Tropical Algebraic Geometry

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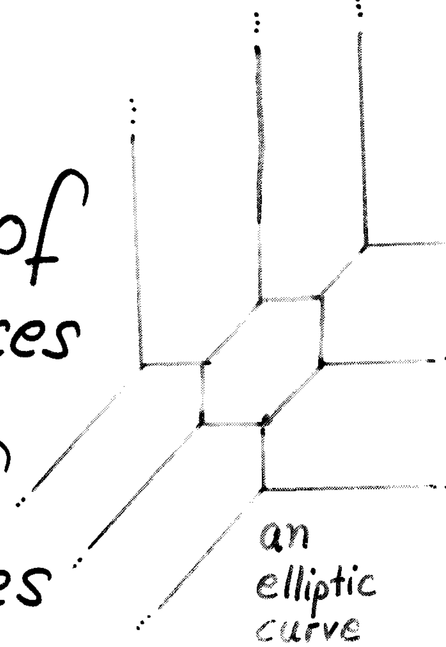
HP-MSRI '03-'04

Outline of the talk



1. Elementary construction of tropical hypersurfaces
2. The general definition of tropical varieties
3. Linear spaces and Grassmannians

References

math.AG/0306366: "First steps in tropical geometry" (w. J. Richter-Gebert, T. Theobald)

math.AG/0304218: "The tropical Grassmannian" (w. David Speyer)

The tropical semiring is $(\mathbb{R}, \oplus, \odot)$

where $x \oplus y = \min(x, y)$

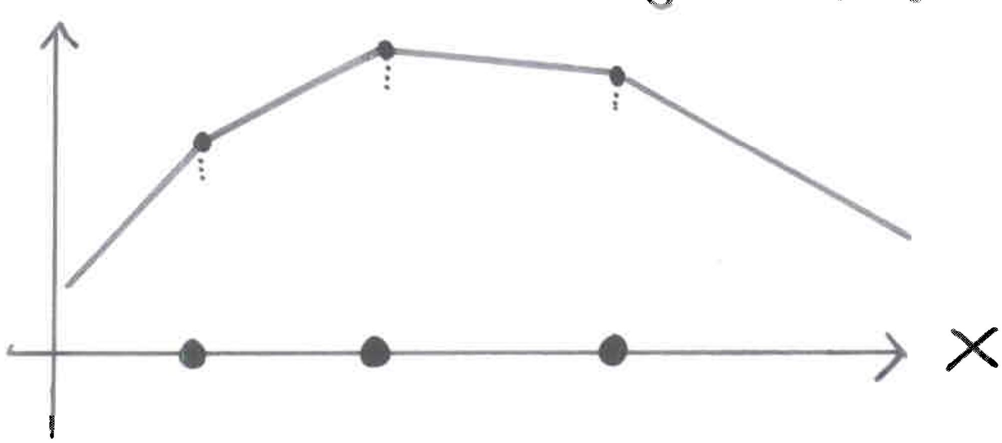
and $x \odot y = x + y$

A tropical polynomial

$$F = \bigoplus_{i=1}^{r} c_i \odot x_1^{\odot a_{i1}} \odot \cdots \odot x_n^{\odot a_{in}}$$

is a piecewise linear concave function

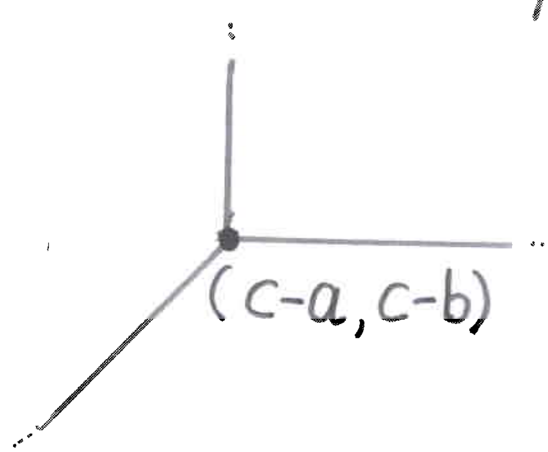
$$F(x_1, \dots, x_n) = \min \left\{ c_i + \sum_{j=1}^{n} a_{ij} x_j \; : \; i = 1, \dots, r \right\}$$



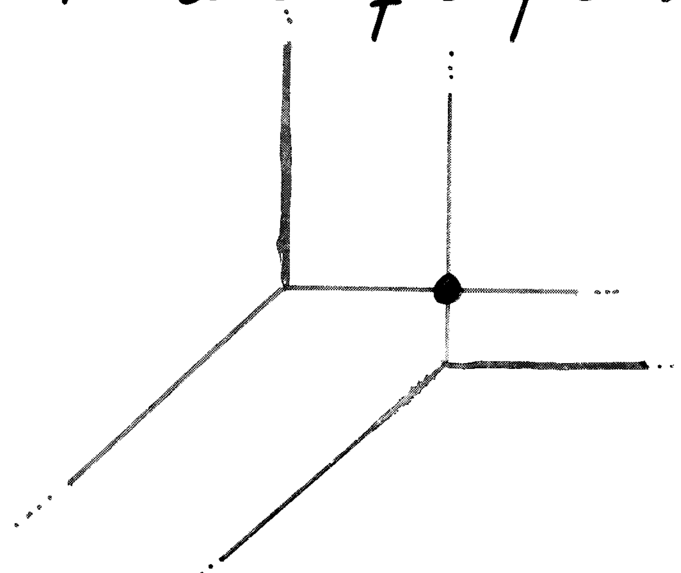
The tropical hypersurface $\mathcal{T}(F)$ is the set of all points $(x_1, \dots, x_n)$ at which $F : \mathbb{R}^n \to \mathbb{R}$ is <u>not linear.</u>

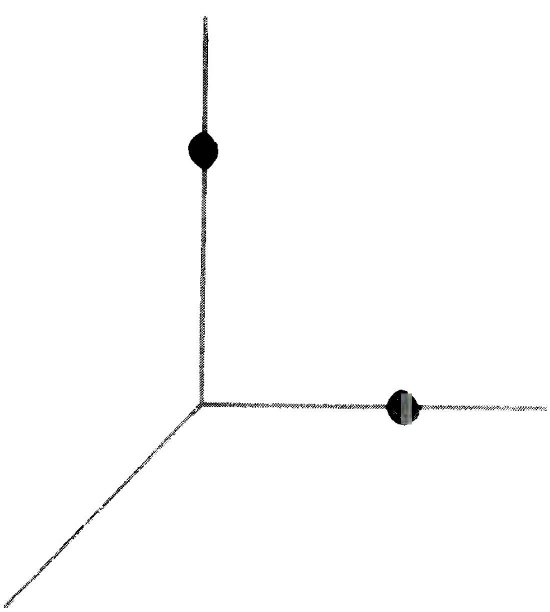
Lines in the plane

$$F(x,y) = a \odot x \oplus b \odot y \oplus c$$



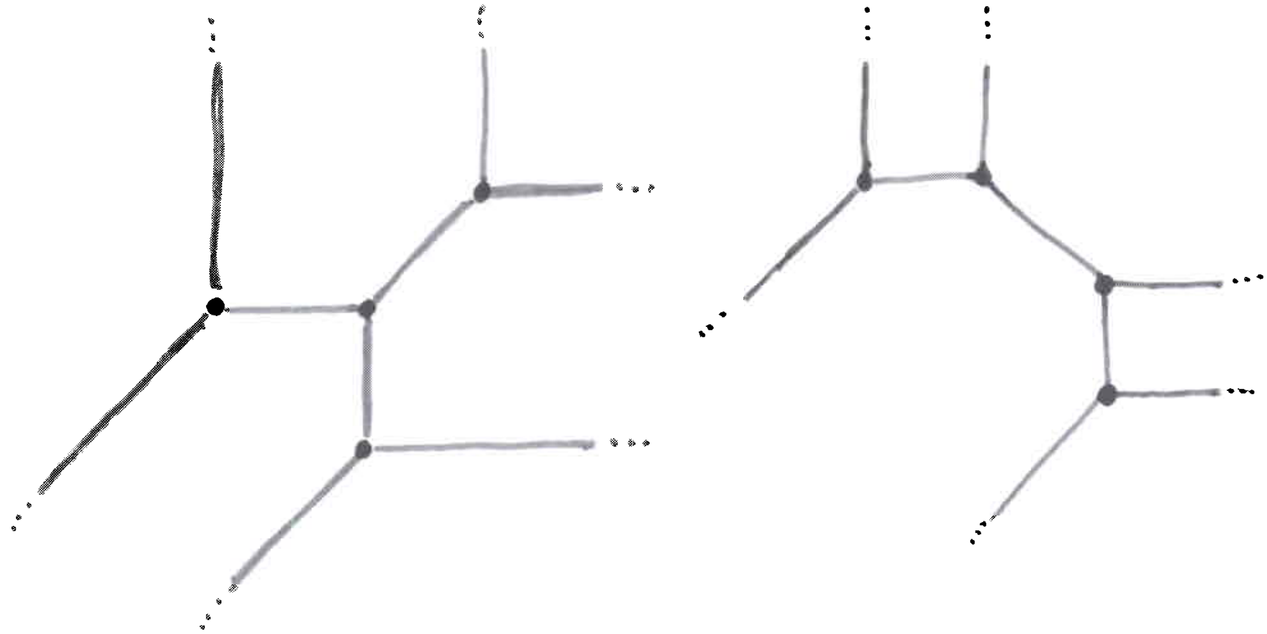
Any two lines meet in a unique point



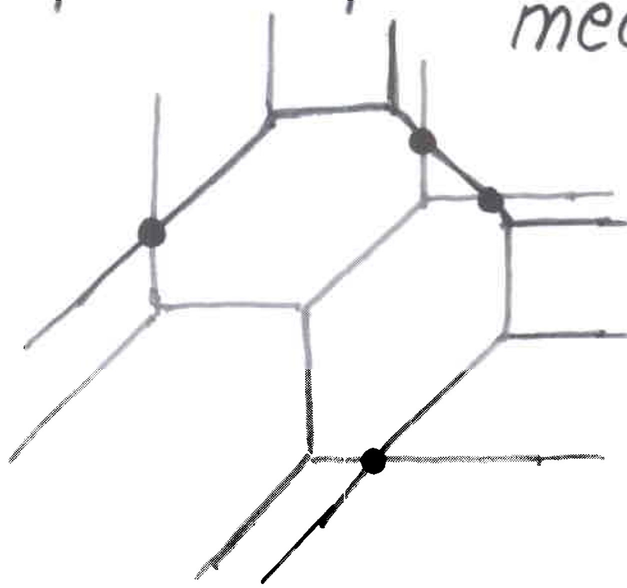


Any two points lie on a unique line

Quadratic curves in the plane



Any two quadratic curves meet in four points



- Bézout's Theorem holds tropically
- Bernstein's Theorem holds tropically
- Gromov-Witten invariants can be computed tropically (Mikhalkin)

"There is no monodromy
in tropical geometry"

Corollary 4.4. *Any two curves of degrees c and d in the tropical projective lane $\mathbb{TP}^2$ intersect stably in a well-defined set of cd points, counting multiplicities.*

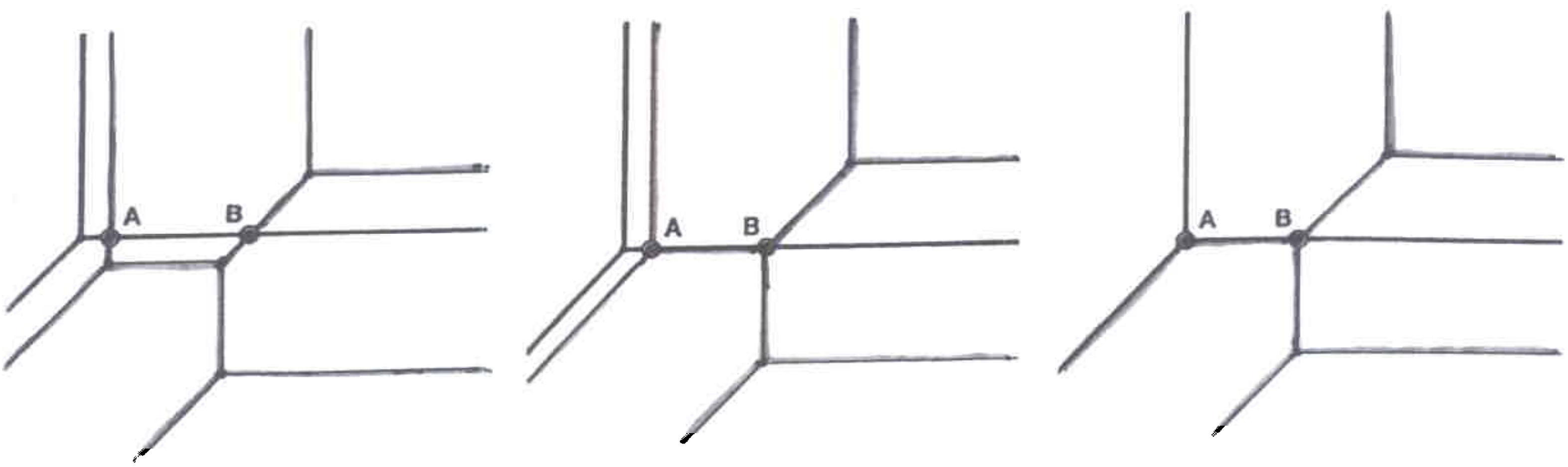


FIGURE 13. Stable intersections of a line and a conic.

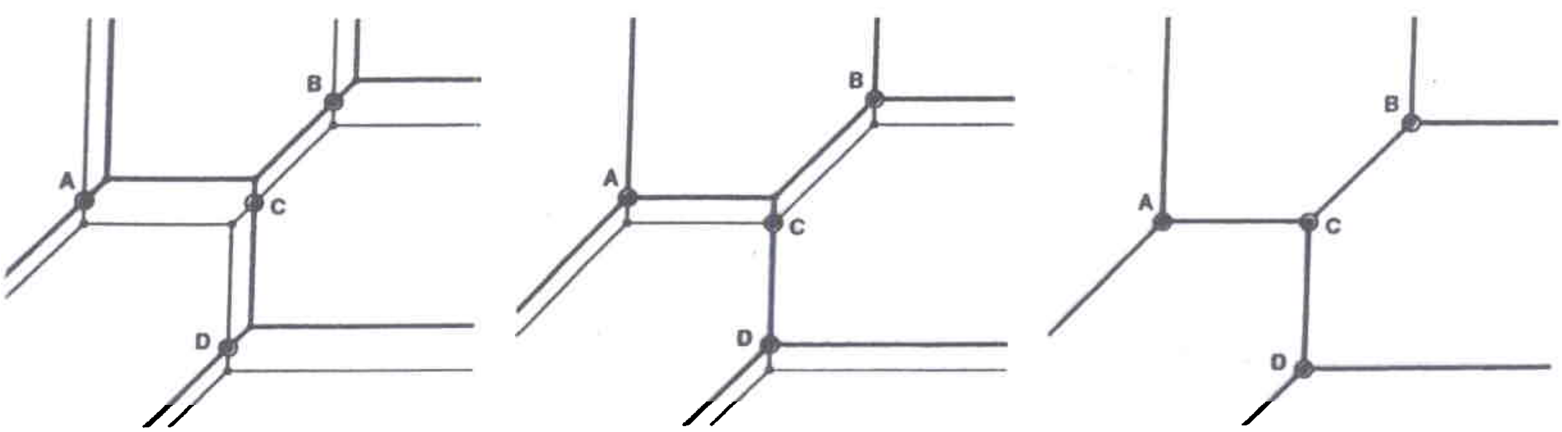


FIGURE 14. The stable intersection of a conic with itself.

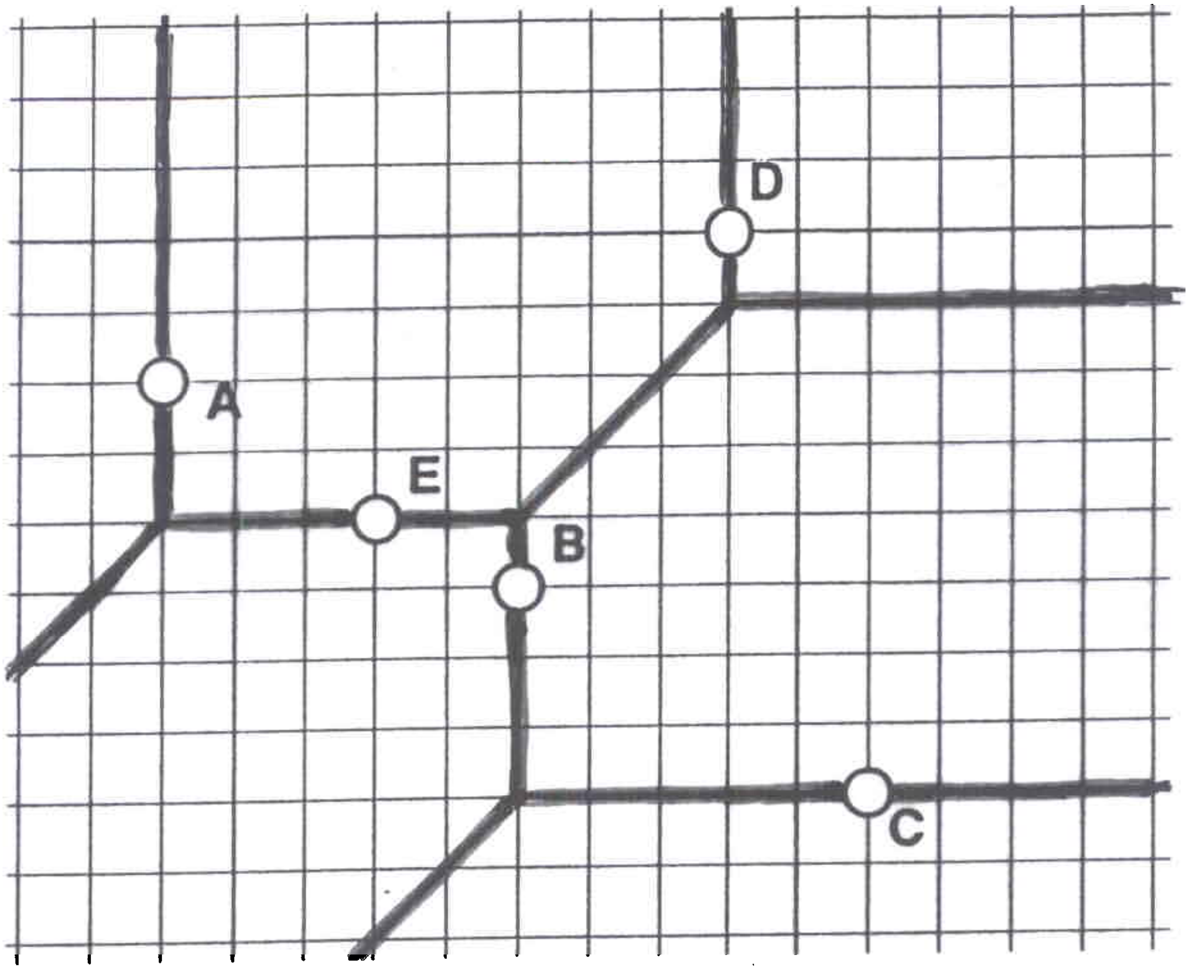


FIGURE 15. Conic through five points.

The equation of this curve is the

tropical determinant of the 6x6-matrix

$$\begin{pmatrix} 0 & a_1 & a_2 & 2a_1 & a_1+a_2 & 2a_2 \\ 0 & b_1 & b_2 & 2b_1 & b_1+b_2 & 2b_2 \\ 0 & c_1 & c_2 & 2c_1 & c_1+c_2 & 2c_2 \\ 0 & d_1 & d_2 & 2d_1 & d_1+d_2 & 2d_2 \\ 0 & e_1 & e_2 & 2e_1 & e_1+e_2 & 2e_2 \\ 0 & x & y & 2x & x+y & 2y \end{pmatrix}$$

The tropical determinant

e.g. for a 3×3-matrix

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

$$\min\{a+e+i,\ a+f+h,\ b+d+i,\ b+f+g,\ c+d+h,\ c+e+g\}$$

$\mathcal{T}(\det) \subset \mathbb{R}^{3\times 3}$ consists of all matrices $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ for which this minimum is attained at least twice.

This happens if and only if the points (a,b,c), (d,e,f) and (g,h,i) lie on a common line in $\mathbb{TP}^2 = \mathbb{R}^3/\mathbb{R}(1,1,1)$

Remark The tropical determinant of an $n\times n$-matrix can be computed fast.

The incidence version of Pappus' Theorem is false in tropical geometry

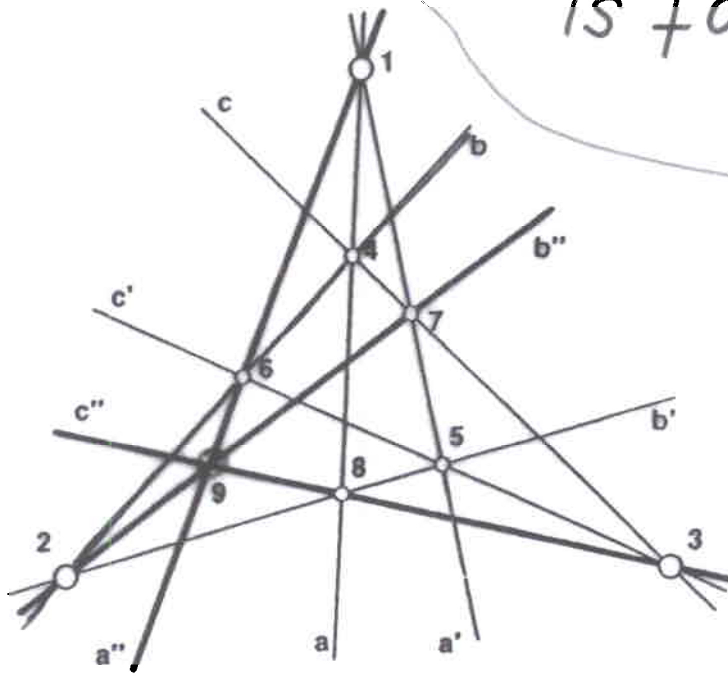


FIGURE 21. Pappus' theorem in classical projective geometry. The lines a'', b'' and c'' are drawn in bold.

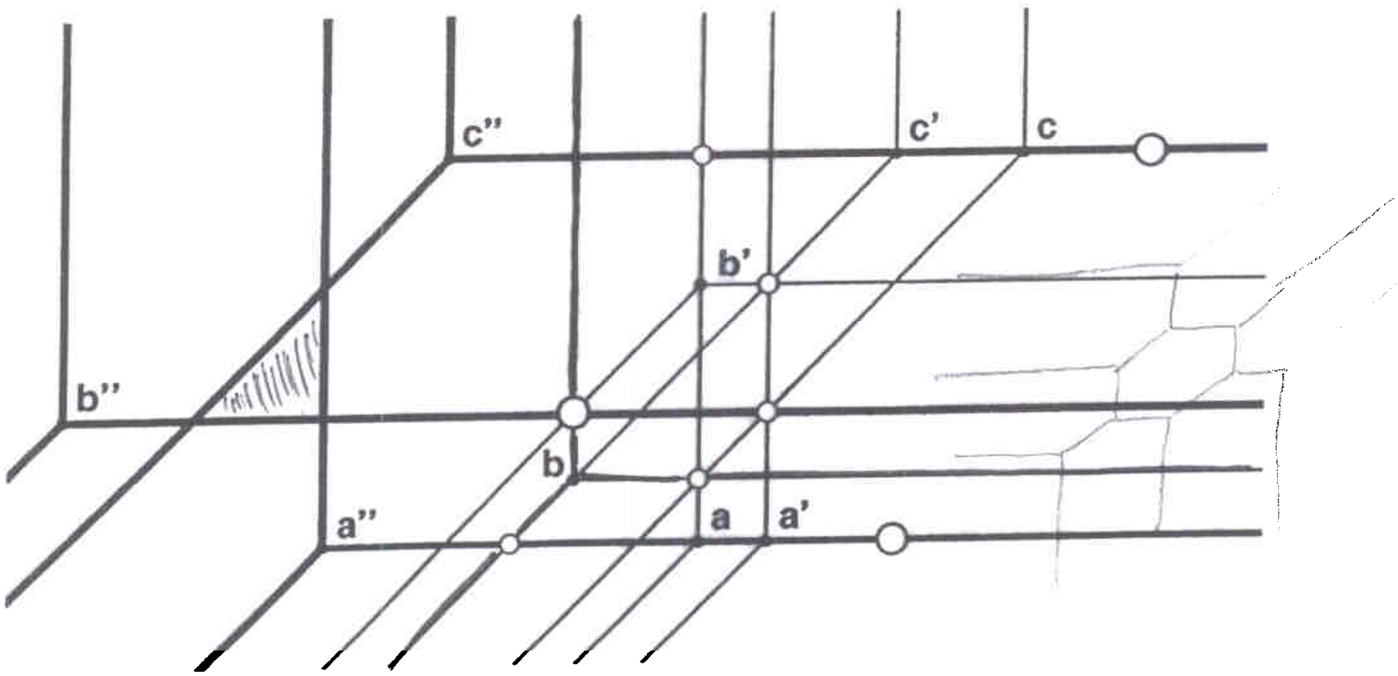


FIGURE 22. A tropical non-Pappus configuration: the triples $[a, a', a'']$, $[b, b', b'']$, $[c, c', c'']$, $[a, b, c]$, $[a', b', c']$, $[a'', b, c']$, $[a', b'', c]$, $[a, b', c'']$ are concurrent, but $[a'', b'', c'']$ is not.

... but the constructive version of Pappus' Theorem is conjectured to be true

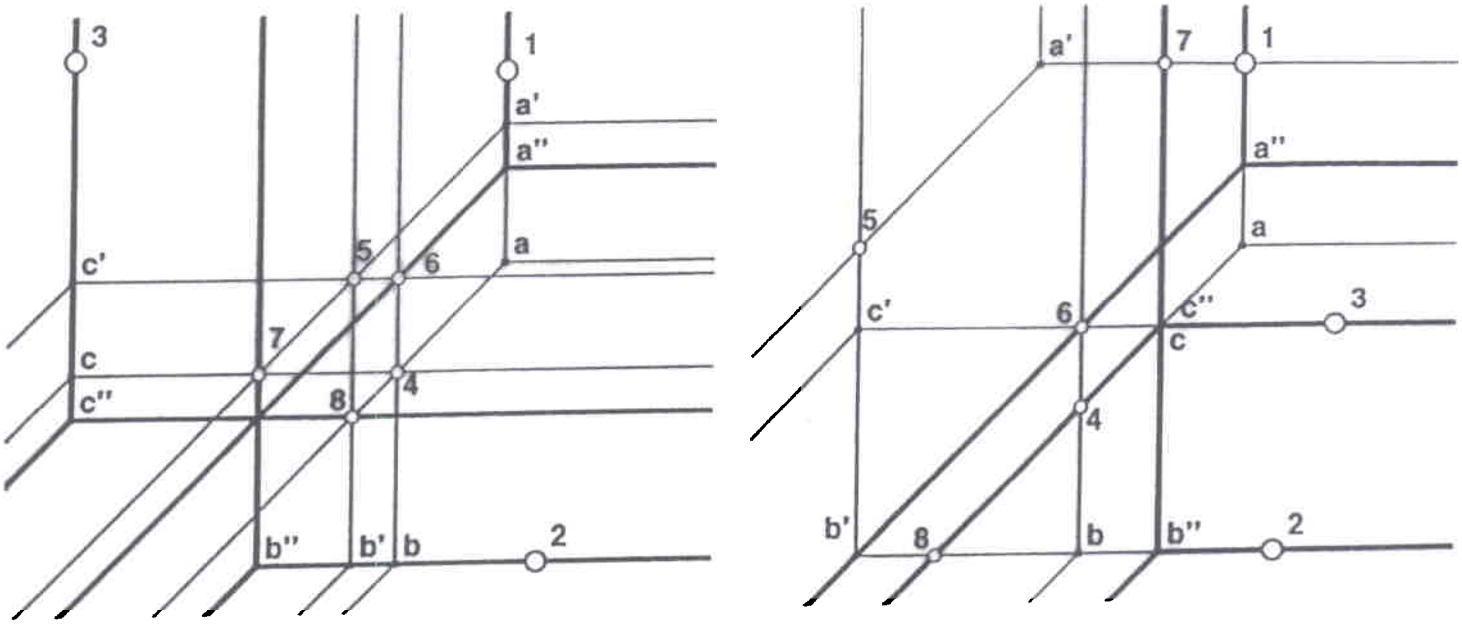


FIGURE 23. The constructive tropical Pappus' Theorem.

PAPPUS' THEOREM, CONSTRUCTIVE VERSION: *Let* $1, 2, 3, 4, 5$ *be five freely chosen points in the projective plane given by homogeneous coordinates. Define the following additional three points and nine lines by a sequence of (stable) join and (stable) meet operations (carried out by cross-products):*

$$a := 1 \otimes 4, \quad b := 2 \otimes 4, \quad c := 3 \otimes 4, \quad a' := 1 \otimes 5, \quad b' := 2 \otimes 5, \quad c' := 3 \otimes 5,$$
$$6 := b \otimes c', \quad 7 := a' \otimes c, \quad 8 := a \otimes b', \quad a'' := 1 \otimes 6, \quad b'' := 2 \otimes 7, \quad c'' := 3 \otimes 8.$$

Then the three tropical lines a'', b'' *and* c'' *are concurrent.*

Thanks to CINDERELLA

Ideals, Varieties and Algorithms

$K = \mathbb{C}\{\{t\}\}$, the field of Puiseux series, has the valuation

$$\text{order} : K^* \to \mathbb{Q}$$
$$c \cdot t^{\alpha} + \cdots \mapsto \alpha$$

Every polynomial in $K[x_1, \ldots, x_n]$,

$$f = c_1(t) \cdot x^{a_1} + \cdots + c_r(t) \cdot x^{a_r},$$

has an associated tropical polynomial

$$F = \operatorname{trop}(f) = \bigoplus_{i=1}^{r} \operatorname{order}(c_i(t)) \odot X^{\odot a_i}$$

For any $w \in \mathbb{R}^n$, the weight of the polynomial f equals $F(w)$. The initial form $\operatorname{in}_w(f)$ is the subsum of all terms $t^{\nu} \cdot x^{a_i}$ of weight $F(w)$.

<u>Example</u> in $K[x,y]$

$$f(x,y) = (t^{1/2}+3t^7)\cdot x^2 y + (t^{-5/2}+11)\cdot xy^5 + 8y^4$$

$$F(x,y) = \tfrac{1}{2}\odot x^{\odot 2}\odot y \oplus \left(-\tfrac{5}{2}\right)\odot x\odot y^{\odot 5} \oplus y^{\odot 4}$$

If $w=(1,1)$ then

$$in_w(f) = t^{1/2}x^2y + t^{-5/2}xy^5$$

For an ideal $I \subset K[x_1,\dots,x_n]$ we have the <u>initial ideal</u>

$$in_w(I) = \langle in_w(f) : f \in I \rangle$$

Two vectors $w, \tilde{w} \in \mathbb{R}^n$ lie in the same cell of the Gröbner complex of I if

$$in_w(I) = in_{\tilde{w}}(I)$$

Theorem For any ideal $I \subset K[x_1, \ldots, x_n]$ the following subsets of $\mathbb{R}^n$ coincide:

(a) The closure of the image of $\mathcal{V}(I) \subset (K^*)^n$ under the map $\text{order}: (K^*)^n \to \mathbb{R}^n$.

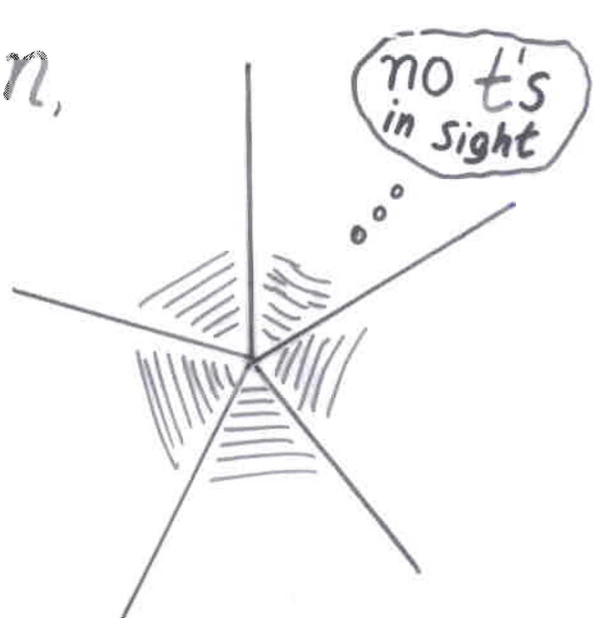
(b) The set of $w \in \mathbb{R}^n$ such that $in_w(I)$ contains no monomial.

(c) The intersection of all the tropical hypersurfaces $\mathcal{T}(\text{trop}(f))$ where f runs over I.

a universal Gröbner basis of

The set (a)=(b)=(c) is a polyhedral complex $\mathcal{T}(I)$, called the tropical variety of I.

Corollary If $I \subset \mathbb{C}[x_1, \dots, x_n]$ then $\mathcal{T}(I)$ is a *fan*, namely, it is a subfan of the Gröbner fan of I.

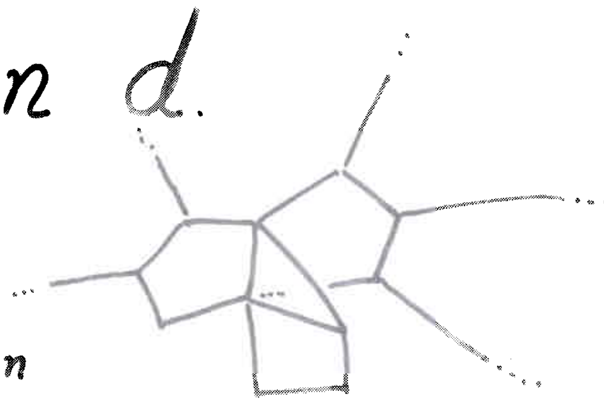


In this case, $\mathcal{T}(I)$ coincides with the *logarithmic limit set* of G. Bergman (1971)

Theorem (R. Bieri & J. Groves 1984; M. Einsiedler, M. Kapranov, D. Lind & T. Ward 2003)

Suppose I is a prime ideal of dimension d. Then $\mathcal{T}(I)$ is connected and pure of dimension d.

all maximal cells have the same dimension



Example: A line in 3-space

$$I = \langle x+y+z+1, \; x+ty+t^2z+t^3 \rangle$$

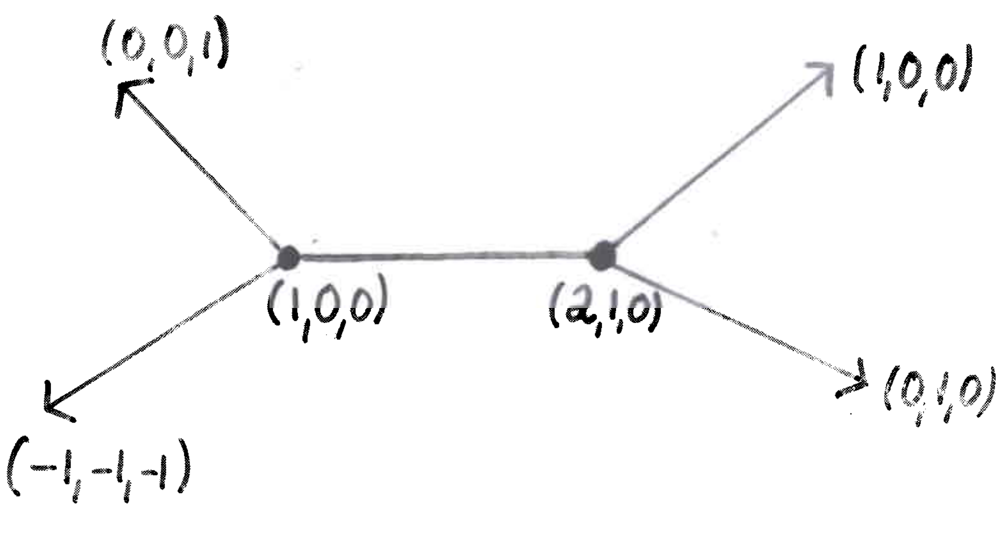
Two points on $\mathcal{T}(I) \subset \mathbb{R}^3$ are $(1,0,0)$ and $(2,1,0)$.

Reason: These are the orders of the following two points on $\mathcal{V}(I)$

$$(x,y,z) = (2t+t^2, -2-2t-t^2, 1)$$

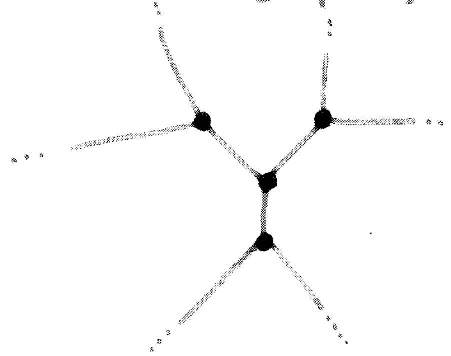
$$(x,y,z) = \left(\frac{t^2}{2+2t}, \frac{t}{2}, -1-\frac{t}{2}-\frac{t^2}{2+2t}\right)$$

The tropical line $\mathcal{T}(I)$ looks like this →



Tropical planes

If I is generated by linear polynomials $c_0(t) + c_1(t)x_1 + \cdots + c_n(t)x_n$ then we call $\mathcal{T}(I)$ a tropical plane.



← a line in 5-space

Theorems

- A tropical d-plane is a contractible polyhedral complex of pure dimension d.
- Most tropical planes are <u>not</u> complete intersections of tropical hyperplanes.
- The tropical d-planes in $\mathbb{R}^n$ are parametrized by the tropical Grassmannian.

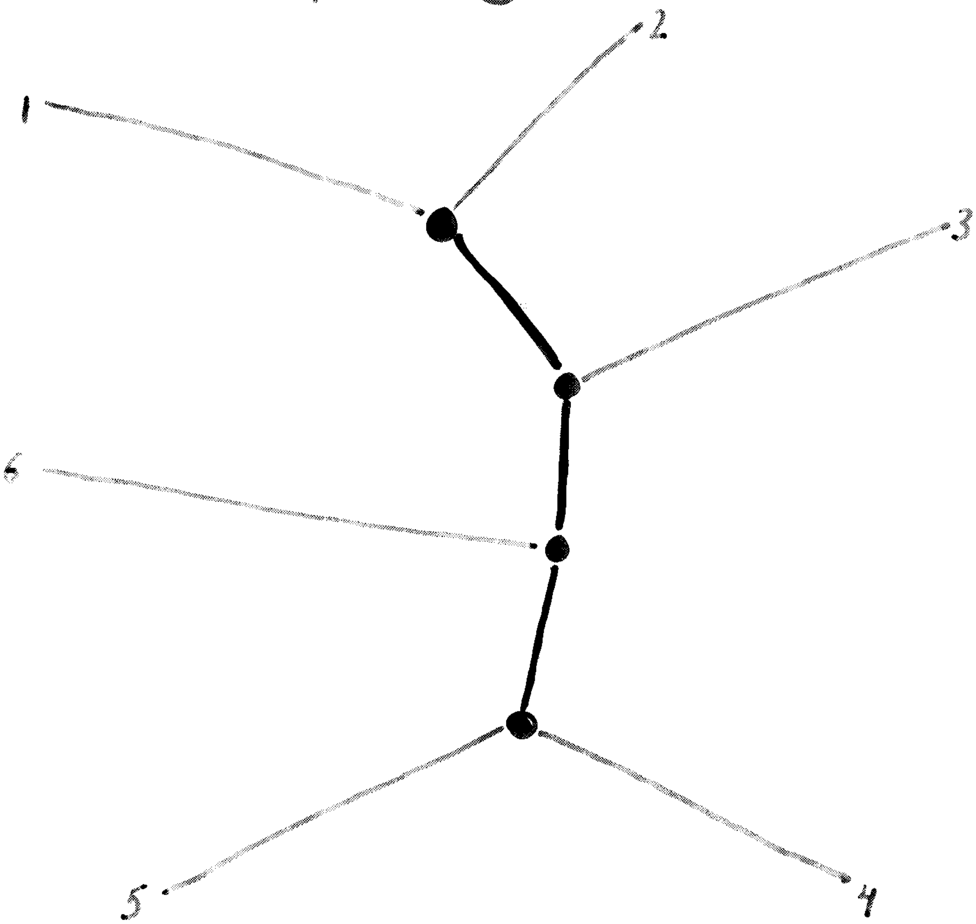
Tropical Lines

in $\mathbb{TP}^{n-1} = \mathbb{R}^n / \mathbb{R}(1,1,\dots,1)$

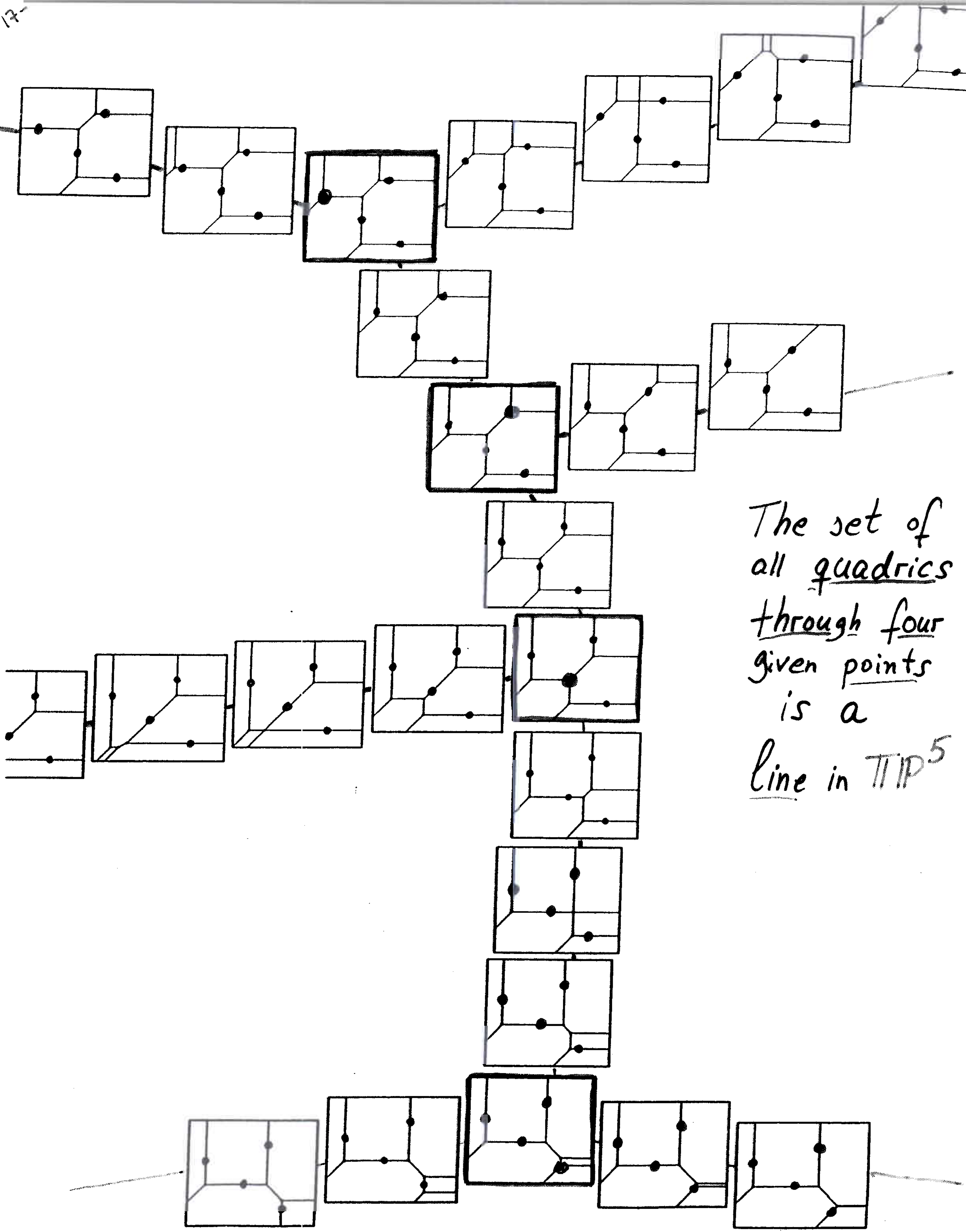
are trees with $n-3$ bounded edges and n half rays.

They are parametrized by the tropical Grassmannian $\mathcal{G}r(2,n)$.

<u>Theorem</u> $\mathcal{G}r(2,n)$ equals the space of phylogenetic trees (Billera Holmes Vogtmann)



← a tropical line in $\mathbb{TP}^5$



The set of all quadrics through four given points is a line in $\mathbb{TP}^5$

The Tropical Grassmannian

$$\mathcal{G}r(d,n) = \mathcal{T}(I_{d,n})$$

is defined by the Plücker ideal

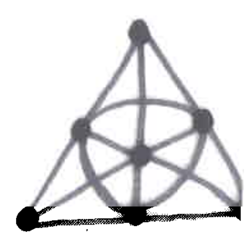
$$I_{d,n} \subset \mathbb{Z}[p_{i_1 i_2 i_3 \cdots i_d} : 1 \le i_1 < \cdots < i_d \le n]$$

of algebraic relations among the $d \times d$-minors of a $d \times n$-matrix.

It is a *fan* of dimension $d(n-d)+1$ in $\mathbb{R}^{\binom{n}{d}}$ but, factoring out the lineality space, we regard $\mathcal{G}r(d,n)$ as a spherical polyhedral complex of dimension $nd-n-d^2$.

Caveat $\mathcal{G}r(3,7)$ depends on the characteristic of K.

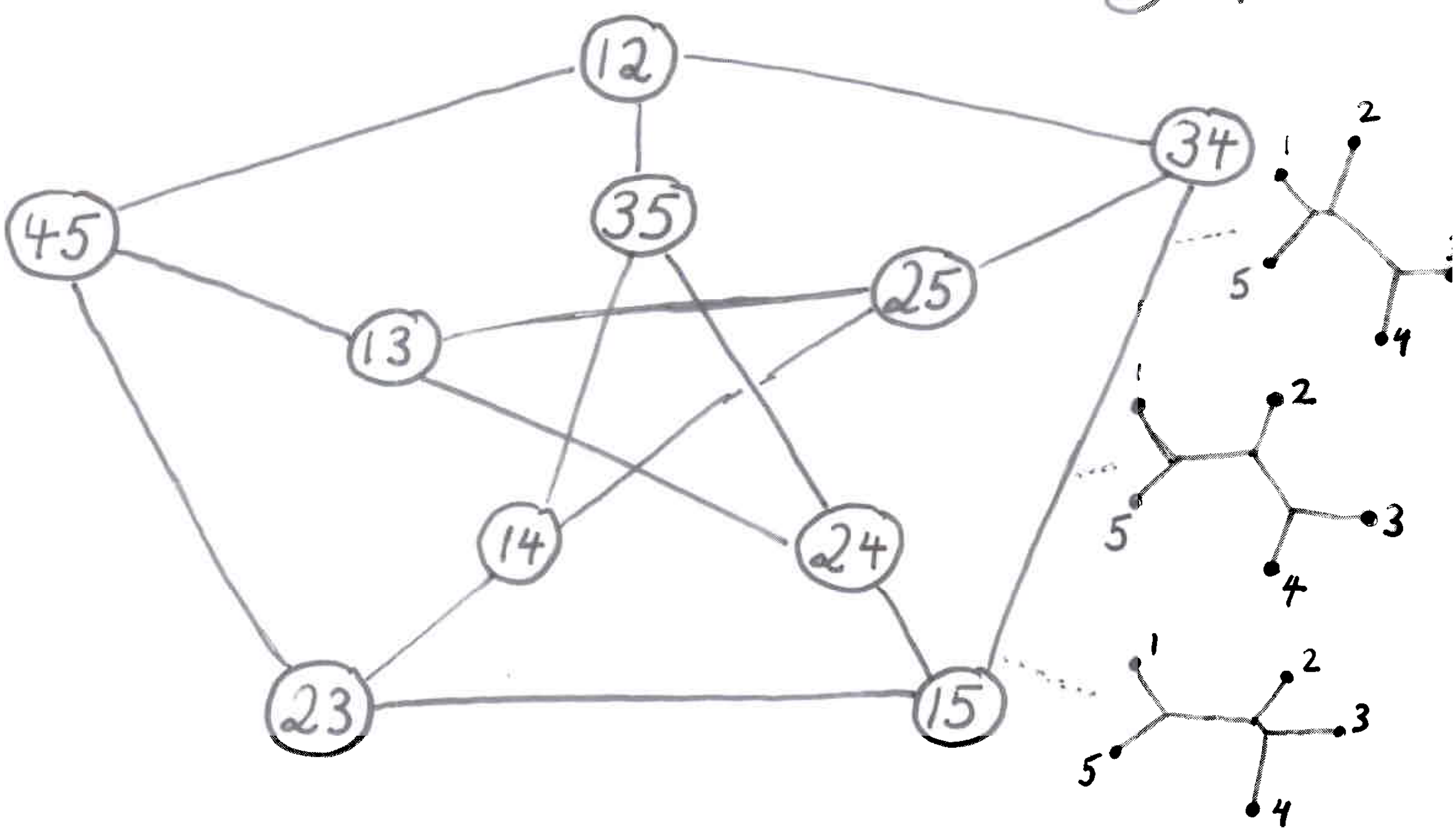
(D. Speyer July '03)



Lines in 4-space

$$I_{2,5} = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23},\ p_{12}p_{35} - p_{13}p_{25} + p_{15}p_{23},\ p_{12}p_{45} - p_{14}p_{25} + p_{15}p_{24},\ p_{13}p_{45} - p_{14}p_{35} + p_{15}p_{34},\ p_{23}p_{45} - p_{24}p_{35} + p_{25}p_{34} \rangle$$

$Gr(2,5)$ is the Petersen graph.



$Gr(2,n)$ is well-understood for all n.

We also computed $Gr(3,6)$.

$f = (65, 550, 1395, 1035,$

Three Conjectures

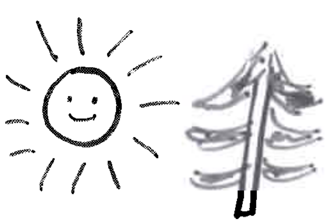
① $Gr(d,n)$ is a simplicial complex.

② Every ridge of $Gr(d,n)$ lies in either 2 or 3 facets.

③ The number of i-faces of a tropical $(d-1)$-plane in $(n-1)$-space is at most

$$\binom{n-i-2}{d-i-1}\binom{2n-d-1}{i}$$

This is tight for trees $(d=2)$ and sagbi planes.



There are many good open problems in tropical geometry.