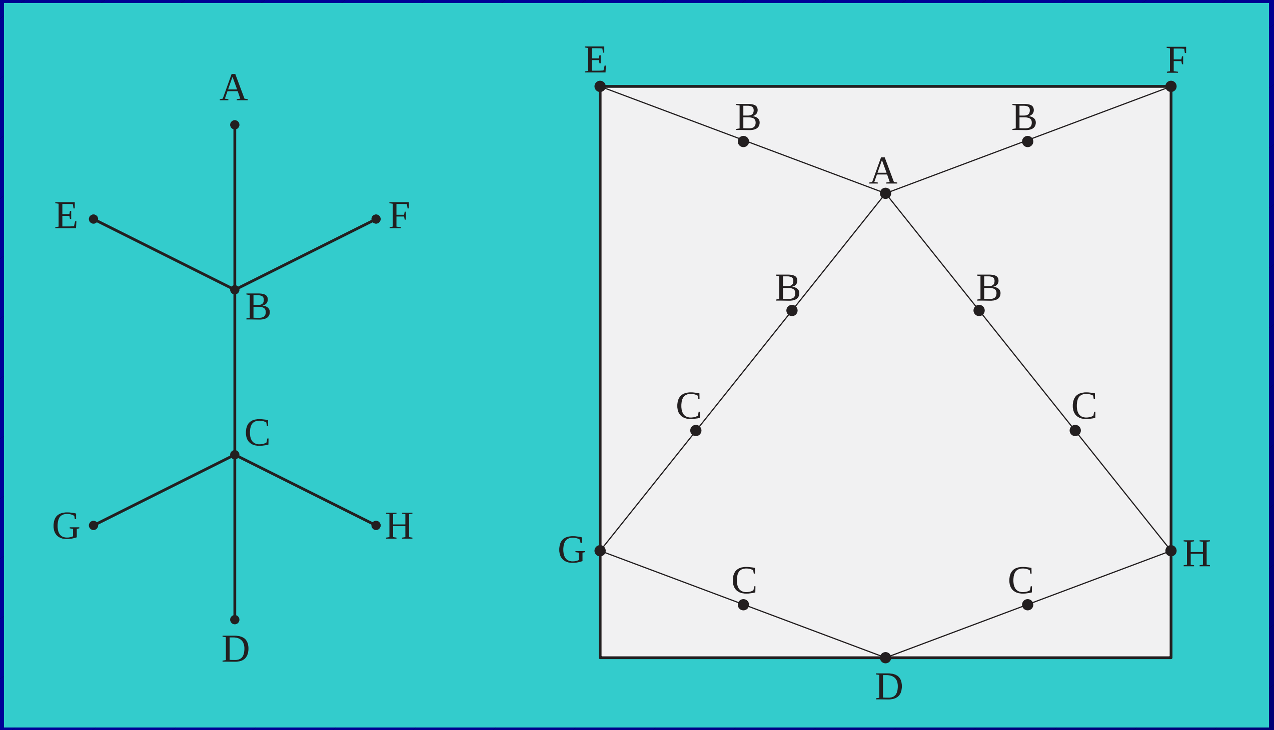

- Tree constraints:
  - unfolded-dist $(leaf_i, leaf_j) = 2$ ?
- $\Rightarrow$ Equal-radius disk packing in square

# Scale Optimization for Lizard



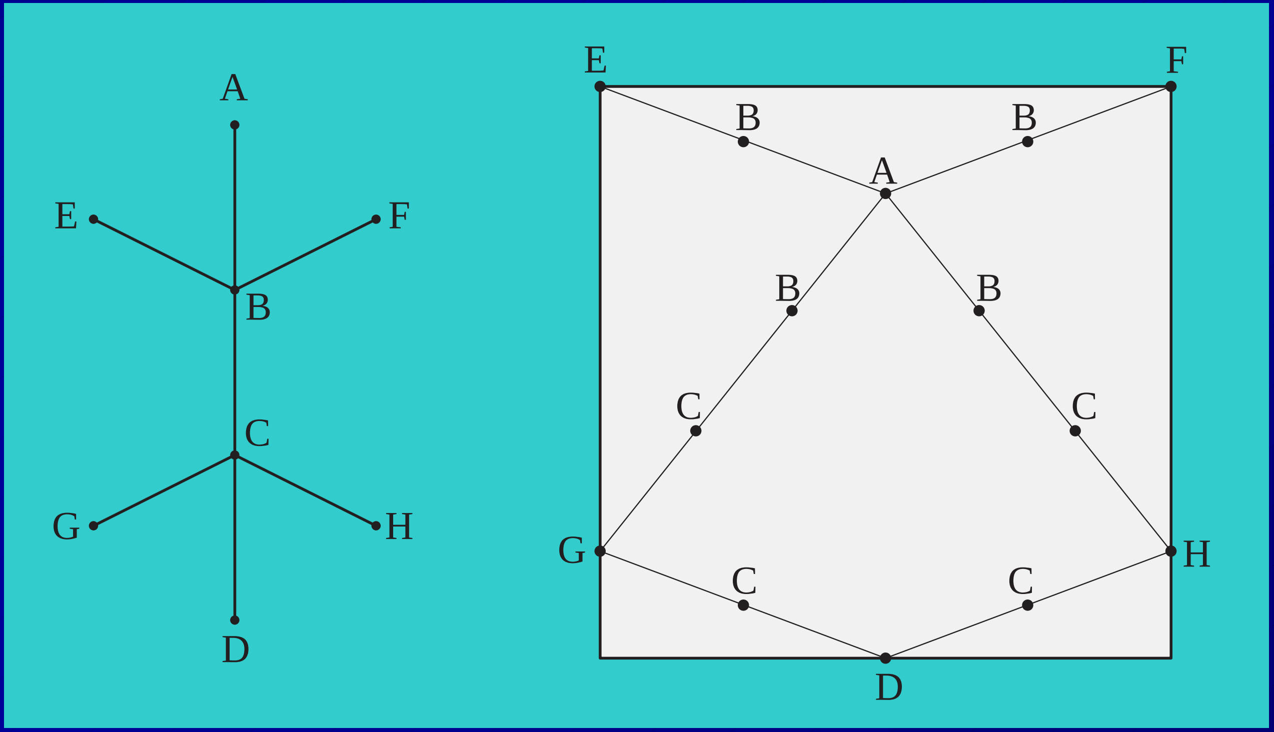
# Finding Other Vertices of the Shadow Tree

- When tree constraint is tight (unfolded distance = shadow distance), must correspond to path in shadow tree



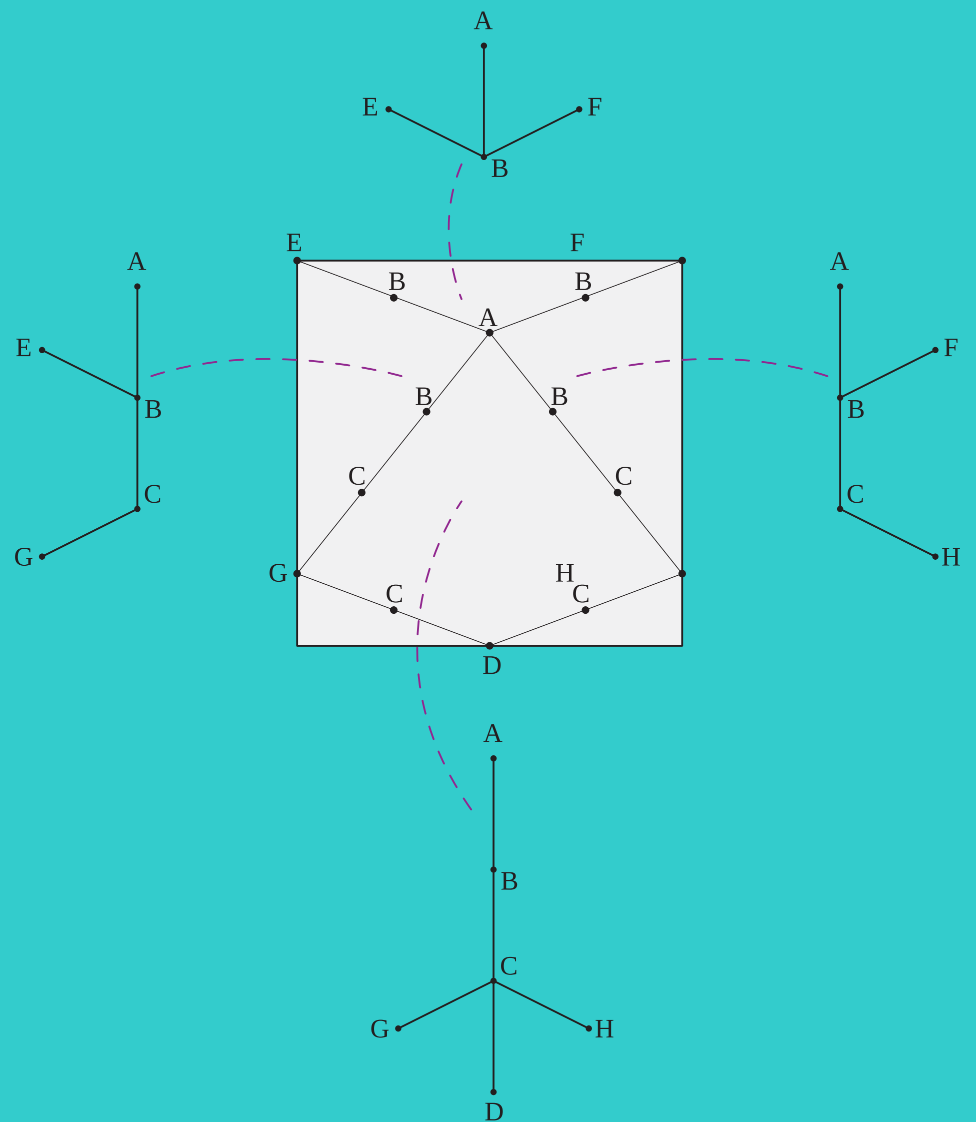
# Convex Decomposition

- These **active paths** "often" decompose the paper into convex regions
  - If not, can modify the tree "somewhat" to fix



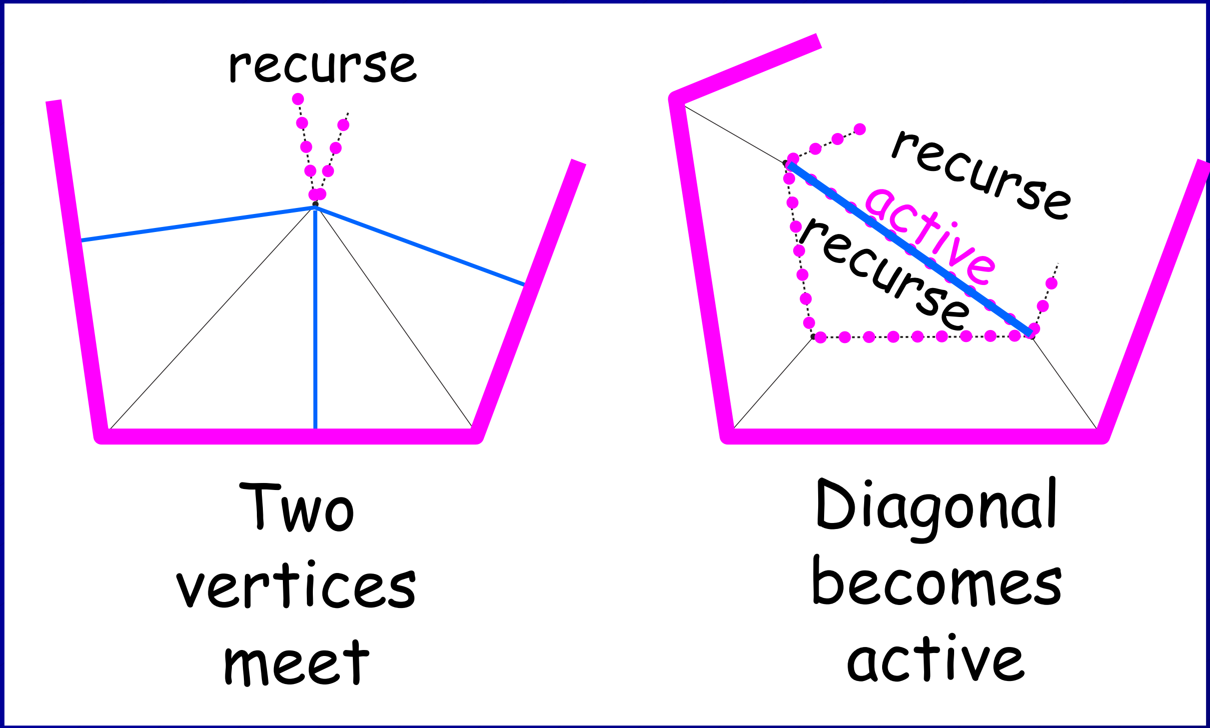
# Convex Subproblems

- Solve each convex region separately
- **Key property:** Because shared boundaries are active paths, creases at these interfaces will always match up

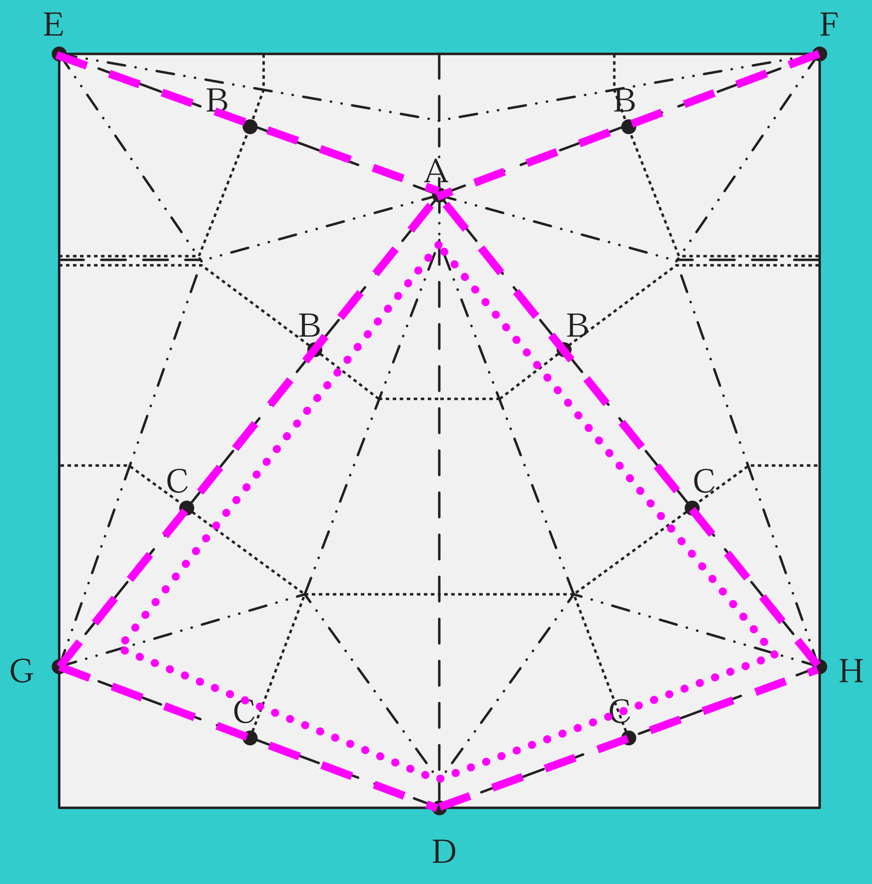


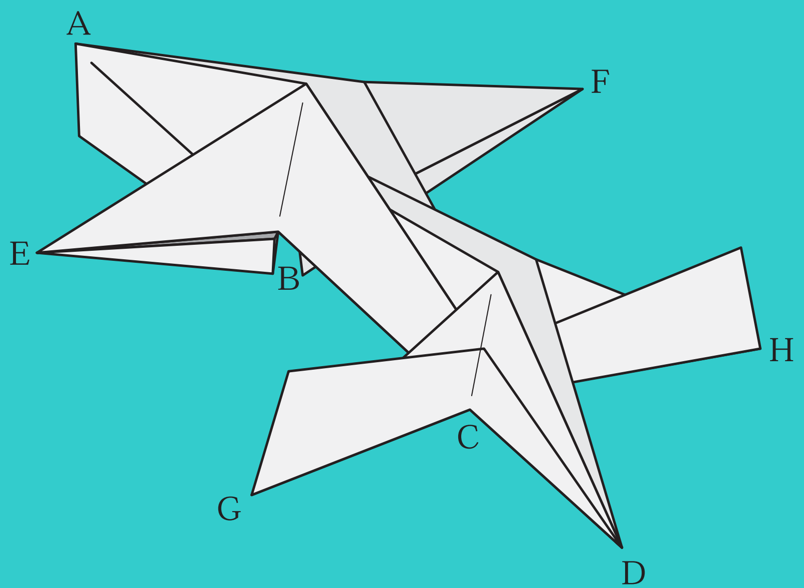
# Universal Molecule

- Shrink convex polygon, tracing vertices
- Two types of events arise:



# Crease Pattern for Lizard



# Outline

- History and Definitions
- Foldability
  - Crease patterns
  - Map folding
- Origami design
  - Silhouettes and gift wrapping
  - Tree method
  - **One complete straight cut**
  - Flattening polyhedra

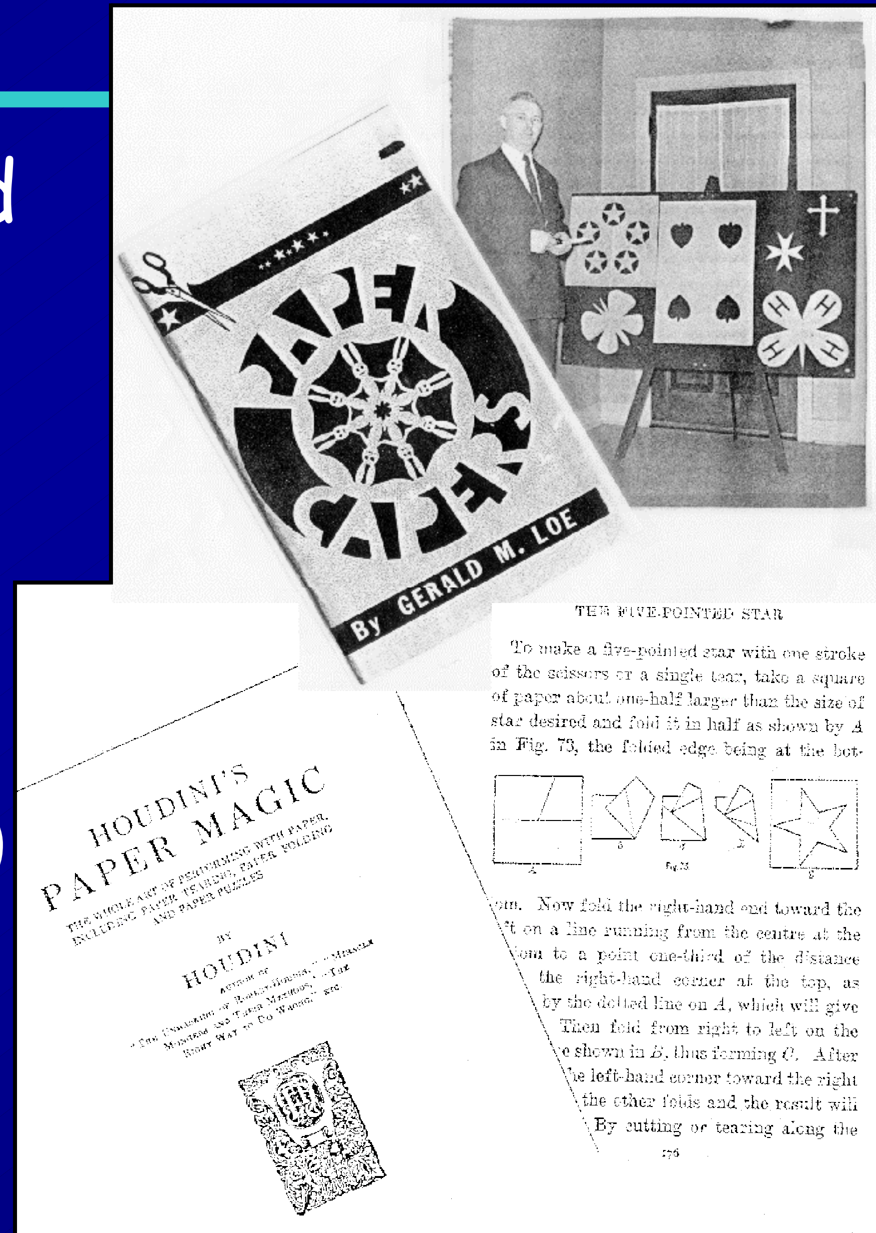
# Fold-and-Cut Problem

- Fold a sheet of paper flat
- Make one complete straight cut
- Unfold the pieces

- What shapes can result?

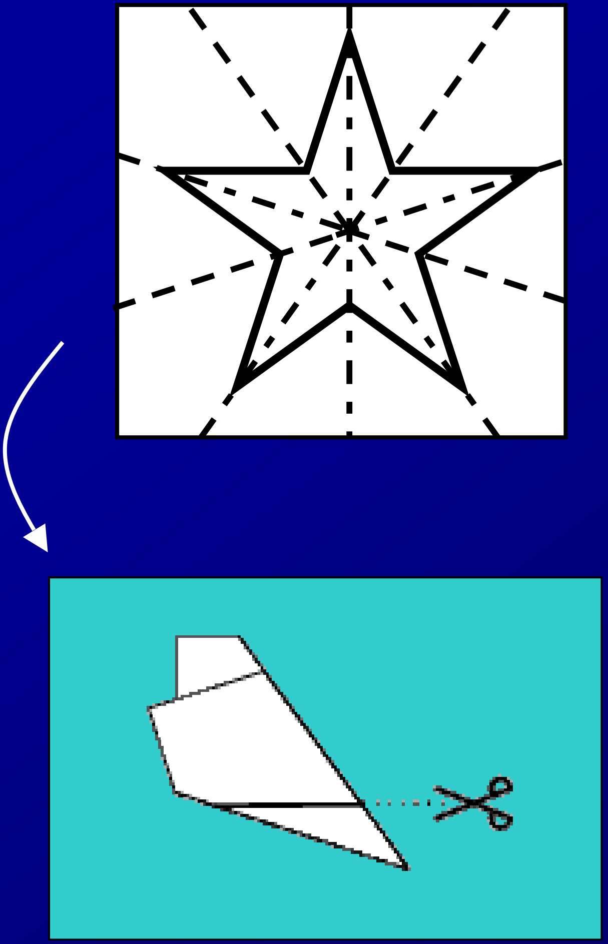
# History of Fold-and-Cut

- Recreationally studied by
  - Kan Chu Sen (1721)
  - Betsy Ross (1777)
  - Houdini (1922)
  - Gerald Loe (1955)
  - Martin Gardner (1960)



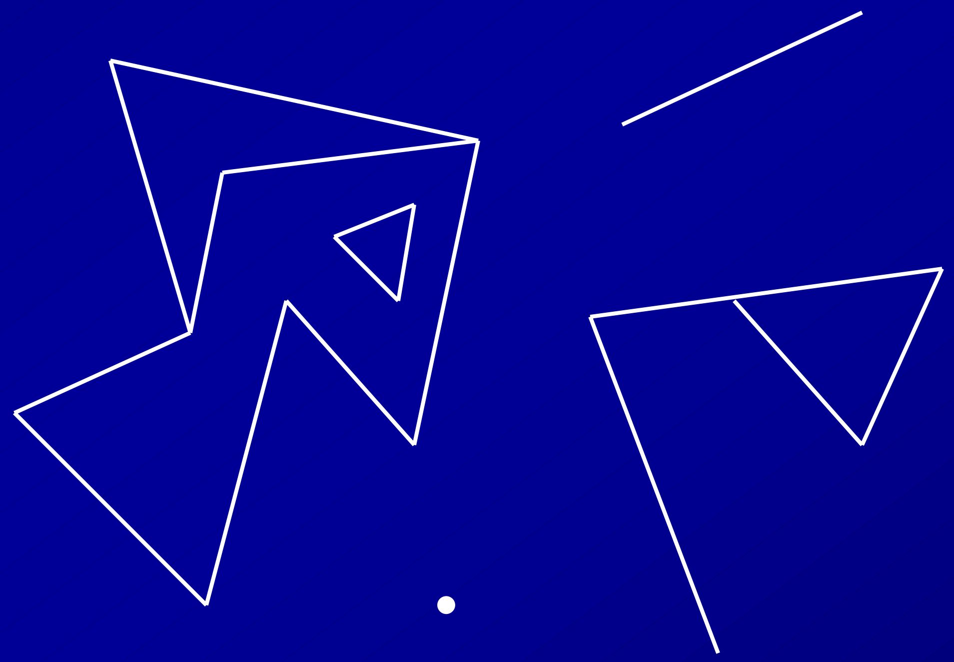
# General Problem

- Given any plane graph (the **cut graph**)
- Can you fold the piece of paper flat so that one complete straight cut makes the graph?
- Equivalently, is there is a flat folding that lines up precisely the cut graph?



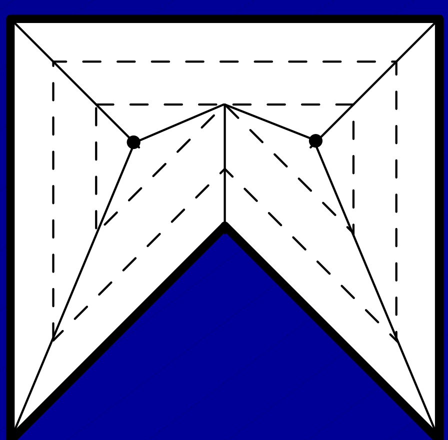
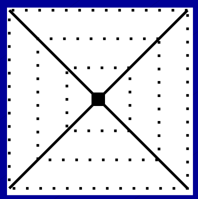
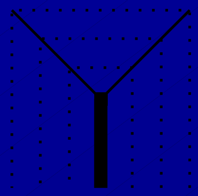
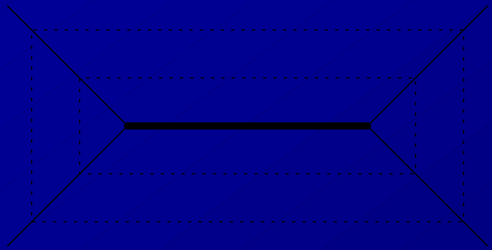
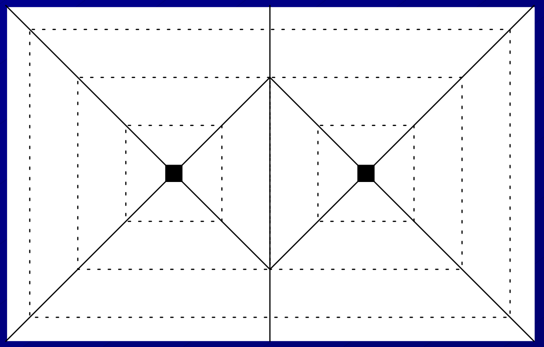
# Theorem [Demaine, Demaine, Lubiw 1998] [Bern, Demaine, Eppstein, Hayes 1999]

- Any plane graph can be lined up by folding flat



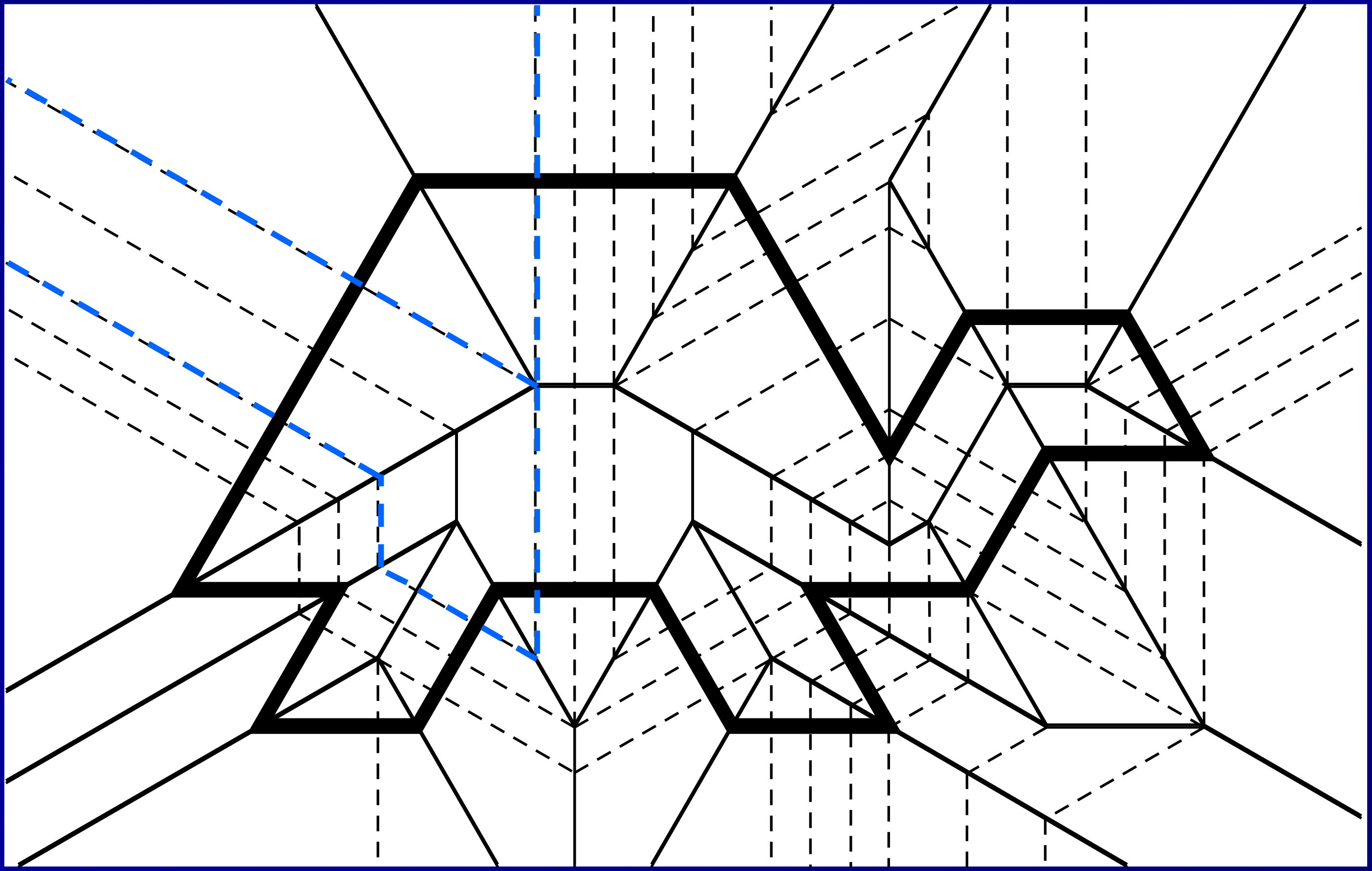
# Straight Skeleton

- Shrink as in Lang's universal molecule, but
  - Handle nonconvex polygons
    $\Rightarrow$ new event when vertex hits opposite edge
  - Handle nonpolygons
    $\Rightarrow$ "butt" vertices of degree 0 and 1
  - Don't worry about active paths

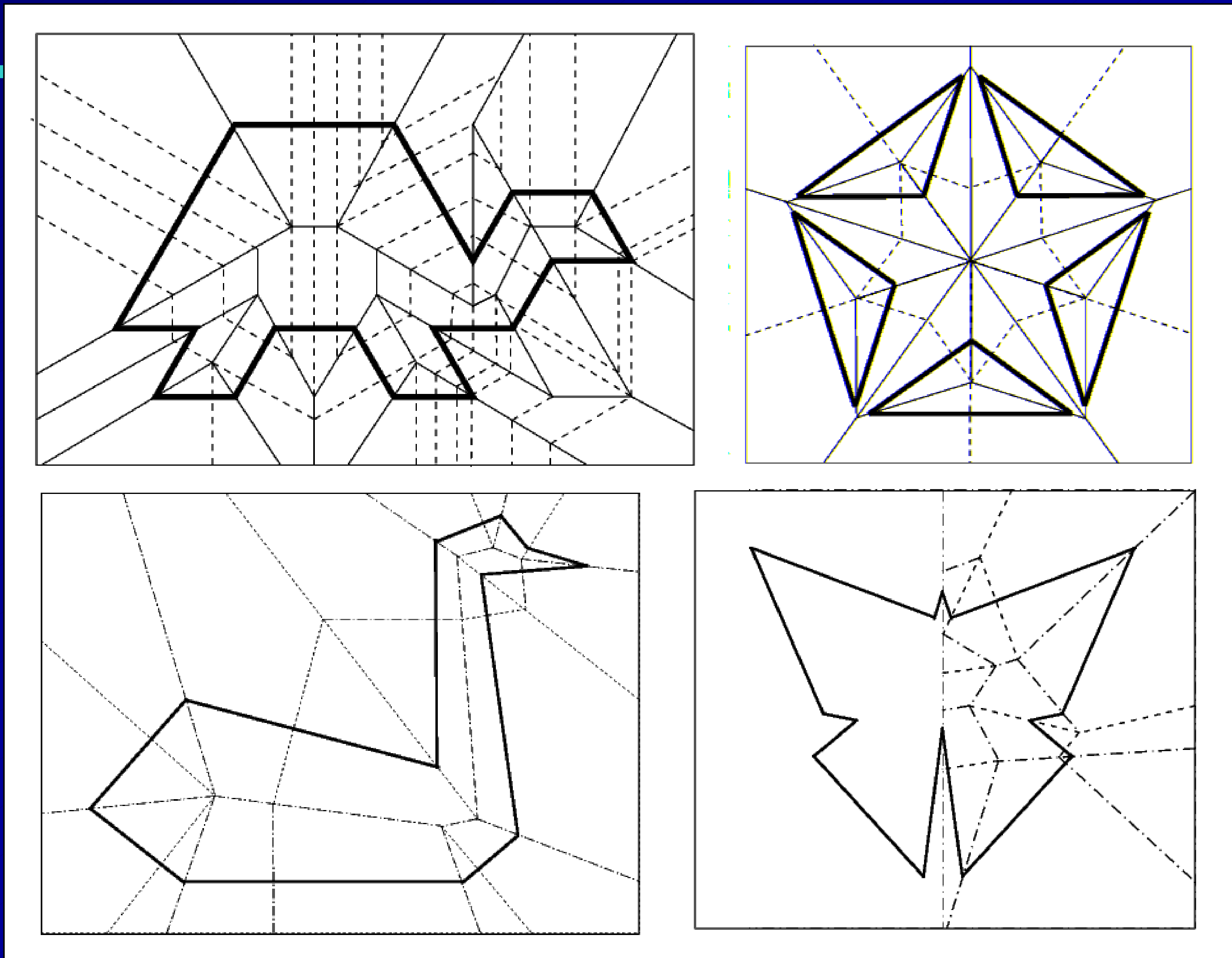






# Perpendiculars

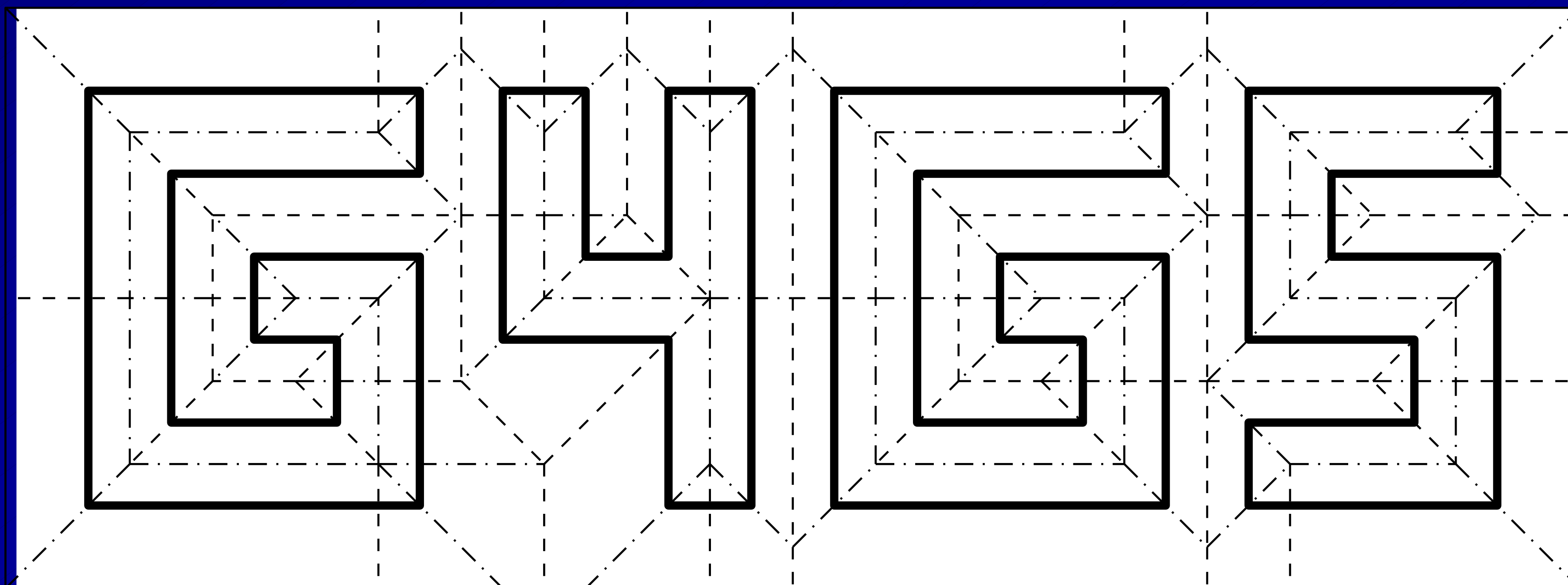
- Behavior is more complicated than tree method



# A Few Examples

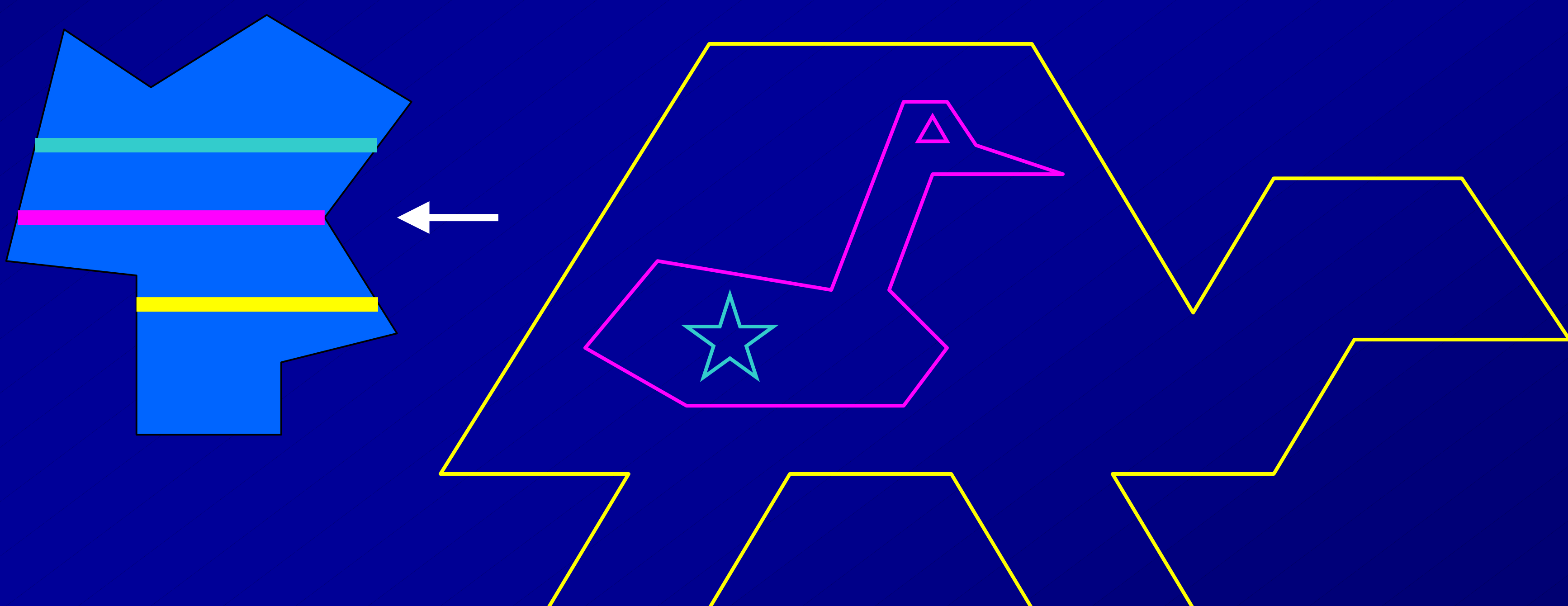


# A Final Example



# Generalization
[Demaine, Hayes, Lang 2001]

- Can fold a piece of paper flat and have a choice between several cut lines, each making a different shape

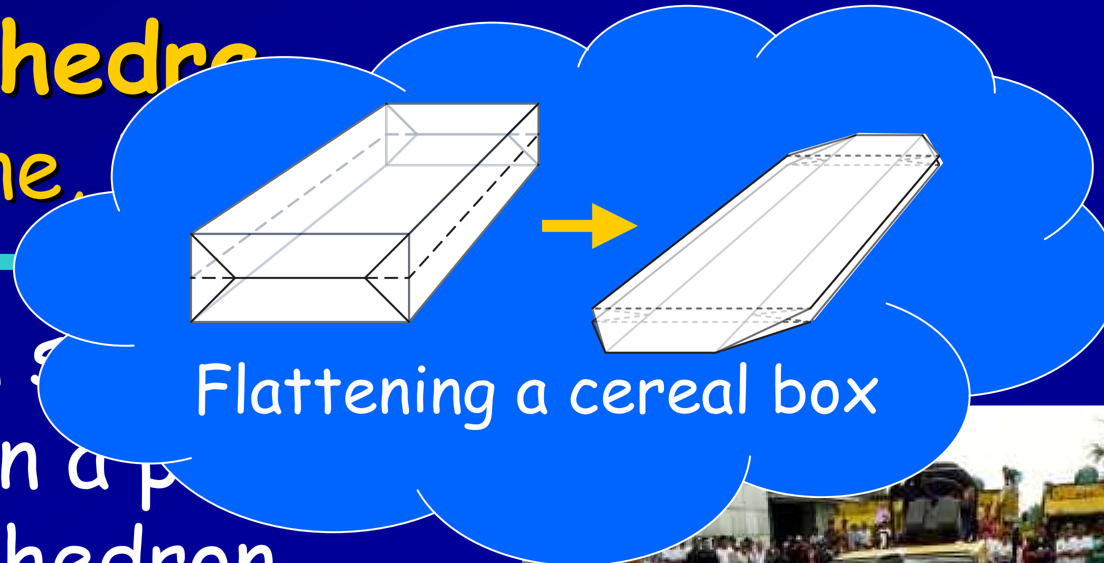


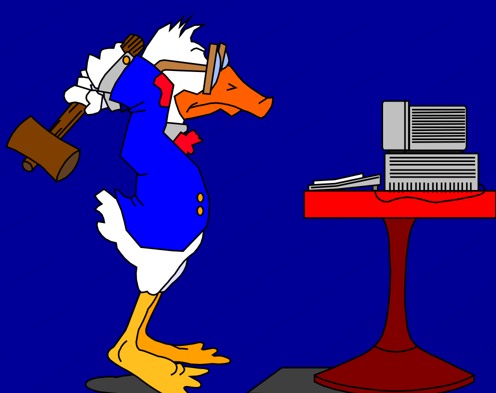
# Outline

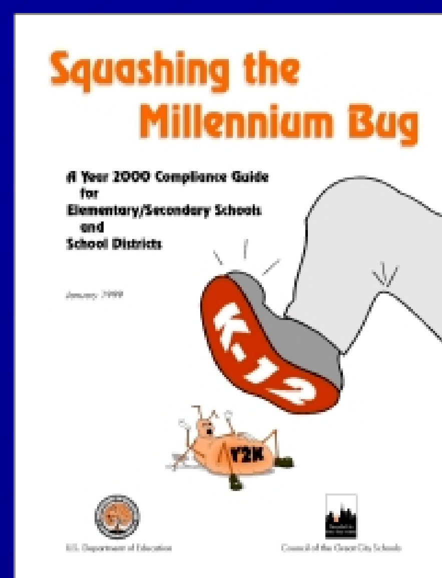
- History and Definitions
- Foldability
  - Crease patterns
  - Map folding
- Origami design
  - Silhouettes and gift wrapping
  - Tree method
  - One complete straight cut
  - **Flattening polyhedra**

# Flattening Polyhedr
## [Demaine, Demaine,

- Intuitively, can s collapse/flatten a p model of a polyhedron
- Problem: Is it possible without tearing?

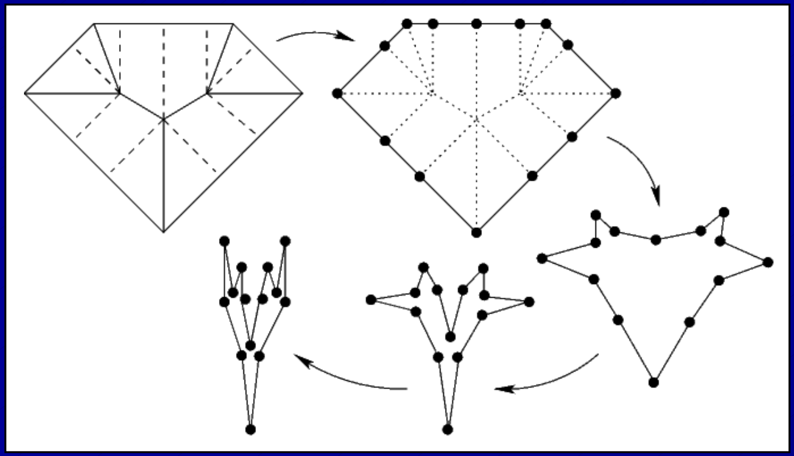


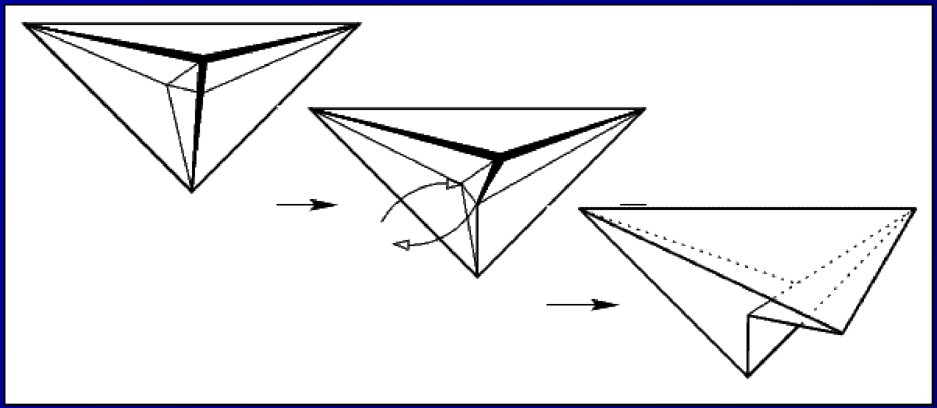






# Connection to Fold-and-Cut

- 2D fold-and-cut
  - Fold a 2D polygon
    - through 3D
    - flat, back into 2D
  - so that 1D boundary lies in a line
- 3D fold-and-cut
  - Fold a 3D polyhedron
    - through 4D
    - flat, back into 3D
  - so that 2D boundary lies in a plane





# Flattening Results

- All polyhedra homeomorphic to a sphere can be flattened (have flat folded states) [Demaine, Demaine, Hayes, Lubiw]
  - ~ Disk-packing solution to 2D fold-and-cut
- **Open:** Can polyhedra of higher genus be flattened?
- **Open:** Can polyhedra be flattened using 3D straight skeleton?
  - Best we know: thin slices of convex polyhedra