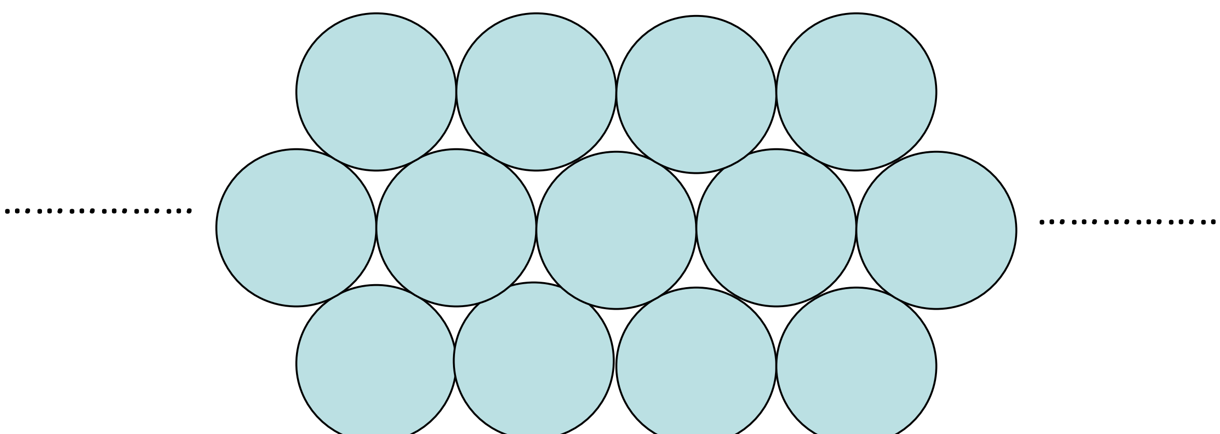
Packing Equal Discs in the Plane

Object: To maximize the density ρ_0 of the covered region

Theorem: (Thue – 1892)

Hexagonal is optimal, with $\rho_0 = \pi/2\sqrt{3}$



Packing Equal Discs in the Plane

Object: To maximize the density ρ_0 of the covered region

Theorem: (Thue – 1892, 1910)   (Fejes Tóth – 1940)

(Lagrange – 1773)

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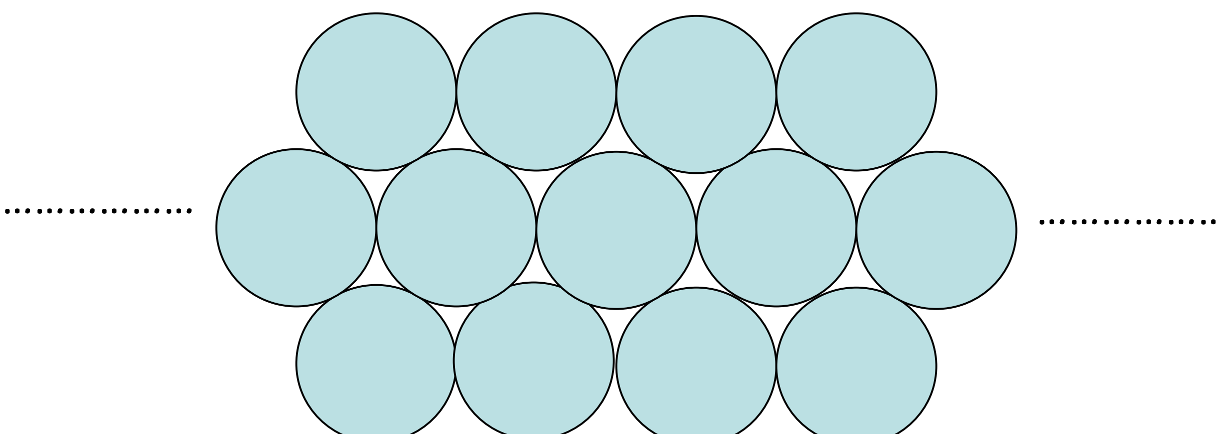
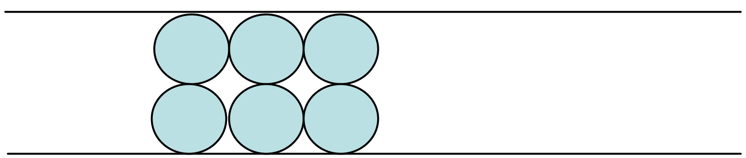


The next simplest case

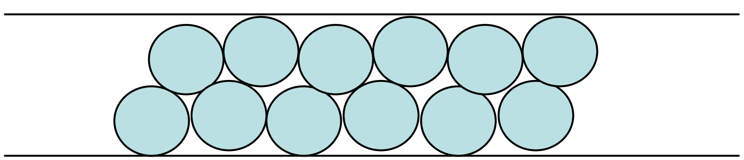
Packing an infinite strip of width w with equal discs.

(We will always assume that our discs have diameter 1.)

For example, take **w = 2**:

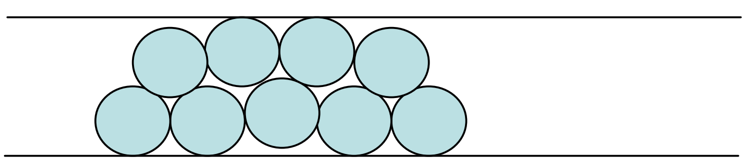


trivial --- **center** density = 1/2



Fejes Tóth (1971)---**center** density =

$$\frac{1}{4}\left(\sqrt{2} + \frac{1}{\sqrt{3}}\right) = 0.49789\ldots$$

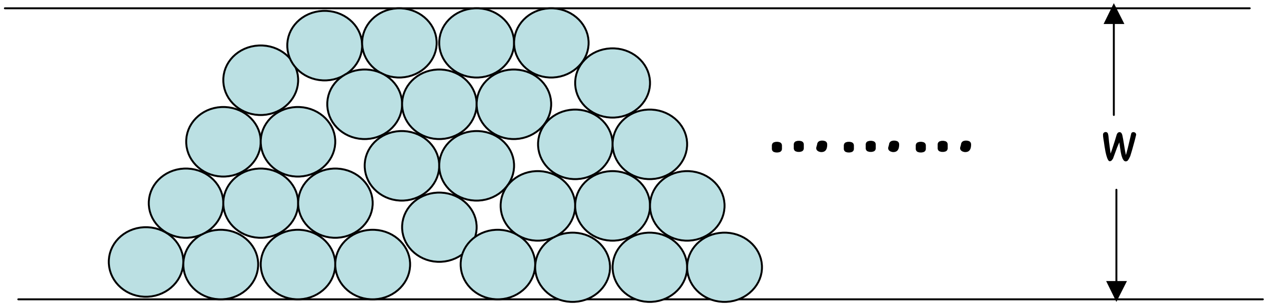


RLG (1971)---**center** density =

$$\frac{1}{6}\left(1 + \sqrt{4\sqrt{3} - 3}\right) = 0.49699\ldots$$

Conjecture:(J. Molnár -- 1970's)

The "alternating triangles" packing of discs is optimal for **every** value of w.



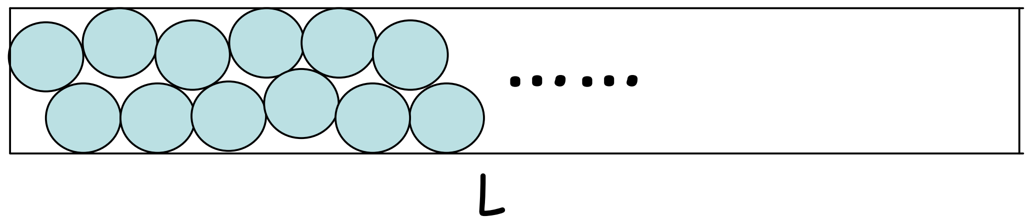
Theorem: (Füredi—1992)

Conjecture is true for $w \leq 1 + \sqrt{3}$

and also for $w = 1 + k\sqrt{3}$ for any positive integer k.

What if the strip is finite (but very long)?

For example, what is the length L_2 of the shortest rectangle of width 2 for which a **non-trivial** packing of units discs is optimal?



If L = 164.992765…, then 332 units discs can be non-trivially packed into a 2-by-L rectangle. The trivial packing requires a 2-by-165 rectangle.

What is the correct value of L_2 ??

How far into the interior do the irregularities penetrate in on optimal packing?

N = 49

49 circles in the 10x1 unit rectangle

radius = 0.068177492110   density = 0.715530035967
distance = 0.142499429277   contacts = 110

N = 50

50 circles in the 10x1 unit rectangle

radius = 0.067466165729   density = 0.714976775101
distance = 0.140946423705   contacts = 110

N = 51

51 circles in the 10x1 unit rectangle

radius = 0.066792996000   density = 0.714793503551
distance = 0.139477929761   contacts = 112

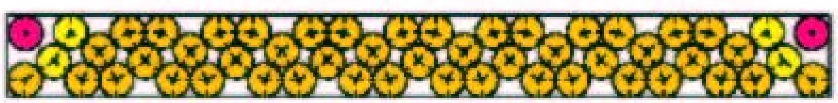
N = 52

52 circles in the 10x1 unit rectangle

radius = 0.066176054501   density = 0.715409918849
distance = 0.138133465924   contacts = 117

N = 53

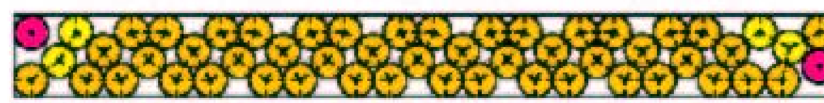
53 circles in the 10x1 unit rectangle



radius = 0.065602609207   density = 0.716505420917
distance = 0.136894663452   contacts = 117

N = 54

54 circles in the 10x1 unit rectangle



radius = 0.065042745209   density = 0.717697375880
distance = 0.135666348710   contacts = 119

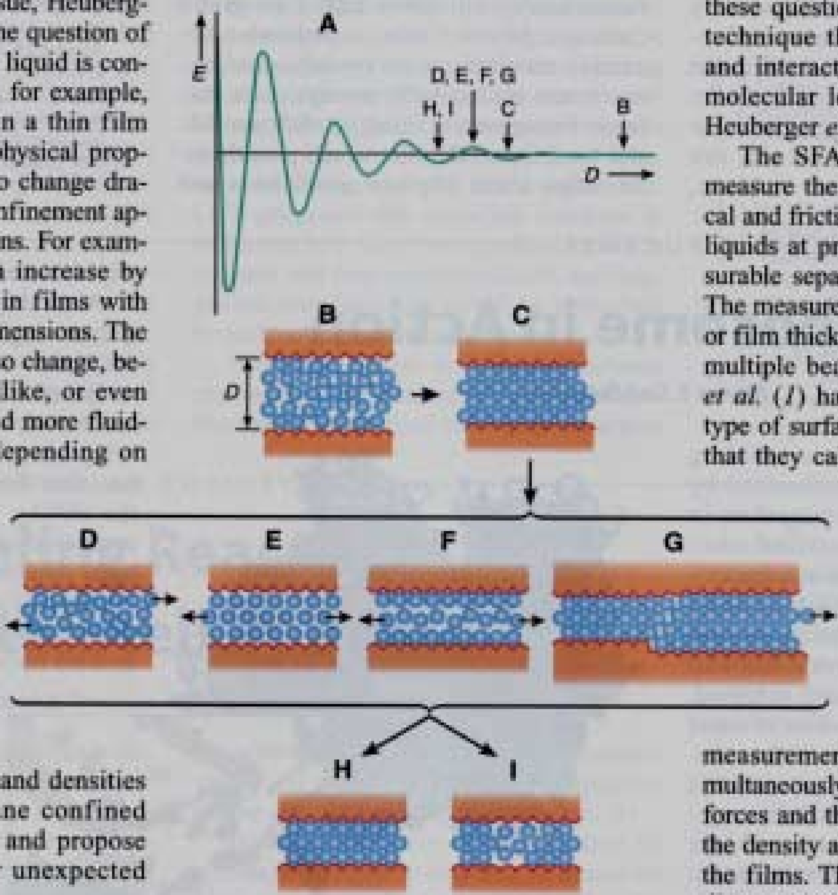
PERSPECTIVES: LIQUIDS

Putting Liquids Under Molecular-Scale Confinement

Jacob Israelachvili and Delphine Gourdon

On page 905 of this issue, Heuberger *et al.* (*1*) address the question of what happens when a liquid is confined within a small volume, for example, in an ultrathin capillary or in a thin film between two surfaces. The physical properties of liquids are known to change dramatically as the degree of confinement approaches molecular dimensions. For example, a liquid's viscosity can increase by several orders of magnitude in films with molecular or "nanoscale" dimensions. The "structure" of a liquid can also change, becoming more ordered, solidlike, or even crystalline or less ordered and more fluidlike than the bulk liquid, depending on how the microscopic shape and atomic structure of the confining walls match that of the liquid molecules (*2*). Many aspects of these changes remain poorly characterized and understood. Heuberger *et al.* (*1*) report unprecedentedly detailed measurements of the forces and densities of thin films of cyclohexane confined between two mica surfaces and propose new explanations for their unexpected observations.

The properties of confined liquids (and solids) are of great interest and importance in areas as diverse as materials science, microfabrication, adhesion and lubrication, biology, geology, and the bud-



How does the confined liquid respond? The short-range oscillatory "solvation" force (also known as the potential of mean force) between two surfaces in a liquid varies between attraction and repulsion with a periodicity

liquid film undergo a succession of liquid-to-solid-to-liquid phase transitions (B → C → D → H), does the film collapse in an ordered fashion (C → E → H), or do individual layers get forced out through dislocations (C → G → H)? And are there both out-of-plane and in-plane (lateral) heterogeneities, such as two-dimensional domains, in the films (F or I)? To answer these questions experimentally requires a technique that can probe both structure and interactions in real time at the submolecular level (<0.1 nm). This is what Heuberger *et al.* (*1*) have achieved.

The SFA (*6*) is traditionally used to measure the normal and lateral (rheological and friction) forces between surfaces in liquids at precisely controllable and measurable separations at the angstrom level. The measurement of the surface separation or film thickness is achieved optically with multiple beam interferometry. Heuberger *et al.* (*1*) have designed and built a new type of surface force–measuring apparatus that they call an extended surface forces apparatus (eSFA). Using fast spectral correlation spectroscopy to record the interference fringes, they were able to measure surface separation D and film refractive index n at least 10 times more accurately than in conventional SFA measurements. This enabled them to simultaneously measure both the interaction forces and the refractive index (and hence the density and, indirectly, the structure) of the films. This allows for the first time a direct correlation between these two intimately related factors.

Using the eSFA, Heuberger *et al.* measured the oscillatory force profile between two molecularly smooth surfaces of mica across liquid cyclohexane, C_6H_{12}, a rough-

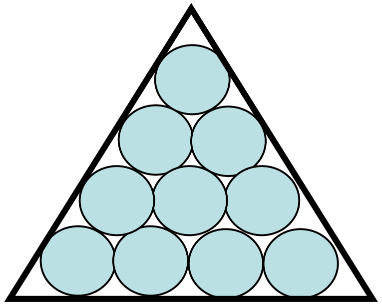
Packing bounded domains

Let T(s) denote an **equilateral triangle** of side s.

What does the densest packing of n unit discs in T(s) look like?

If $n = \binom{m}{2}$, the answer is “obvious”.

For example, for n = 10:



Conjecture (Zassenhaus) This is always optimal for $n = \binom{m}{2}$

Theorem (Oler--1961) This conjecture is true.

One of the simplest proofs is based on the following result.

Let K denote a simplicial complex in the plane, and let

$a_i(K)$ denote the number of i-simplices of K, i = 0, 1, 2.

As usual, let $\chi(K) = a_0(K) - a_1(K) + a_2(K)$

denote the Euler characteristic of K

Let A(K) and P(K) denote the **area** and **perimeter** of K, respectively

Theorem (Folkman, RLG—1969)

If K is a simplicial complex in the plane, and $d(x,y) \geq \mathbf{1}$ for $x \neq y$ in K,

then $$a_0(K) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K)$$

Let X be a compact convex subset of the plane.

By a packing of X, we mean a subset S of X such that

$$x, y \in S \Rightarrow d(x, y) \geq 1$$

The **packing number** $\rho(X)$ is defined to be max $\{|S|: S \text{ is a packing of } X\}$

Corollary (Oler – 1961)

$$\rho(X) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + 1$$

Example: X = T(n), an equilateral triangle of side n.

Then, $A(T(n)) = \frac{\sqrt{3}}{4} n^2$, and $P(T(n)) = 3n$

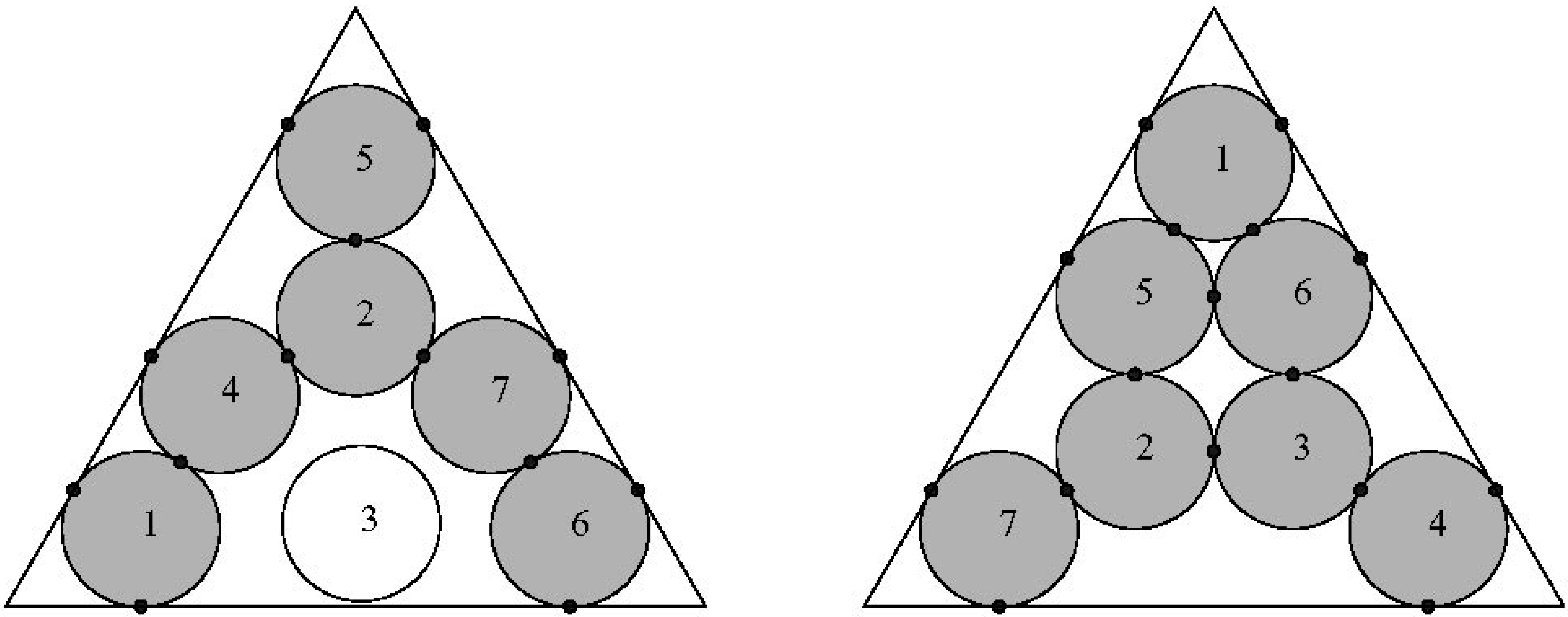
Thus, $\rho(T(n)) \leq \frac{2}{\sqrt{3}} \frac{\sqrt{3}}{4} n^2 + \frac{3}{2} n + 1 = \frac{1}{2}(n^2 + 3n + 2) = \binom{n+2}{2}$

Conjecture (D. J. Newman) - $100

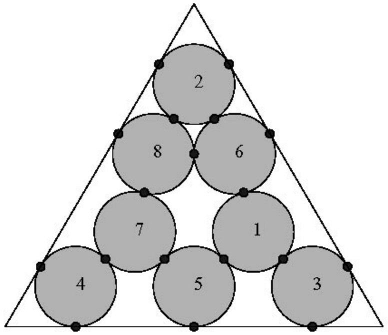
The smallest equilateral triangle into which $\binom{n+2}{2} - 1$

points can be packed is (still) n !

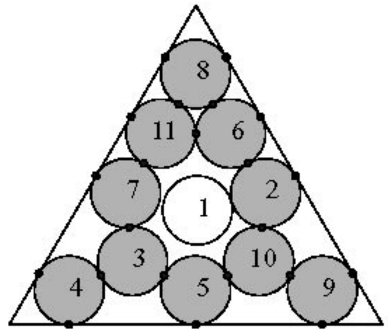
This is no longer true for $\binom{n+2}{2} - 2$ points.



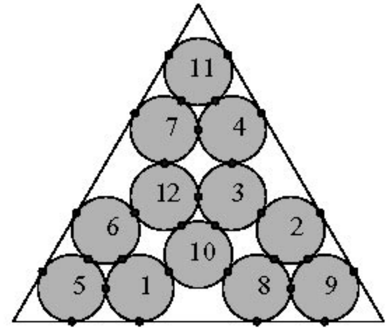
Two different optimal packings of 7 discs in an equilateral triangle



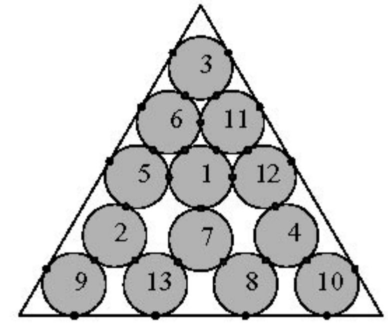
8



11



12

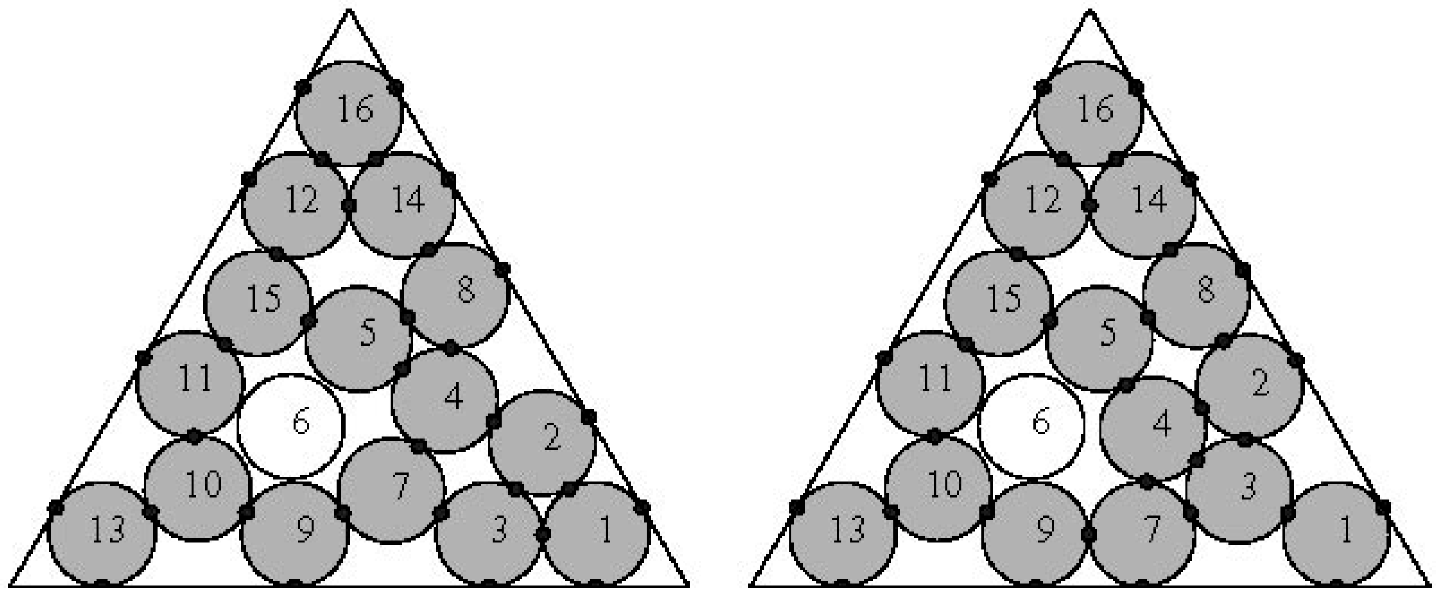


13

Optimal packings for n = 8, 11, and 12 (H. Melissen – 1995)

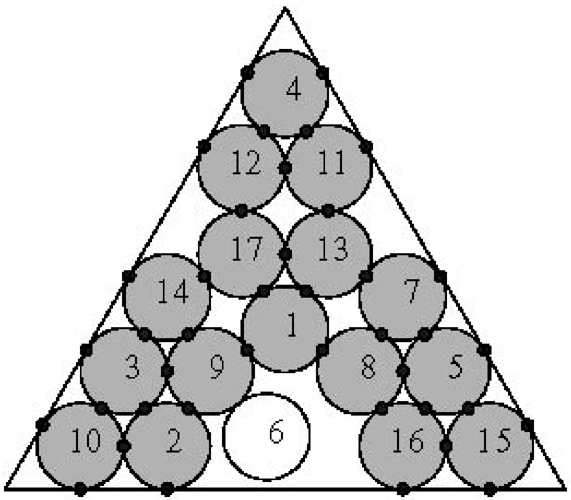
Packing for n = 13 is conjectured to be optimal.

n = 12 is the last known optimal value when $n \neq \Delta$

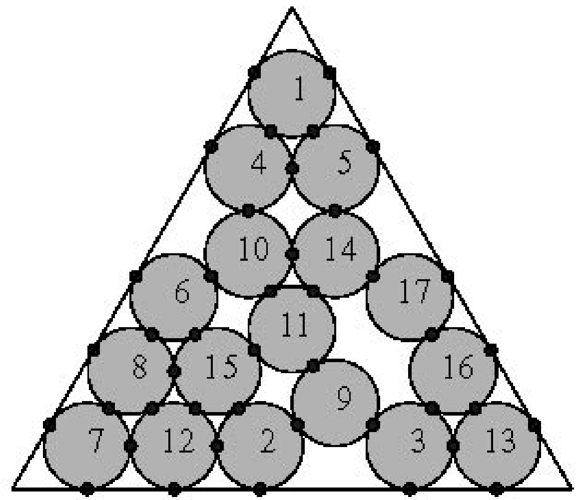


Two equally good packings for n = 16, conjectured to be optimal.

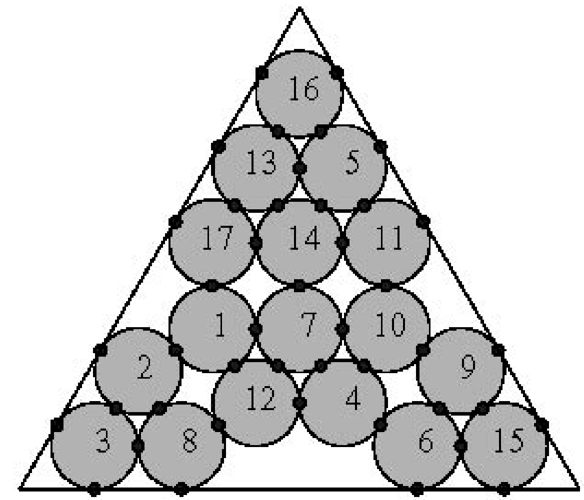
Especially hard cases seem to be when n = Δ + 1.



40 “bonds”



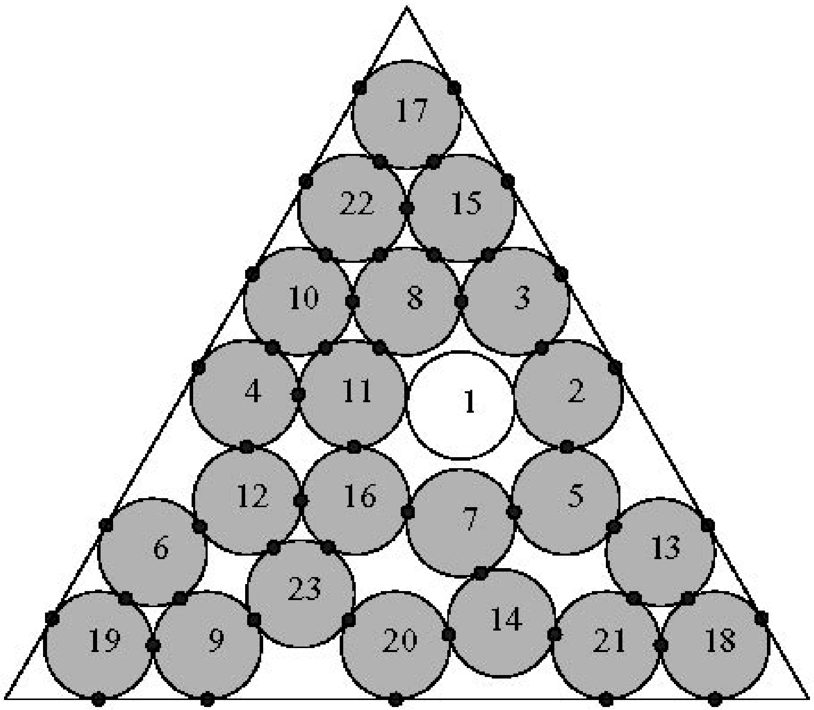
42 “bonds”



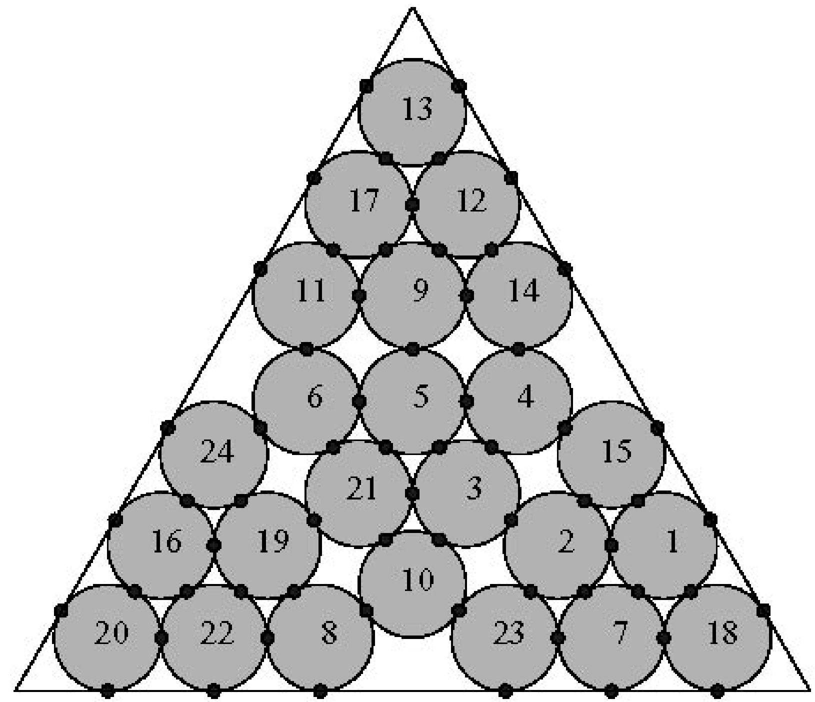
43 “bonds”

Three equally good conjectured optimal packings of 17 discs

.Note the “rattler” in the first packing



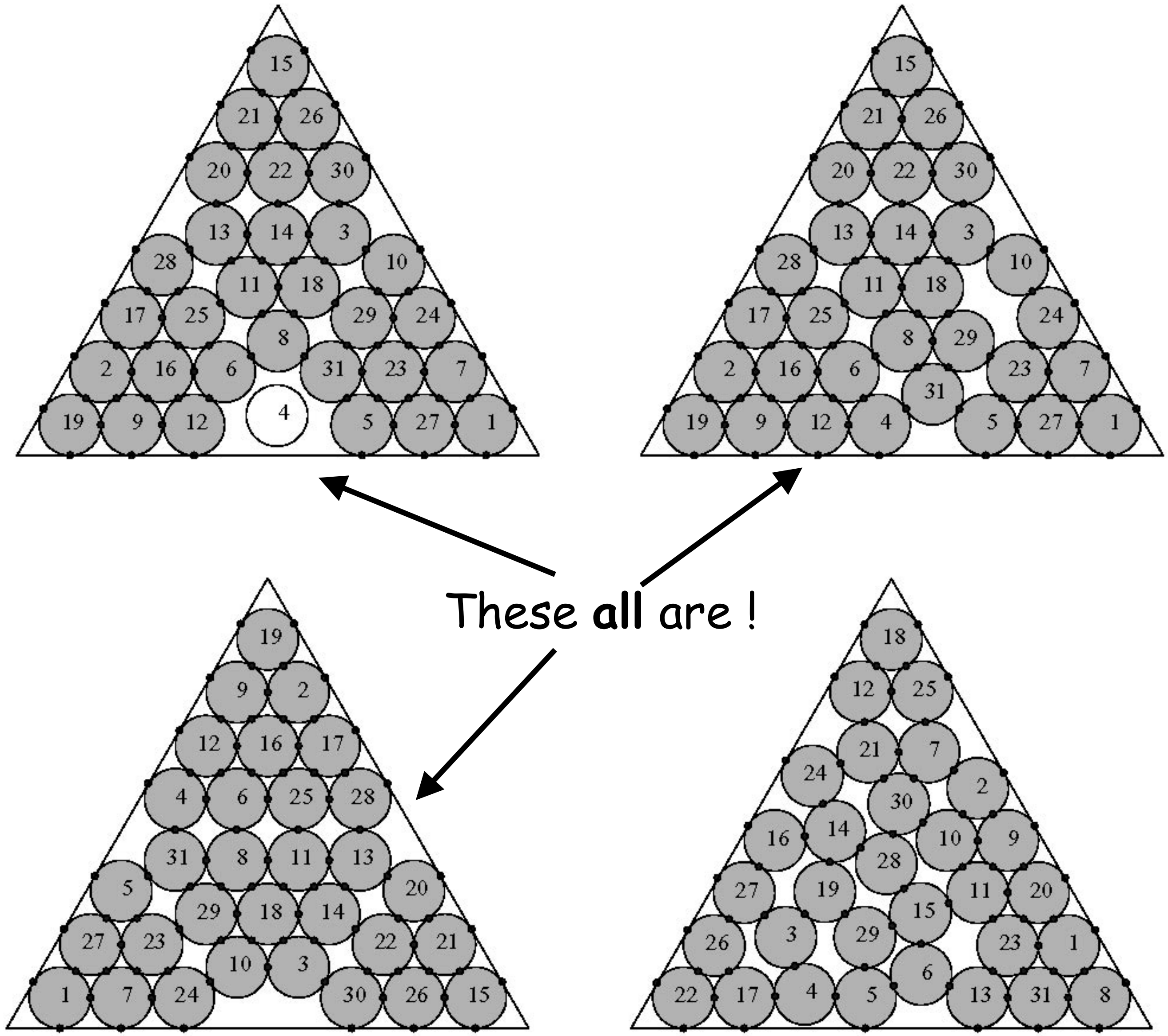
23



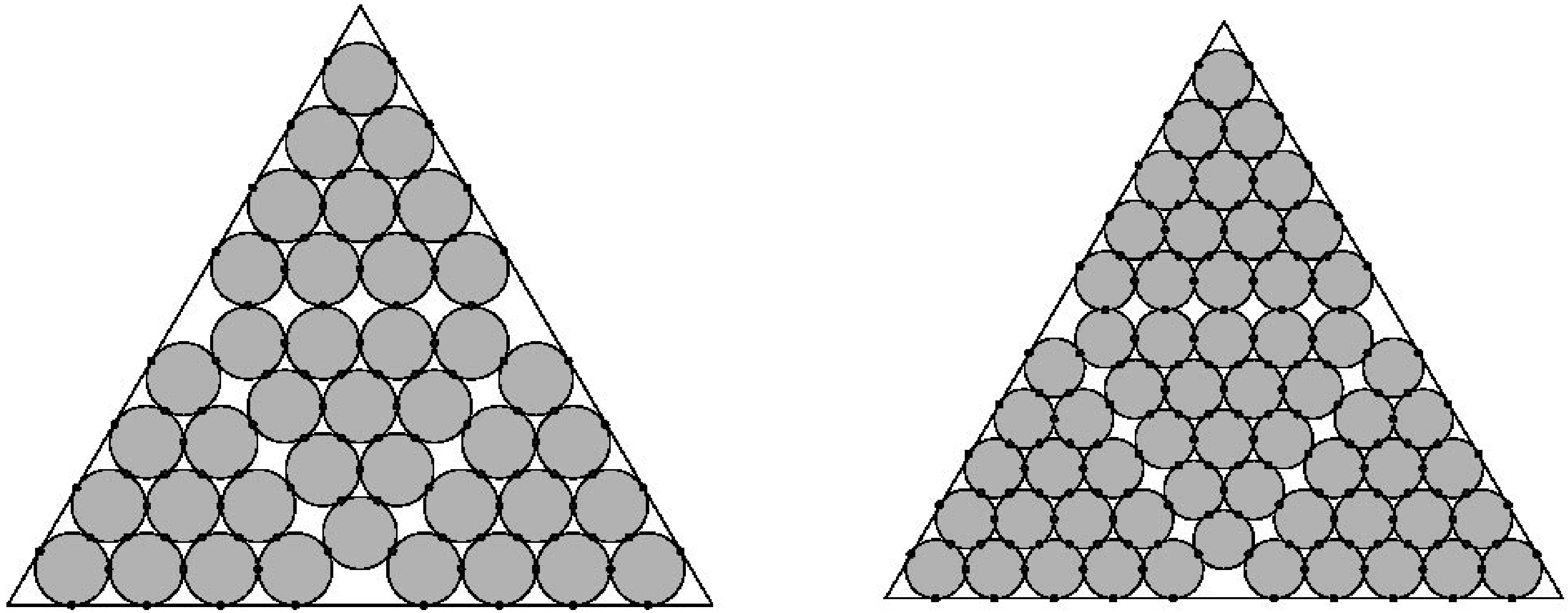
24

Conjectured optimal packings of 23 and 24 discs.

Which is nicer?   Any conjectures??

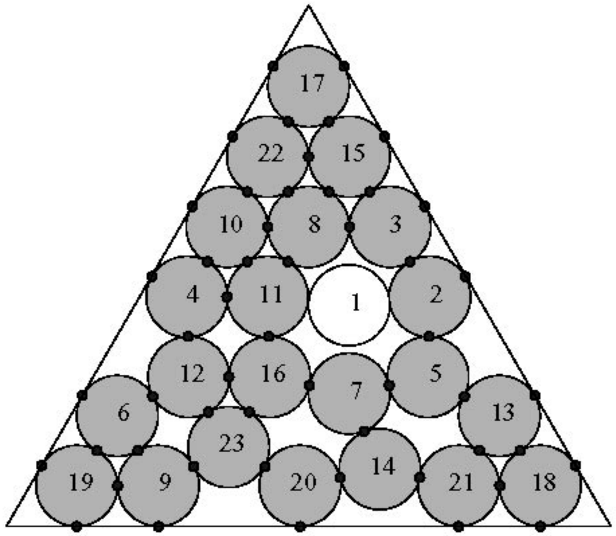


Four packings of 31 discs. Which is the best?



40 60

Conjectured optimal packings when n = 4Δ


17
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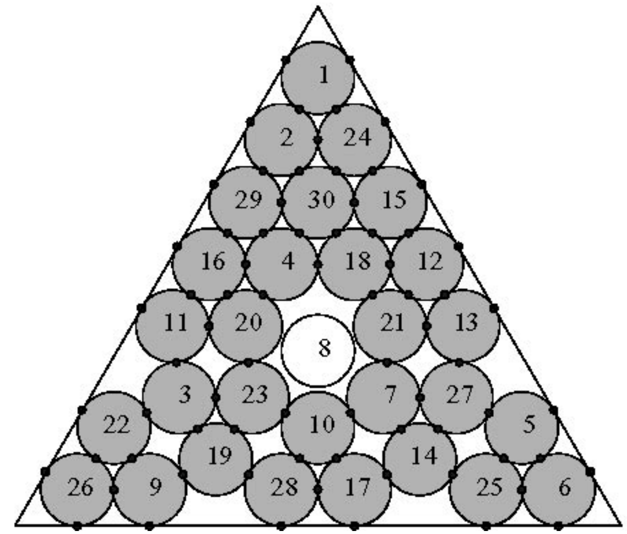
23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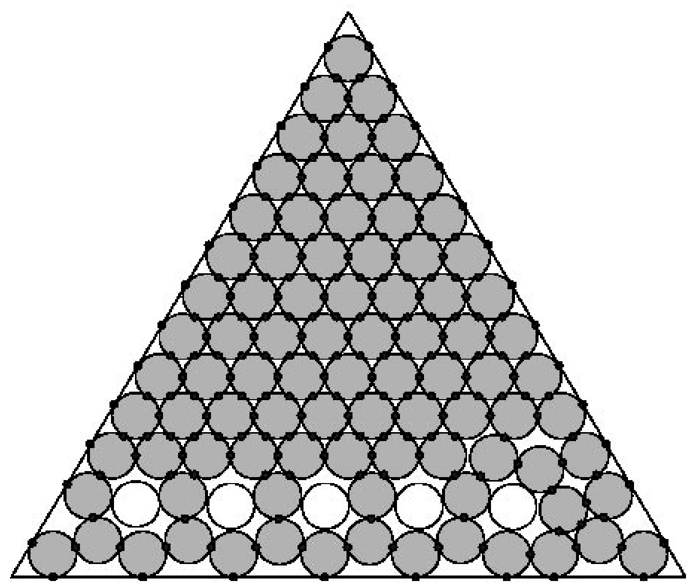
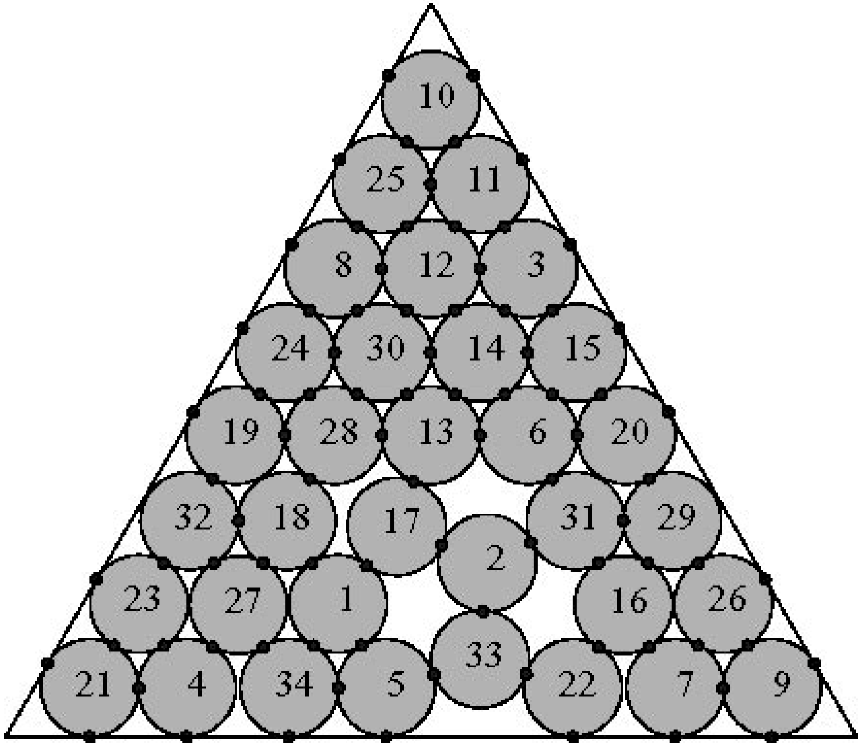
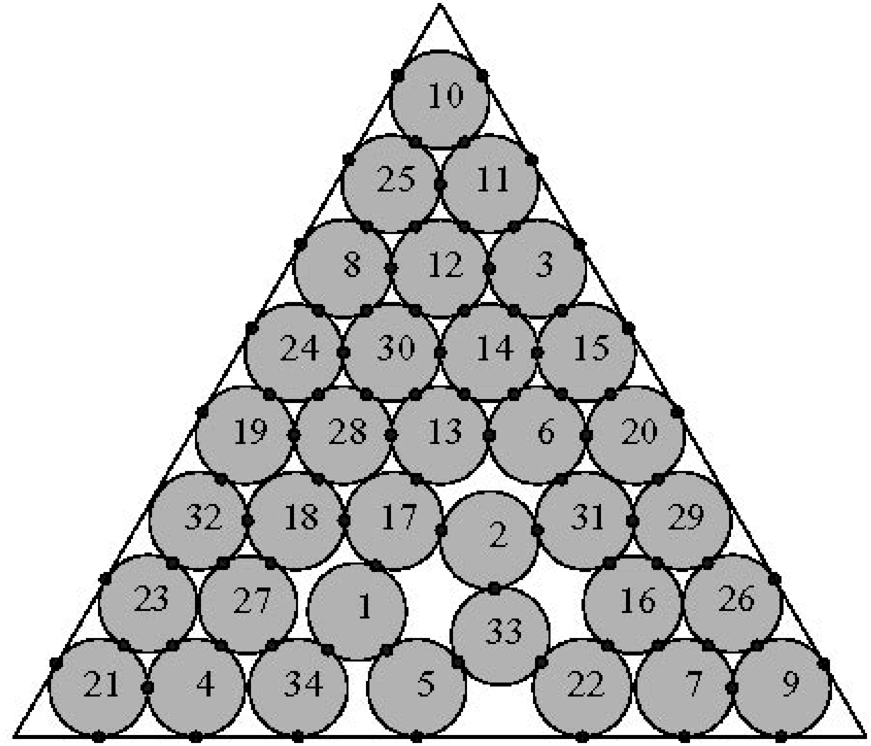


92



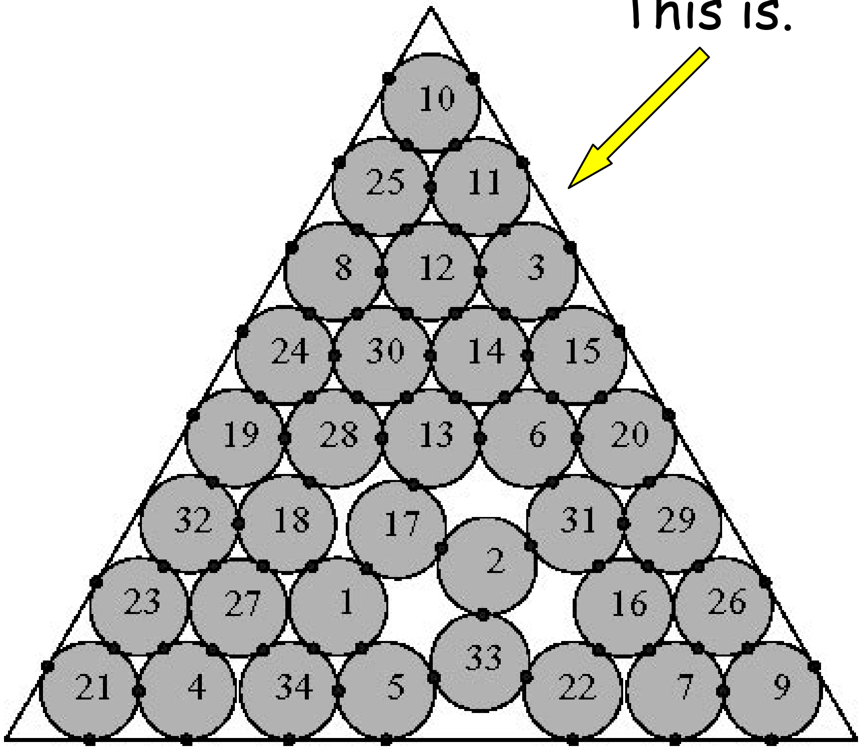
34



34

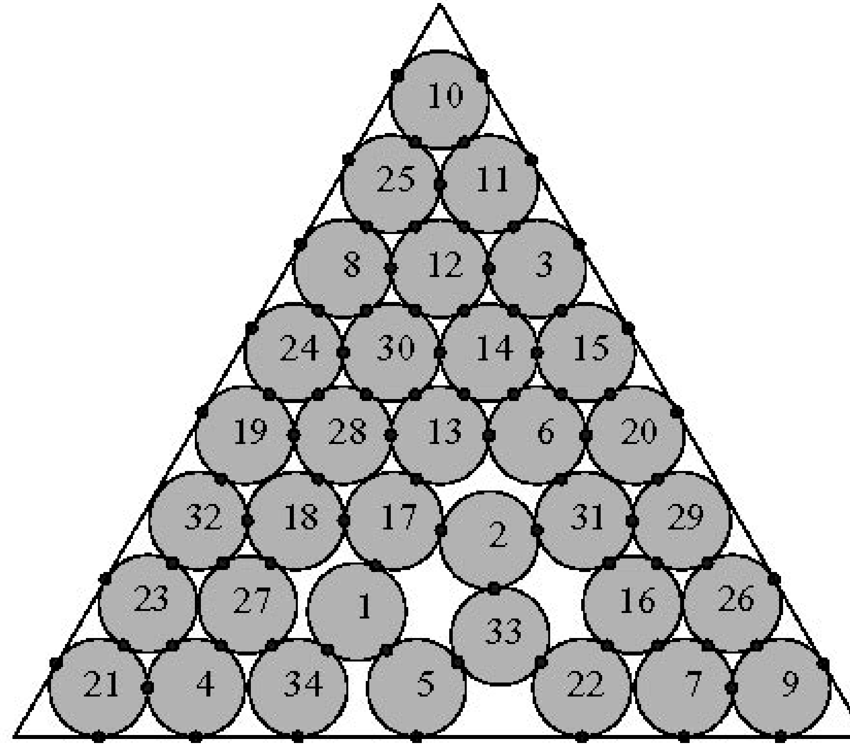
Which is better ??

This is.



34

0.142869646754496 84 bonds



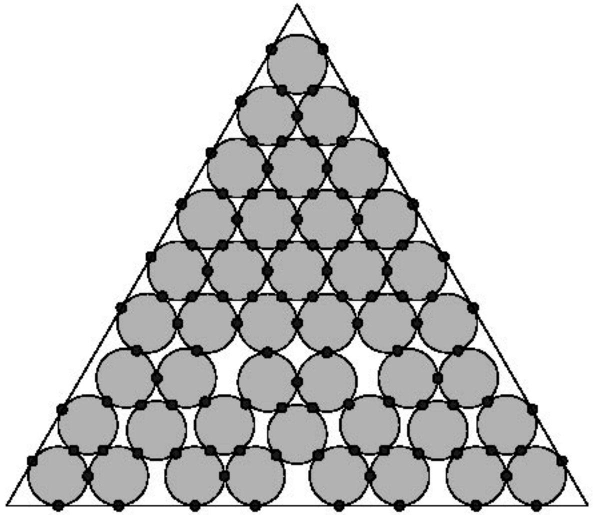
34

0.142867647681844 83 bonds

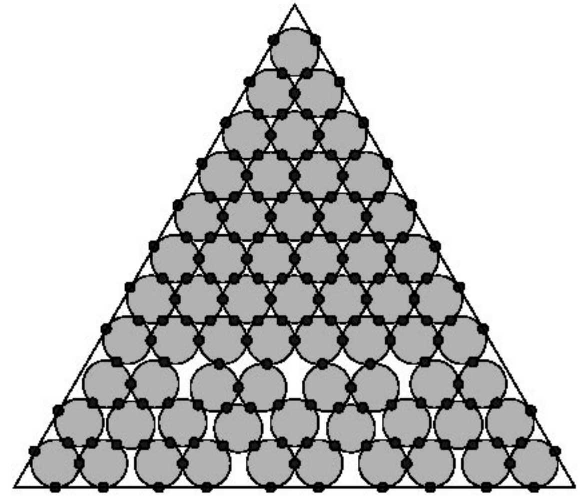
Which is better ??

(radius of discs if reduced triangle has side 1)

Patterns ?



42



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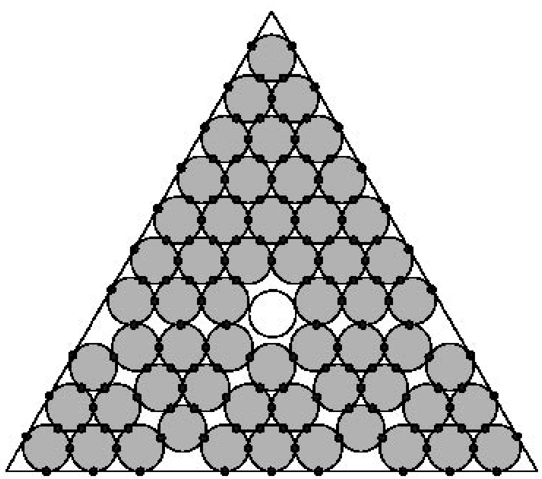

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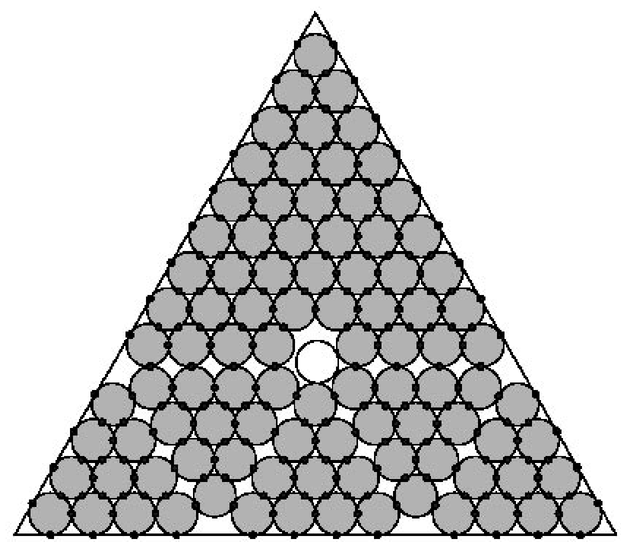
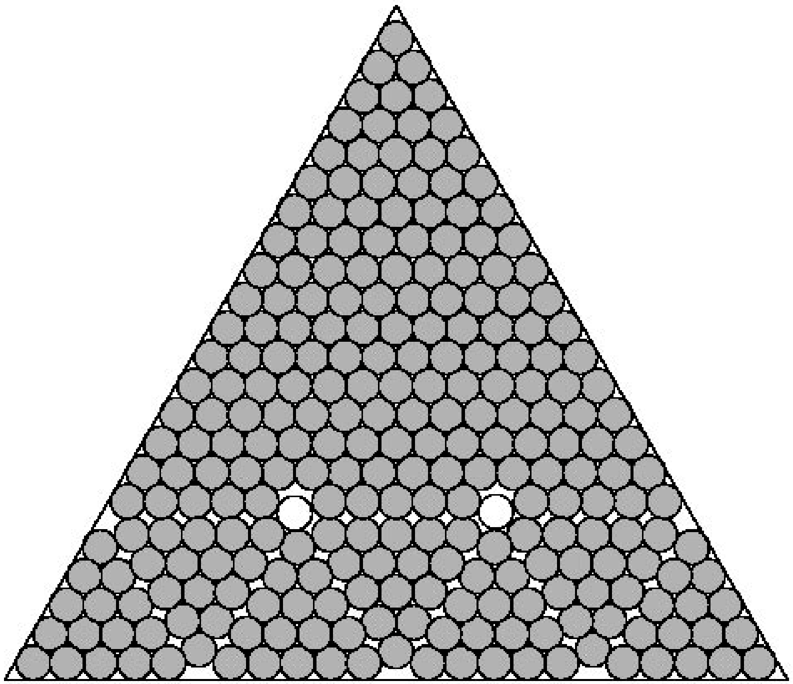
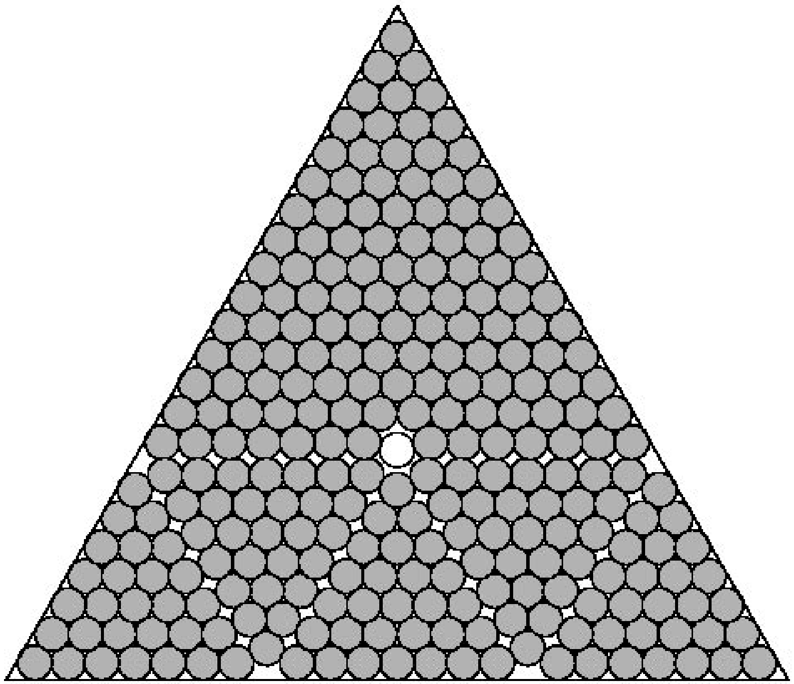


95



n = 258



n = 260

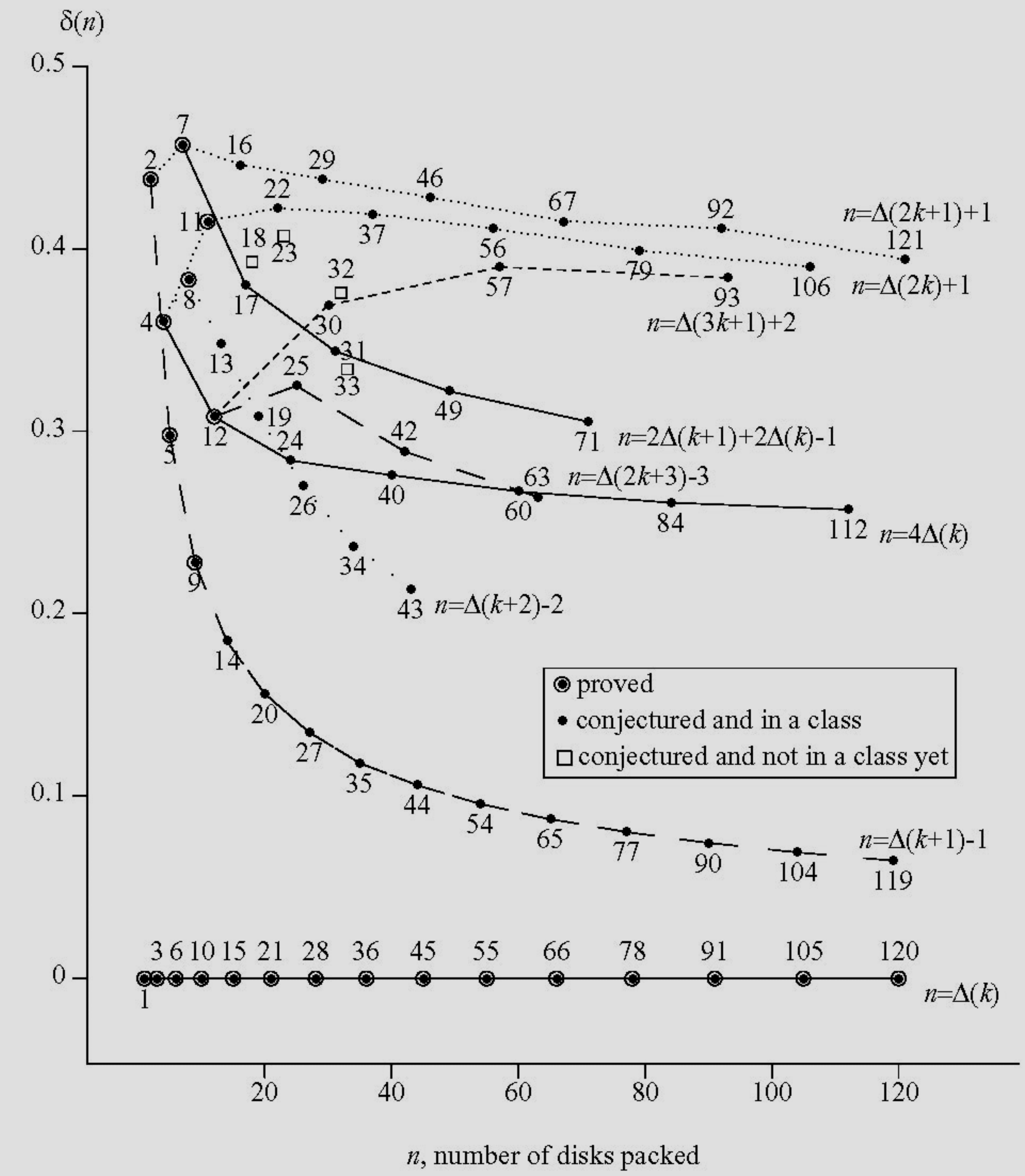
Part of general patterns?

T(n) :=minimum side length of equilateral triangle into which n unit discs can be packed

$$\delta(n) := T(n) - \tfrac{1}{2}\left(-3 + \sqrt{8n+1}\right)$$

$$\Delta(n) := \tfrac{1}{2}n(n+1)$$

δ(n)



Conjecture:

$T(\Delta(n) + 1) - T(\Delta(n)) > \varepsilon$

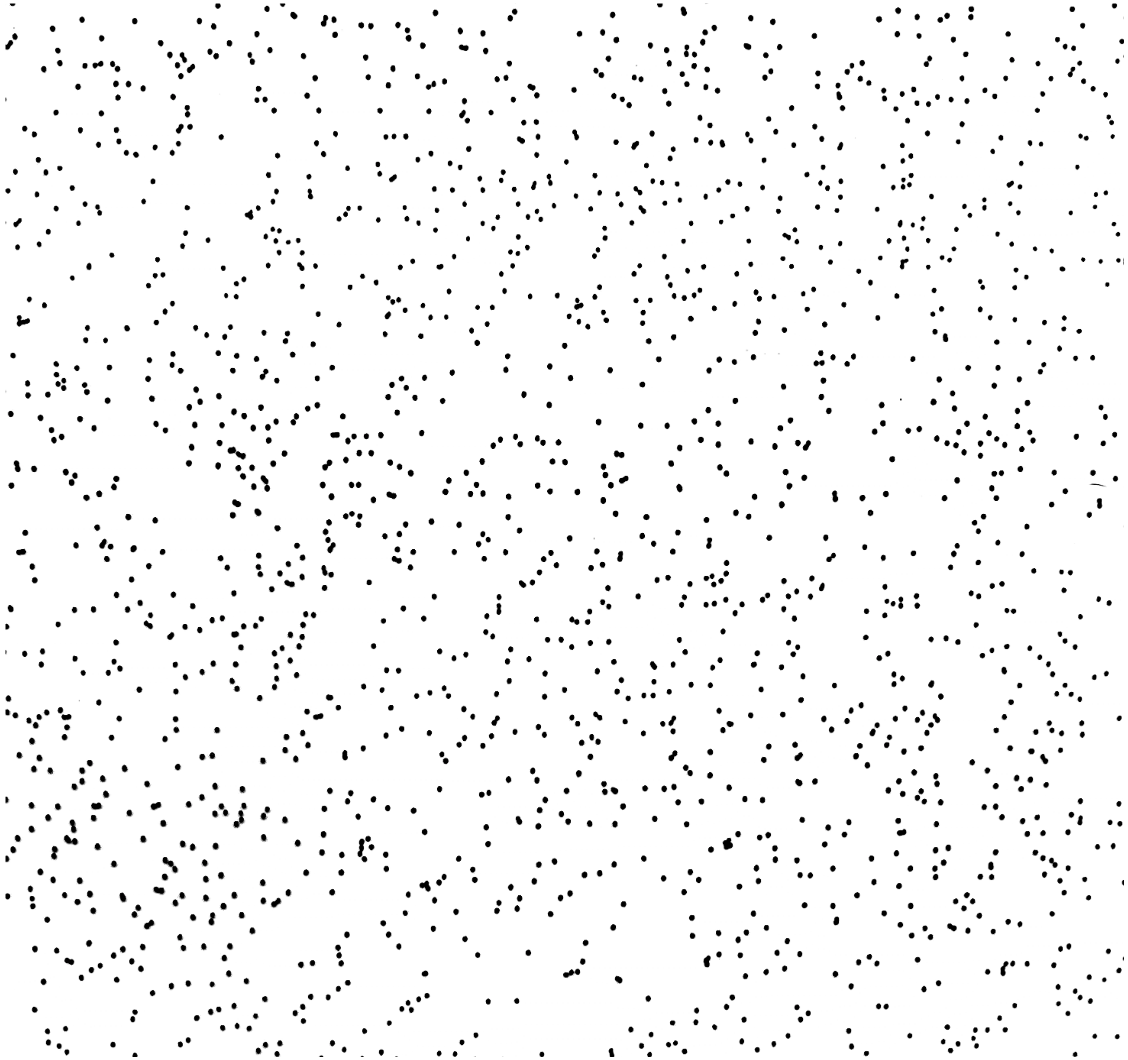
for some $\varepsilon > 0$.

Packings were generated by an "event driven" billiards simulation algorithm written by Boris Lubachevsky (formerly of Bell Labs). They were designed in part to understand crystal growth in the presence of irregularities.

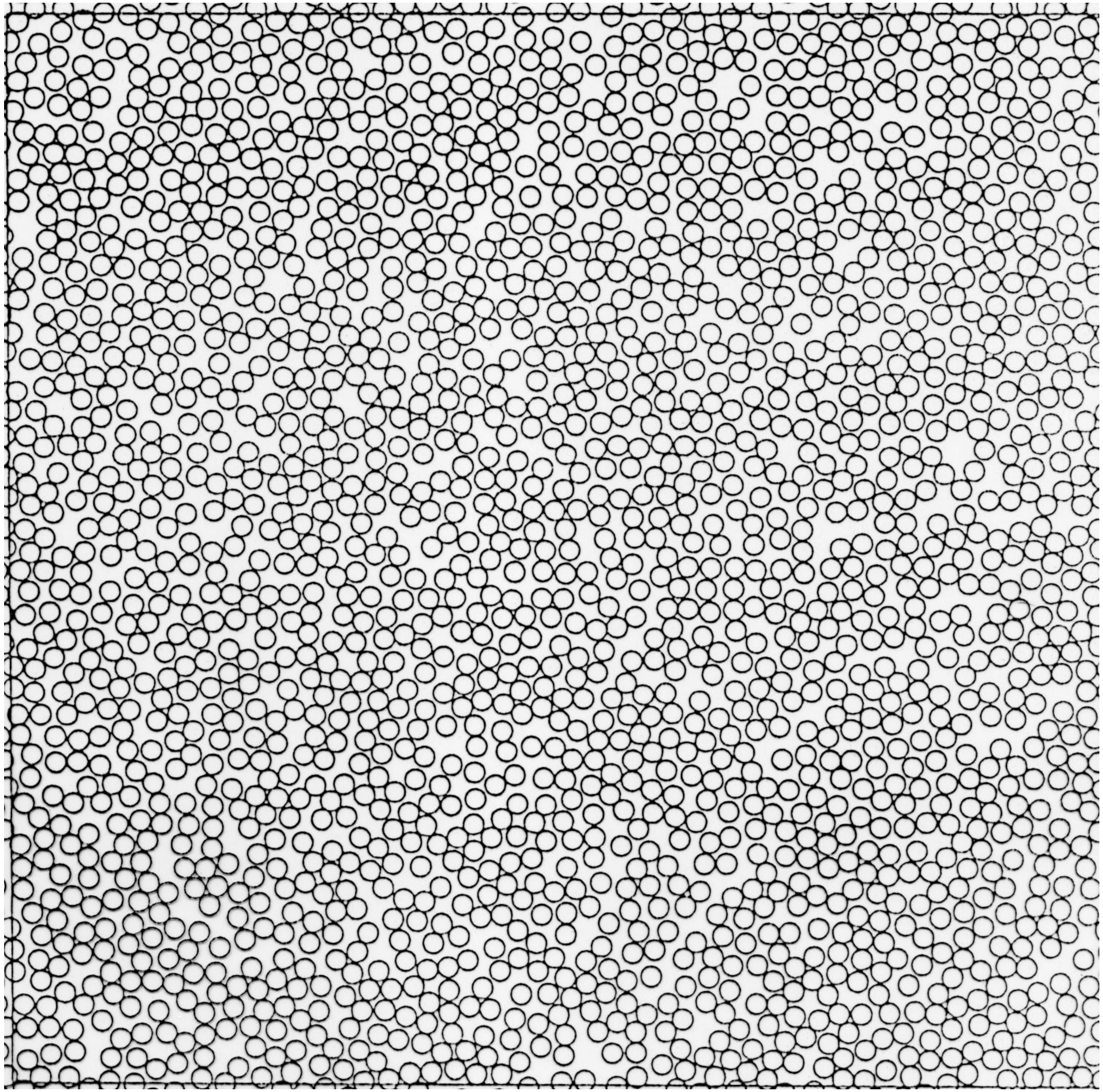
We start with very small discs with random positions and velocities and them let bounce around elastically while slowing increasing in size, until after many millions of bounces, they become "stuck".

You then repeat this process thousand of times!

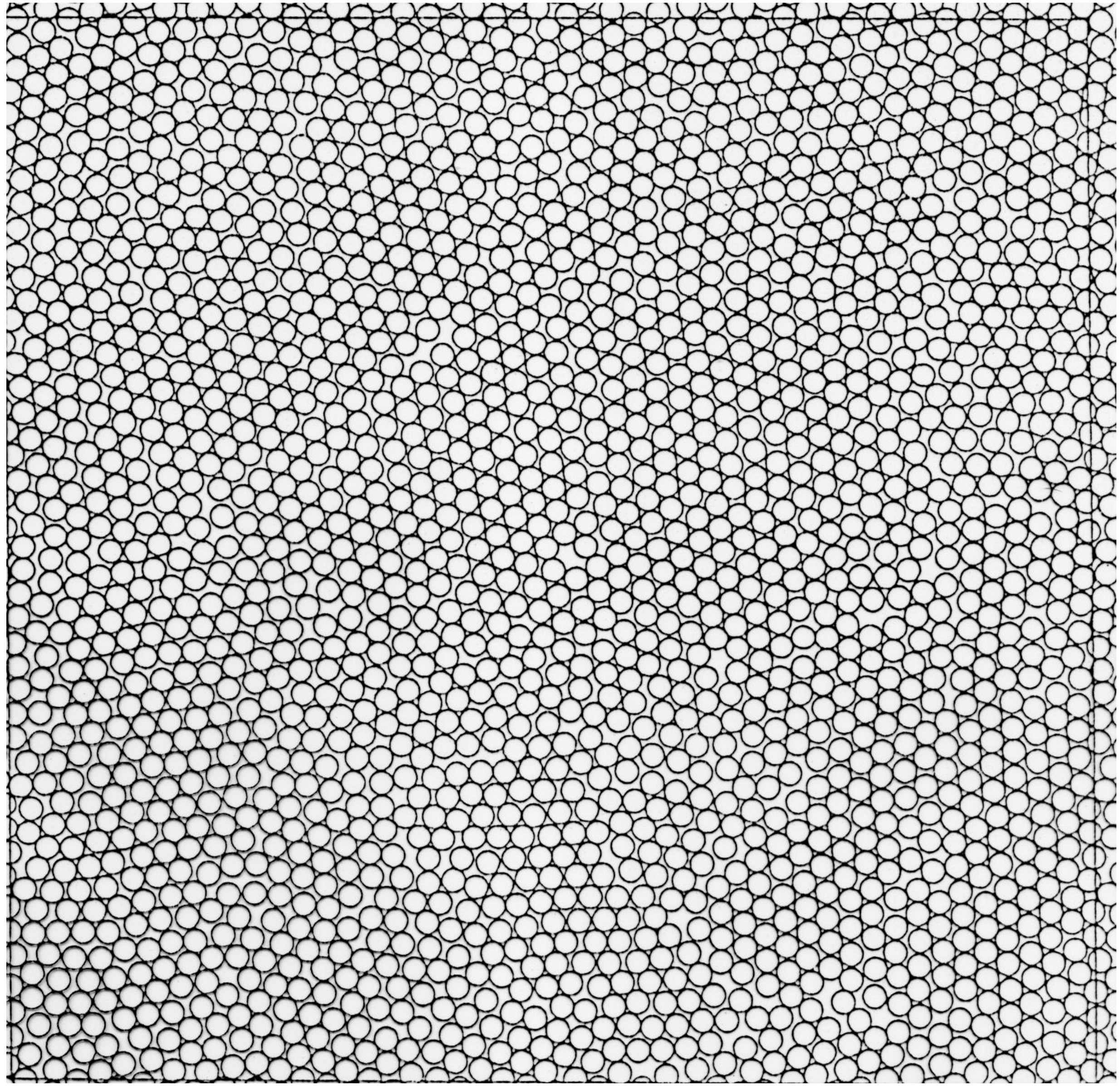
Stillinger, Frank H.; Lubachevsky, Boris D. Crystalline-amorphous interface packings for disks and spheres. *J. Statist. Phys.* **73** (1993), no. 3-4, 497--514



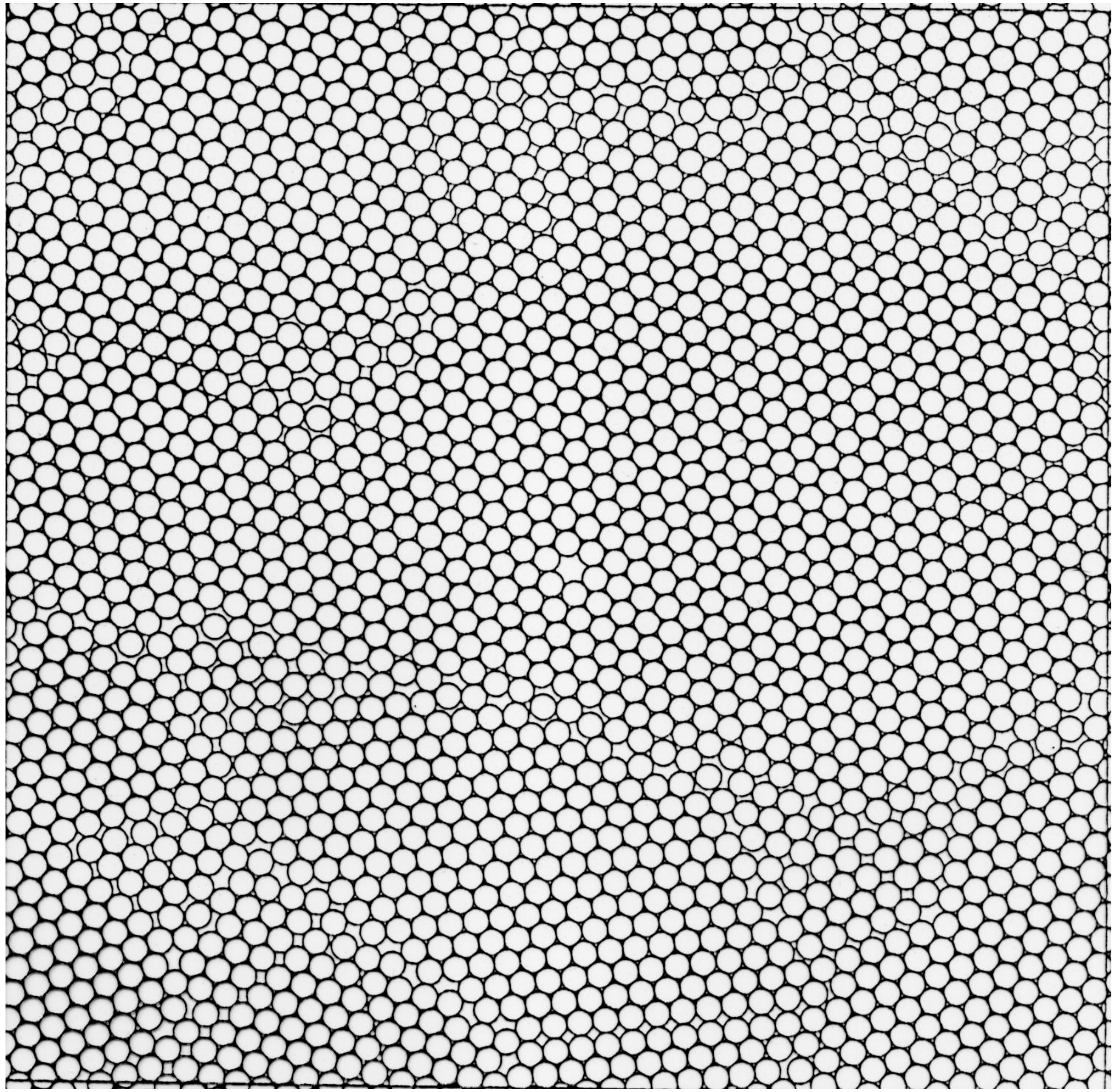
2000 random points



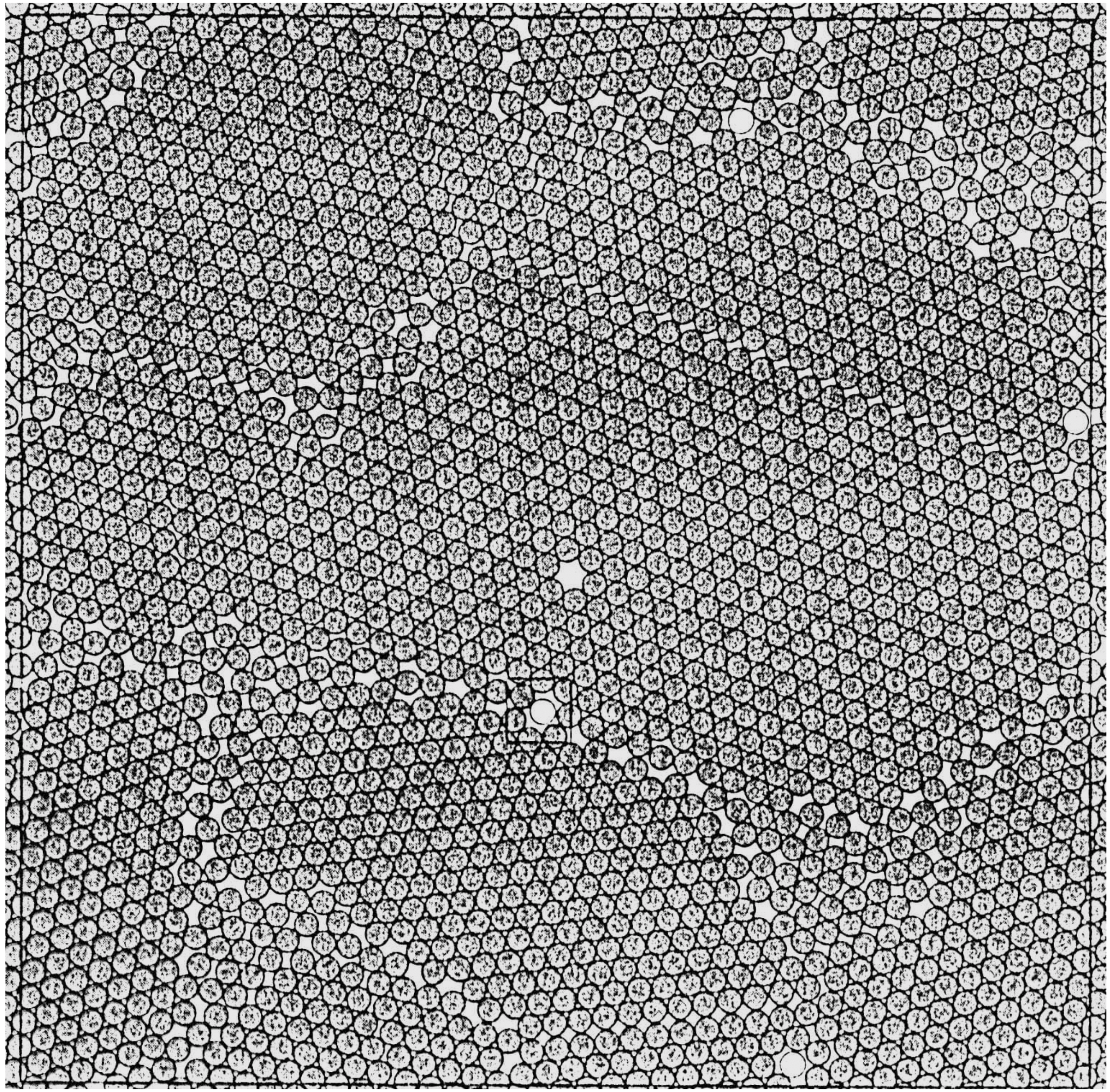
After 4×10^4 collisions



After 4 x 10^5 collisions



After 4 x 10^6 collisions



After 4×10^7 collisions

H. Melissen, Densest packings of congruent circles in an equilateral triangle, Amer. Math. Monthly 100 (1993), 916-925

B. D. Lubachevsky and RLG, Dense packings of equal discs in an equilateral triangle: from 22 to 34 and beyond, Electronic J. Combinatorics 2 (1995), #A1

H. Melissen, Packing and covering with circles, Ph.D. dissertation, Utrecht University, 1997, viii + 180 pp.

Packing discs into squares

This is one of the classic disc packing problems.

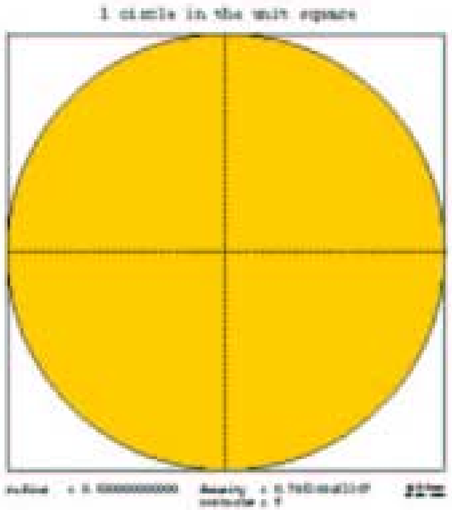
2. Old Results

The problem of packing circles into different geometrical shapes has received much attention since the seminal work of Fejes Tóth [7]. A recent survey of results and problems still open can be found in [3]. One of the most natural and most studied of these problems is that of packing circles in a square.

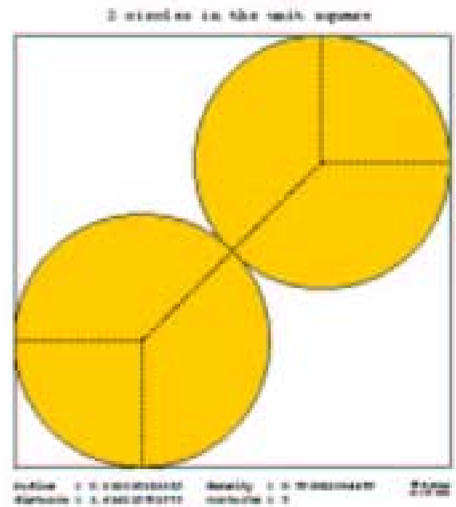
This problem was solved for up to nine circles in the 1960s by Graham, Meir and Schaer; the proofs of these cases have been reported in [12], [18], [20], and [22]. The proofs for $n \leq 5$ are easy, whereas the cases $6 \leq n \leq 9$ require more elaborate mathematical tools. For example, for $n = 5$ we can divide the square into four subsquares as indicated in Fig. 1. Now at least one square must contain two points due to the pigeon-hole principle, so the length of the diagonals in the subsquares ($\sqrt{2}/2$) upper-bounds d_5. This is also a lower bound, since in the solution in Fig. 1 (which is the only possible optimal solution), this is the smallest distance between two points. Thereby $d_5 = \sqrt{2}/2$.

For $n \geq 10$, only the optimal packings of 14 [26], 16 [24], 25 [25], and 36 [10] circles have been proved by hand.

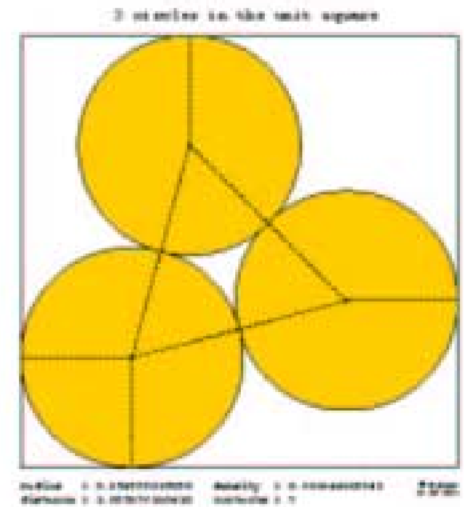
N = 1 *



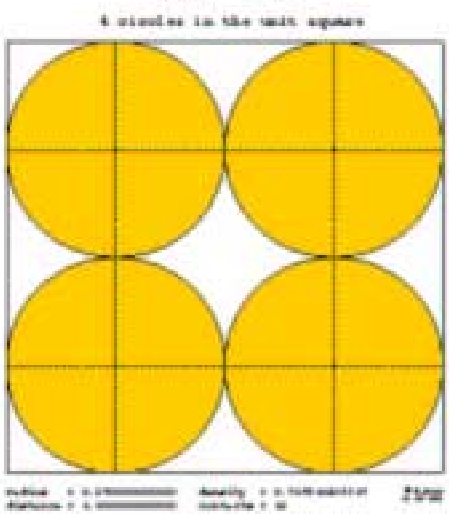
N = 2 *



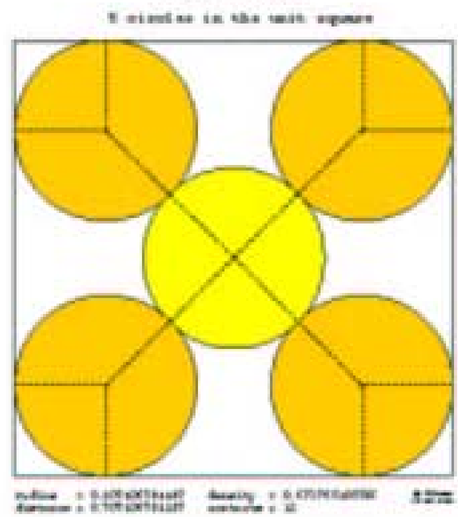
N = 3 *



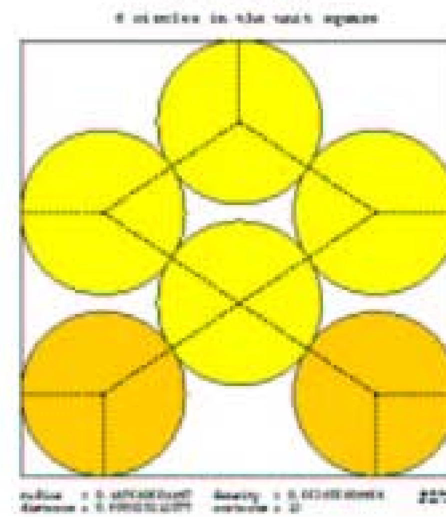
N = 4 *



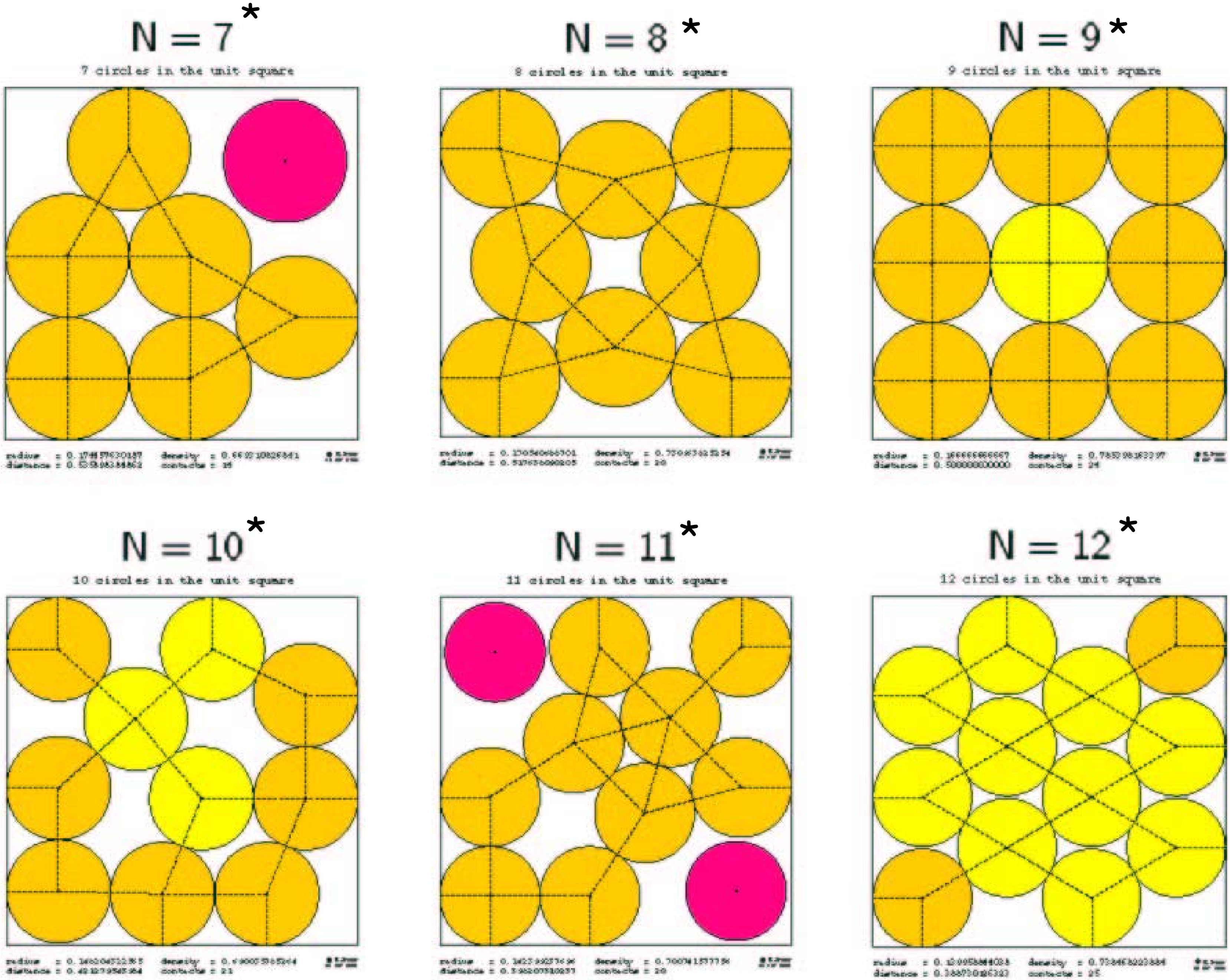
N = 5 *



N = 6 *

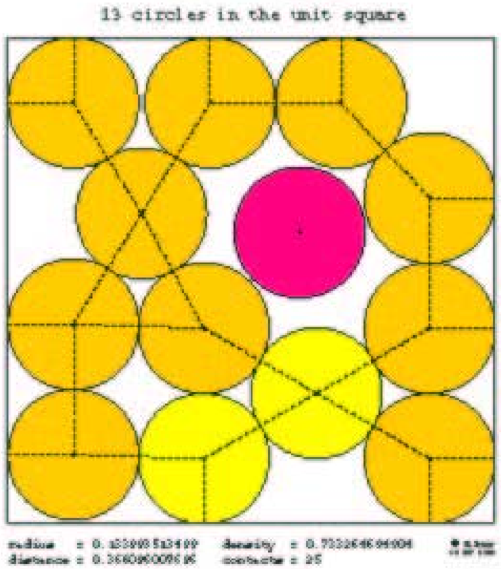


* $\Rightarrow$ optimal

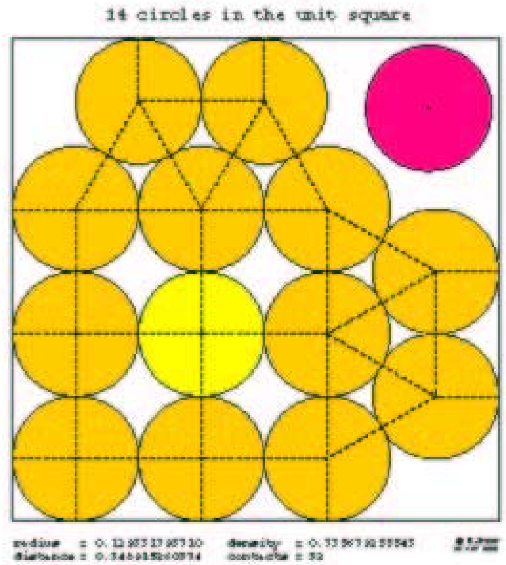


N = 7*
7 circles in the unit square
N = 8*
8 circles in the unit square
N = 9*
9 circles in the unit square
N = 10*
10 circles in the unit square
N = 11*
11 circles in the unit square
N = 12*
12 circles in the unit square

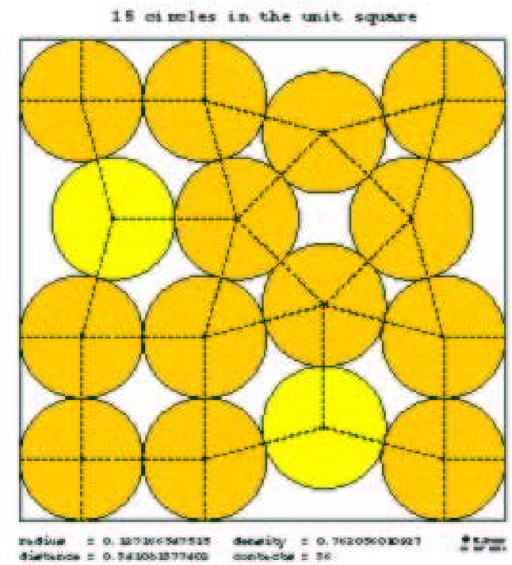
N = 13*



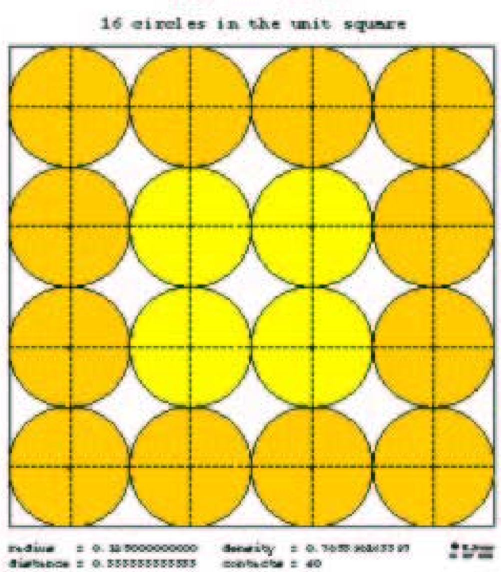
N = 14*



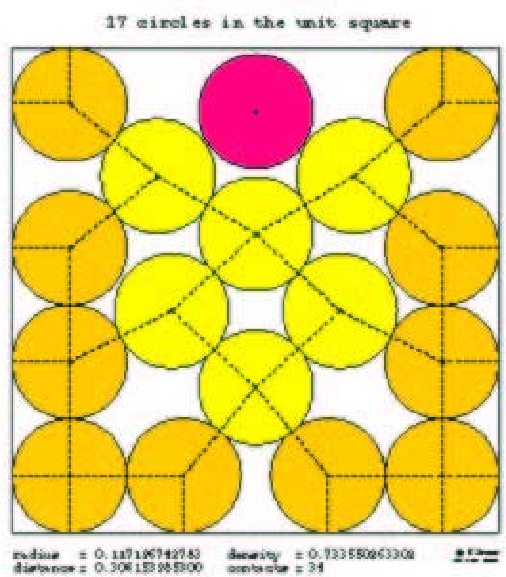
N = 15*



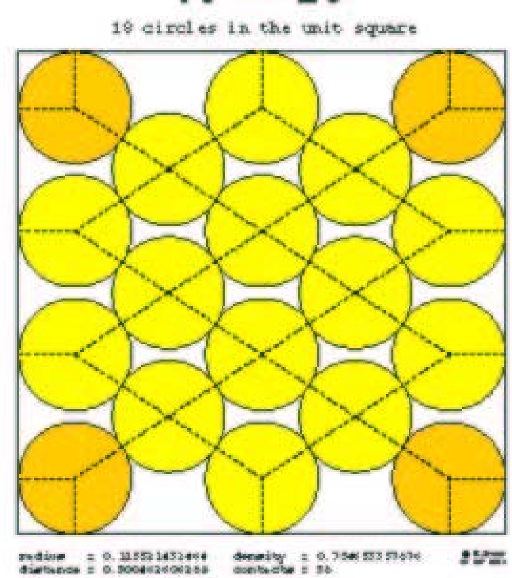
N = 16*

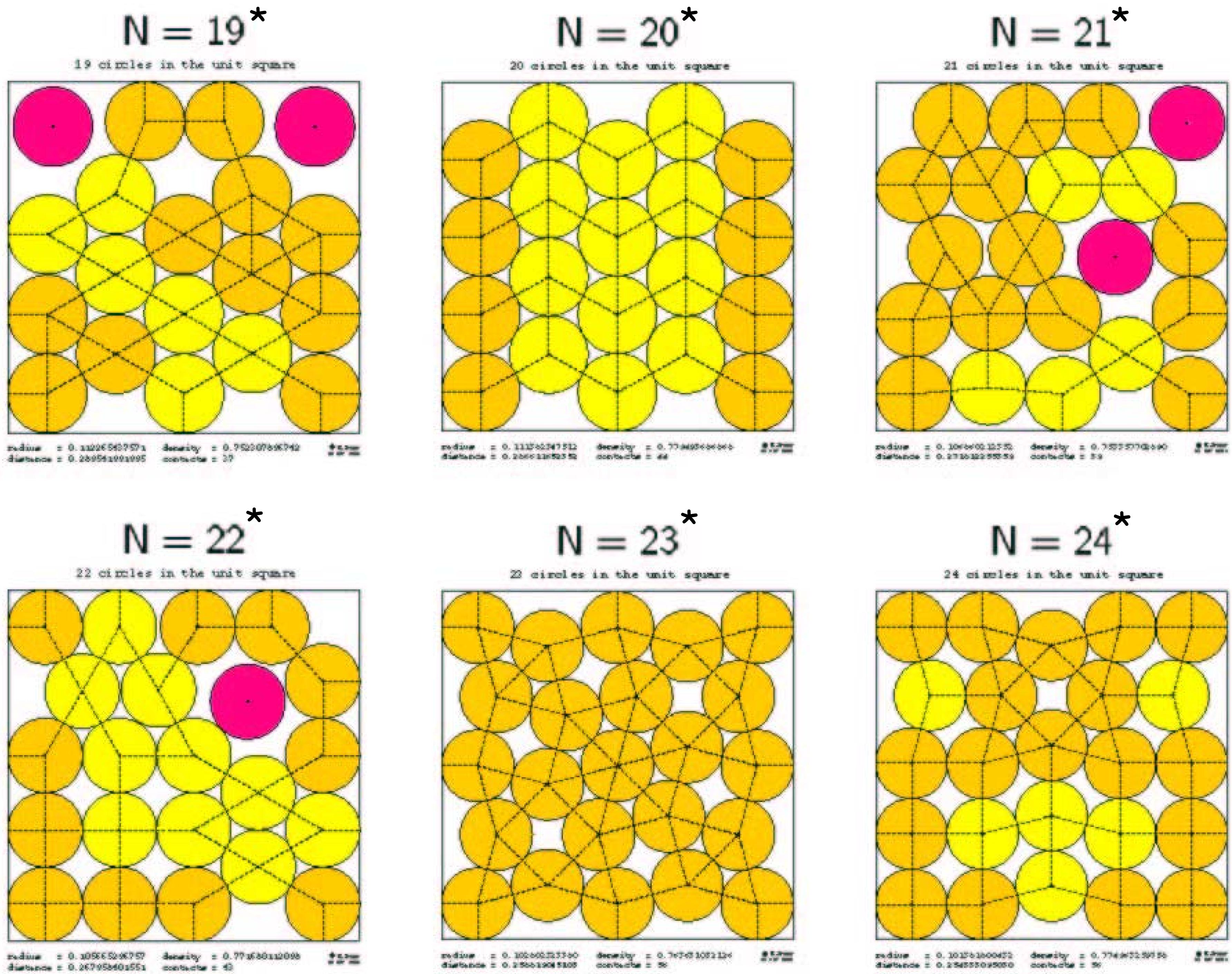


N = 17*



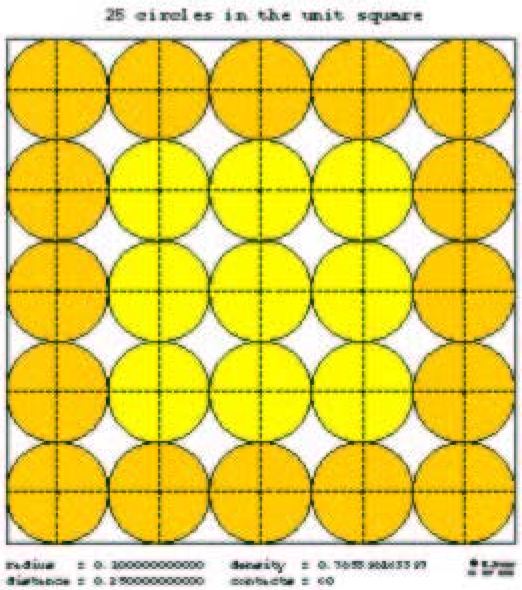
N = 18*



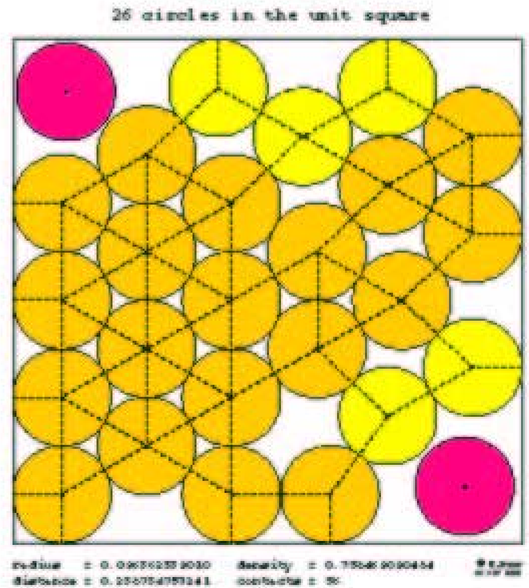


N = 19*
19 circles in the unit square
N = 20*
20 circles in the unit square
N = 21*
21 circles in the unit square
N = 22*
22 circles in the unit square
N = 23*
23 circles in the unit square
N = 24*
24 circles in the unit square

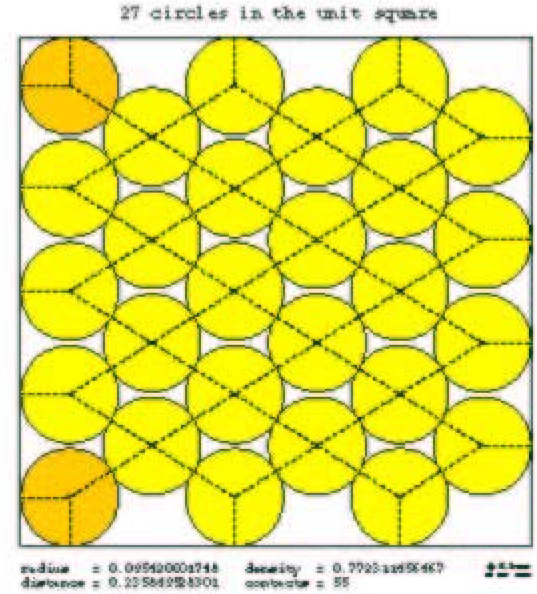
$N = 25^*$



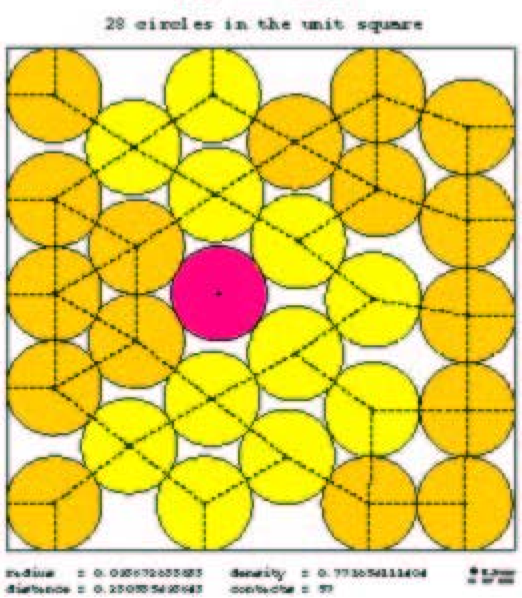
$N = 26^*$



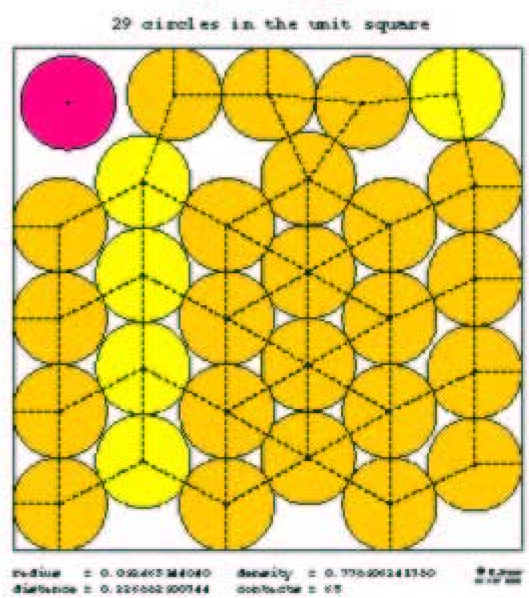
$N = 27^*$



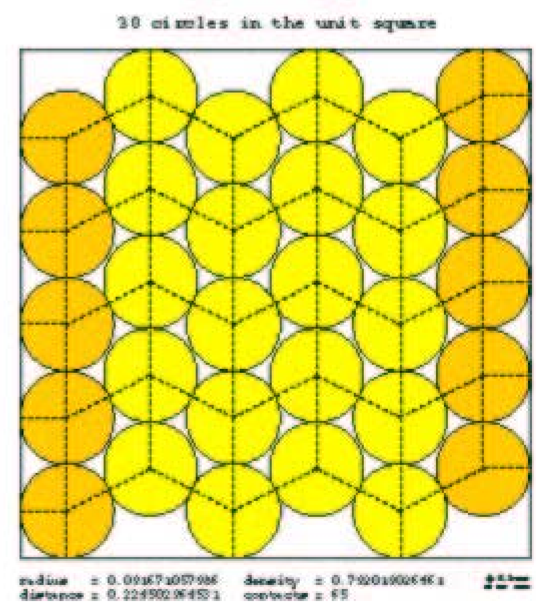
$N = 28$



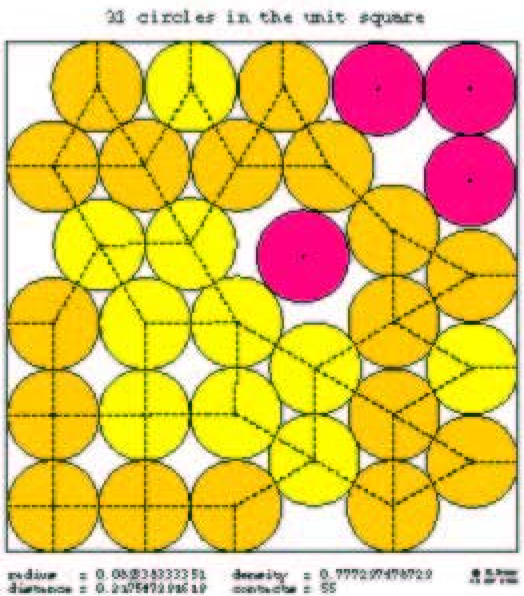
$N = 29$



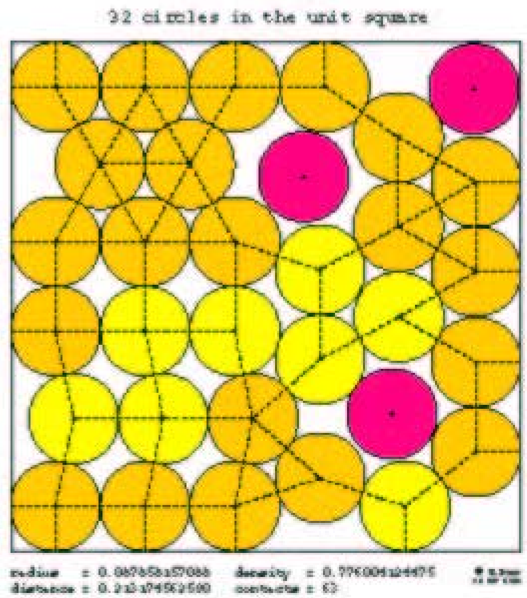
$N = 30$



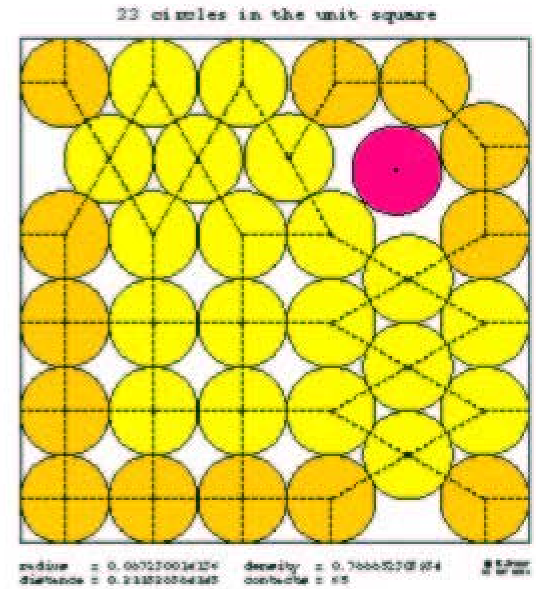
N = 31



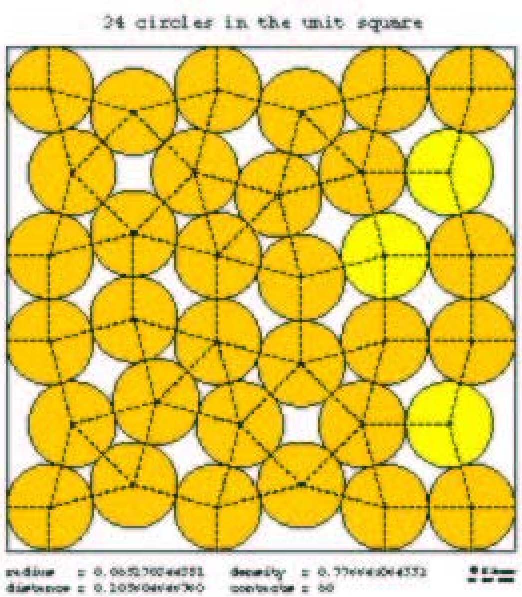
N = 32



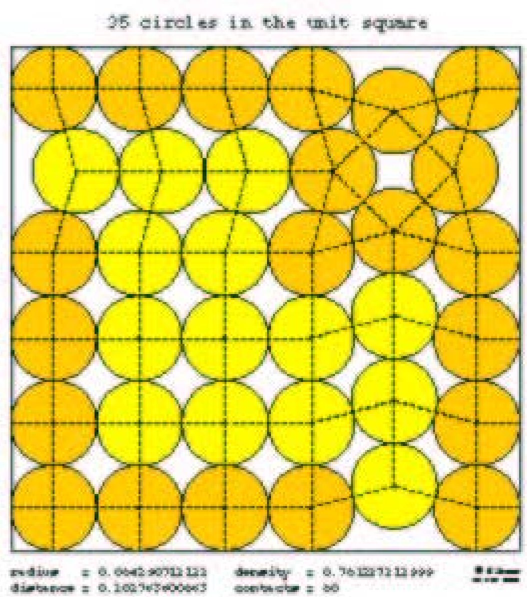
N = 33



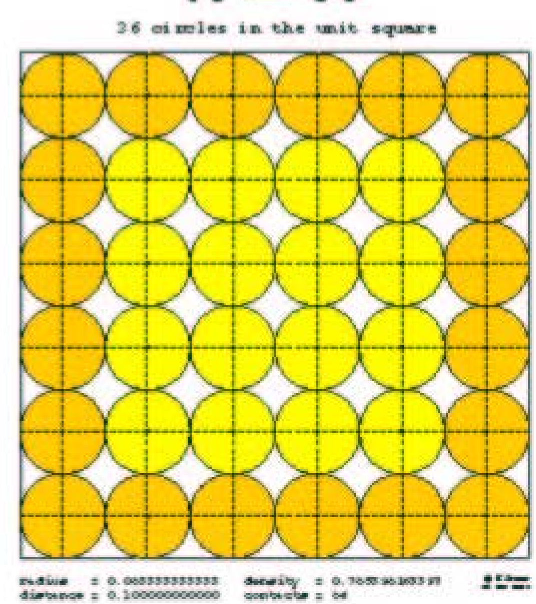
N = 34



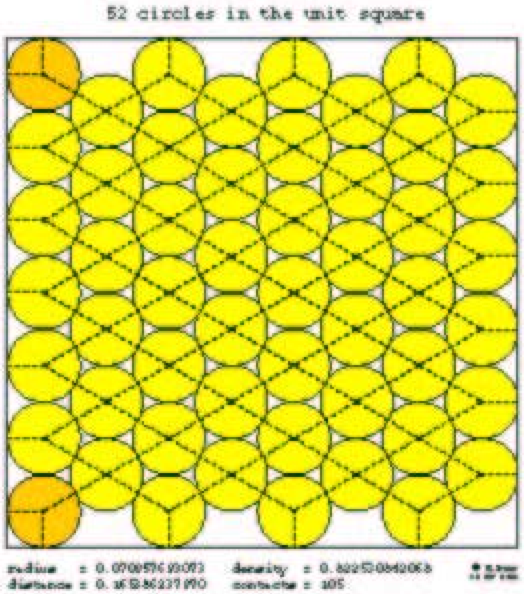
N = 35



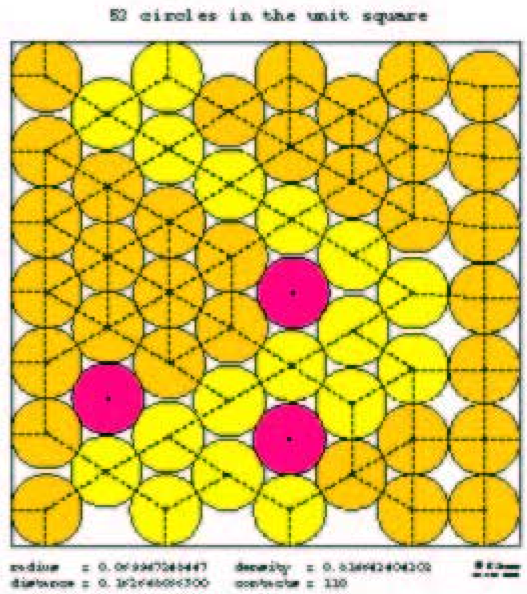
N = 36*



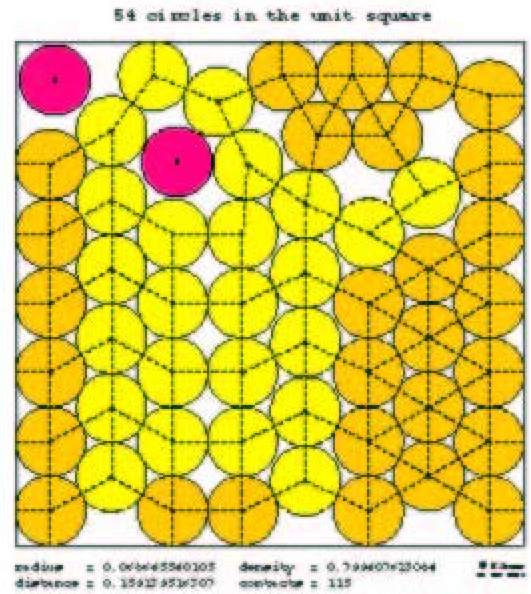
N = 52



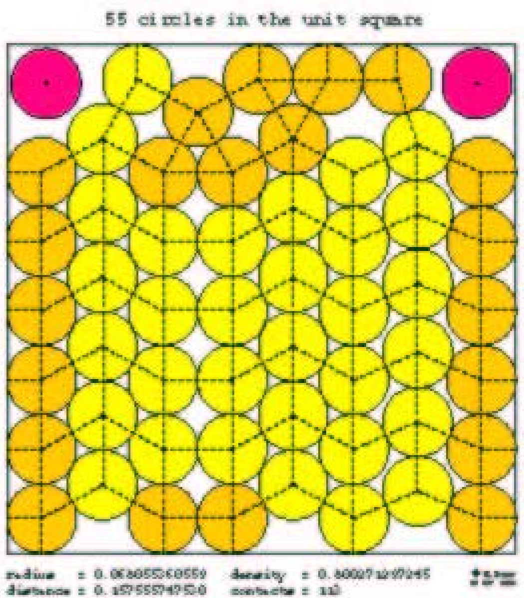
N = 53



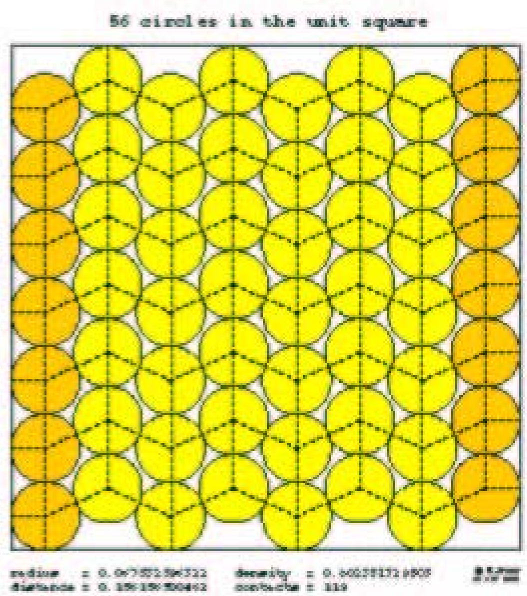
N = 54



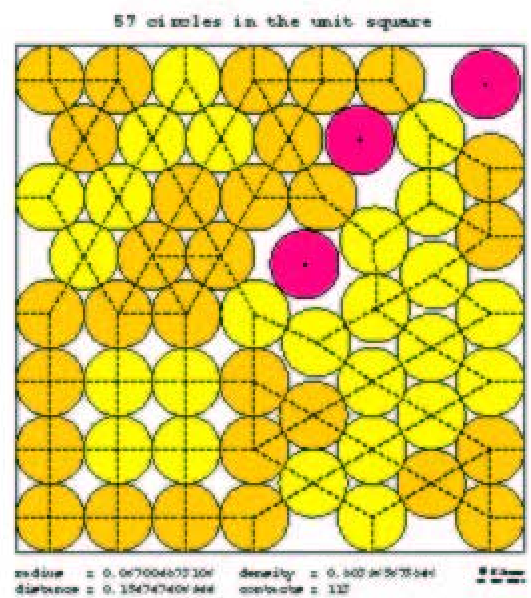
N = 55



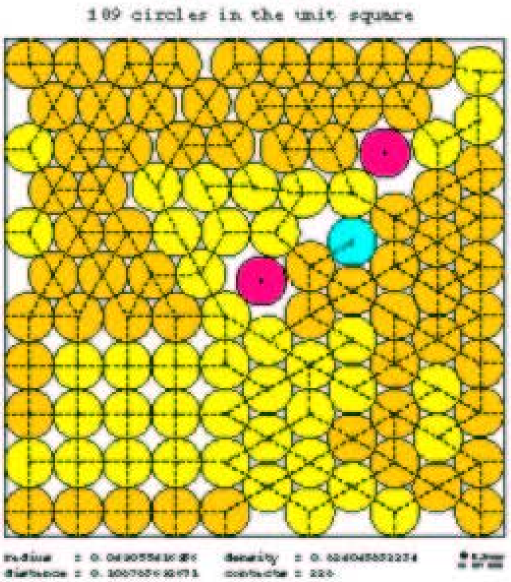
N = 56



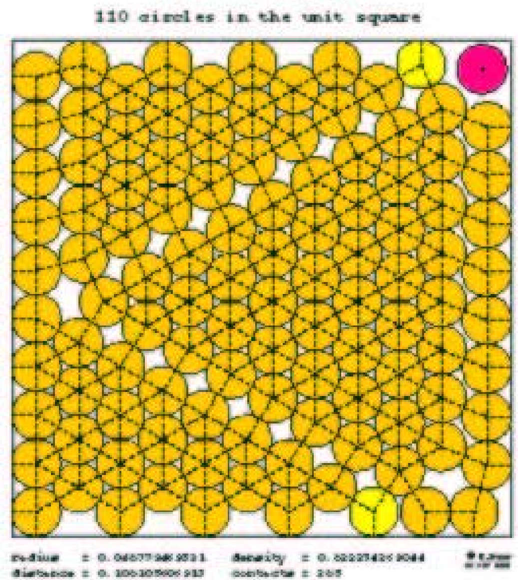
N = 57



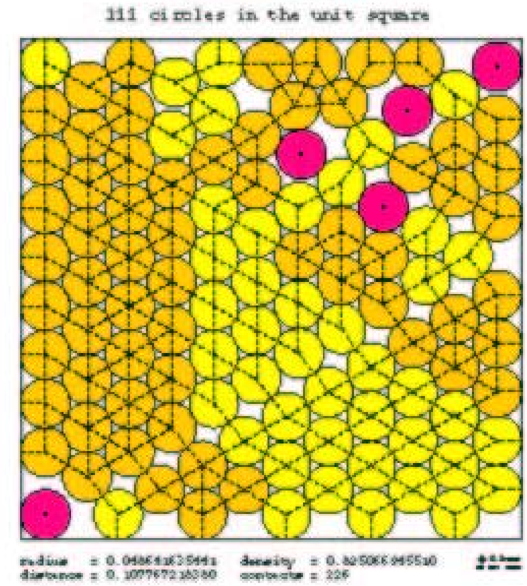
N = 109



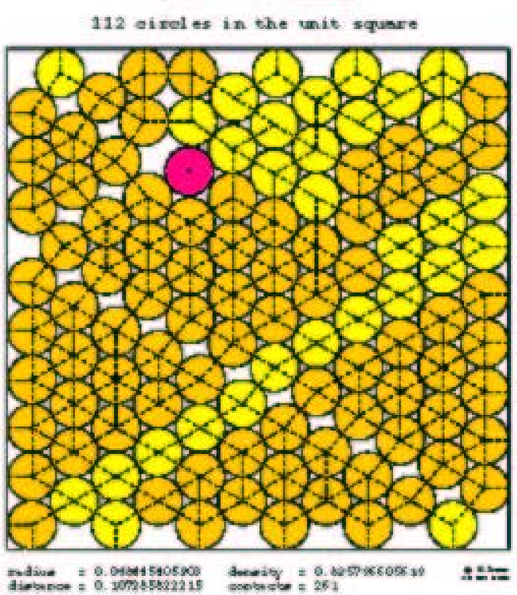
N = 110



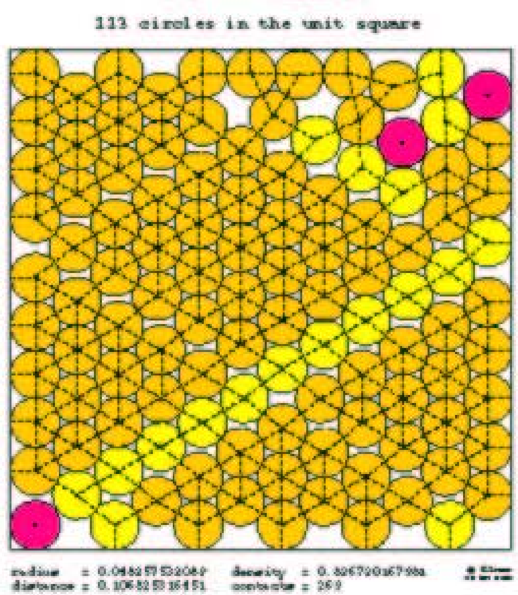
N = 111



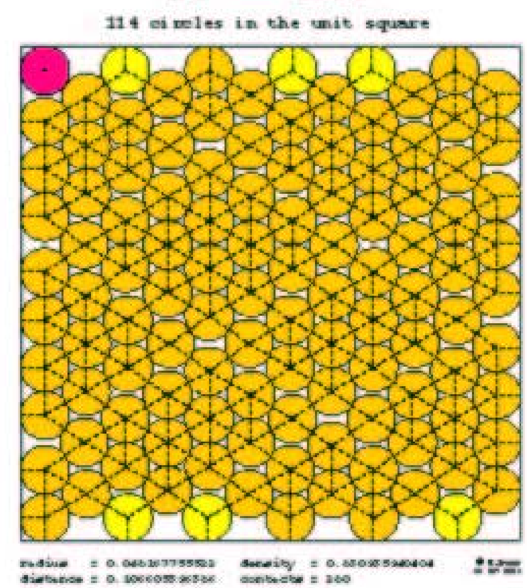
N = 112



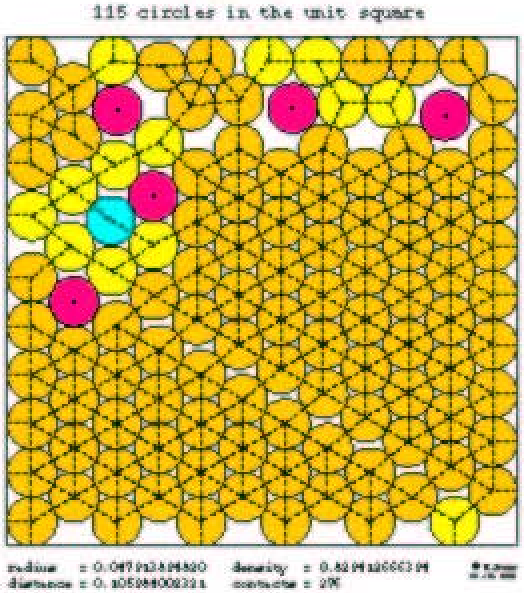
N = 113



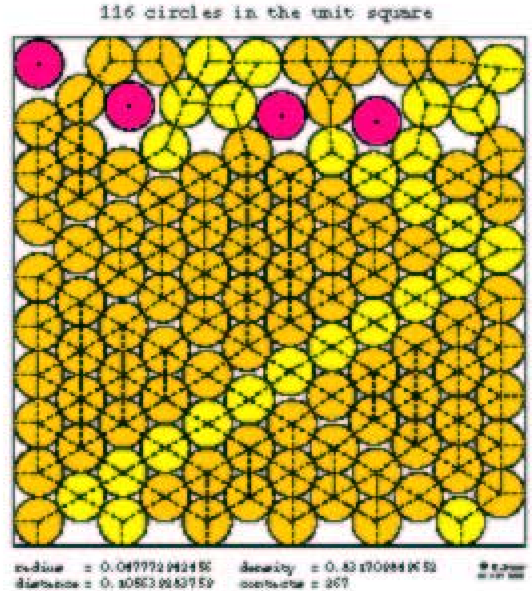
N = 114



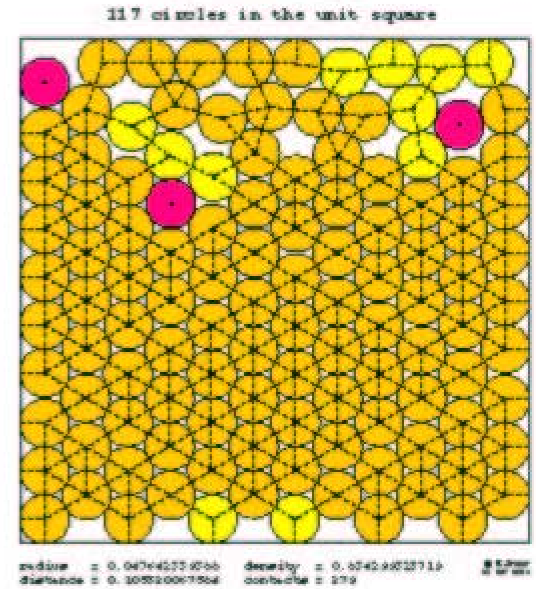
N = 115



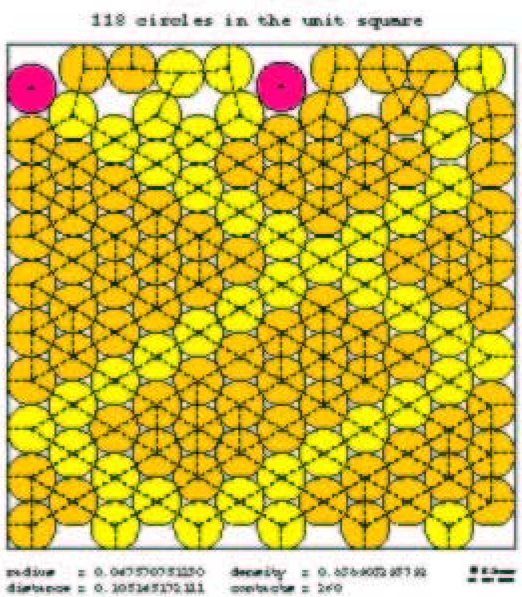
N = 116



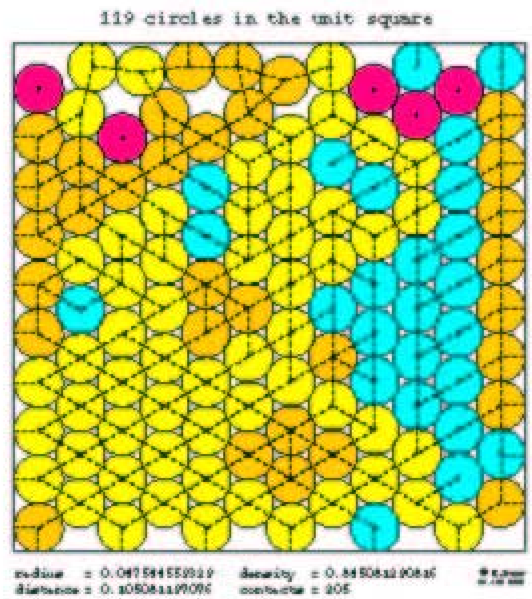
N = 117



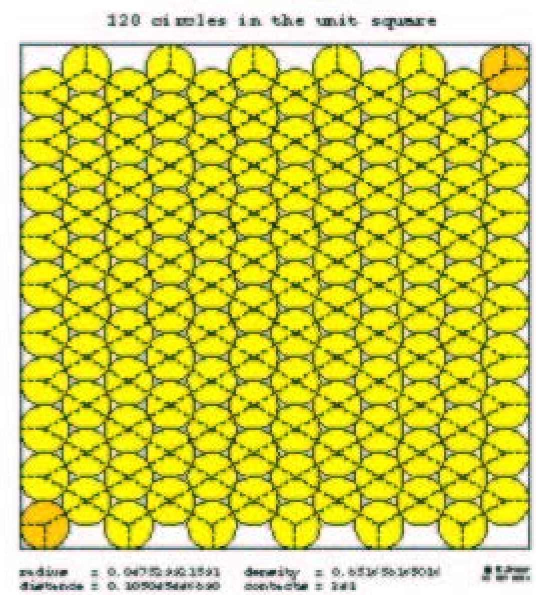
N = 118



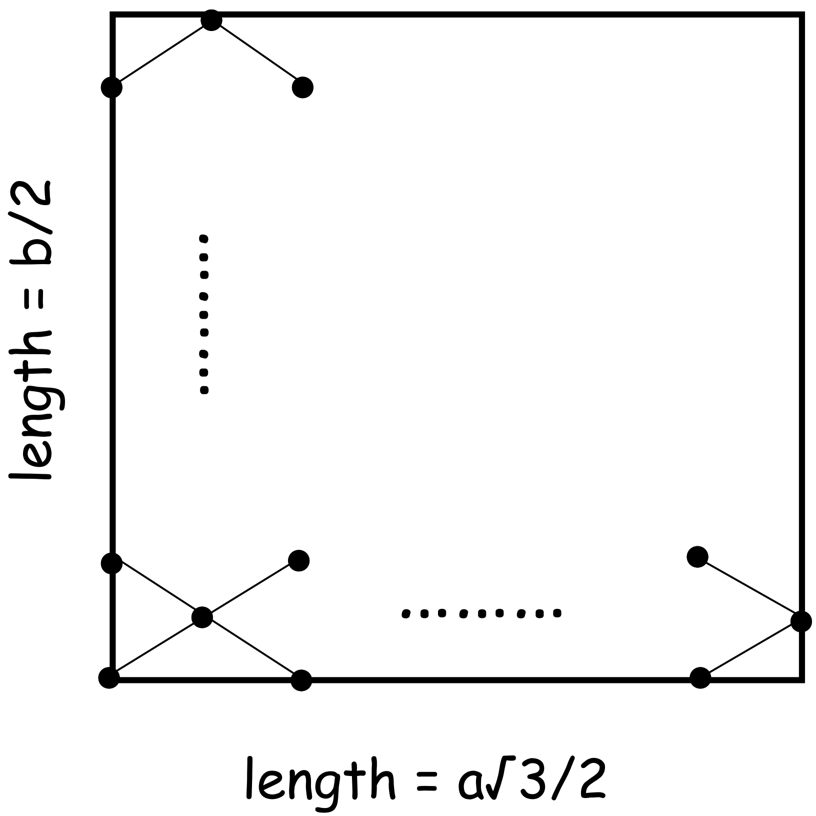
N = 119



N = 120



Why is the packing for 120 discs so good?



a+1 columns (an even #)

b+1 rows

number of points (= disc centers) is (a+1)(b+1)/2

If $b = a\sqrt{3}/2 - \varepsilon$ then can pack a slightly distorted copy of this disc arrangement into a square.

So we need to have a/b slightly less than √3.

Use the (under-)convergents to √3.

These are: $\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}, \frac{265}{103}, \ldots, \frac{b_n}{a_n}$

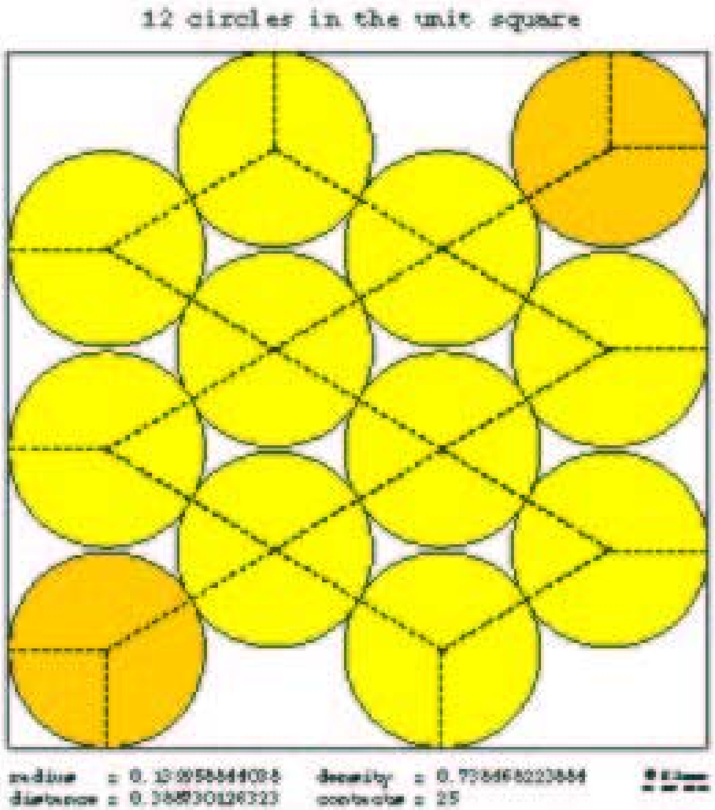
The corresponding values of the number of discs $N = \frac{1}{2}(a_n + 1)(b_n + 1)$

are 2, 12, 120, 1512, 13832,……

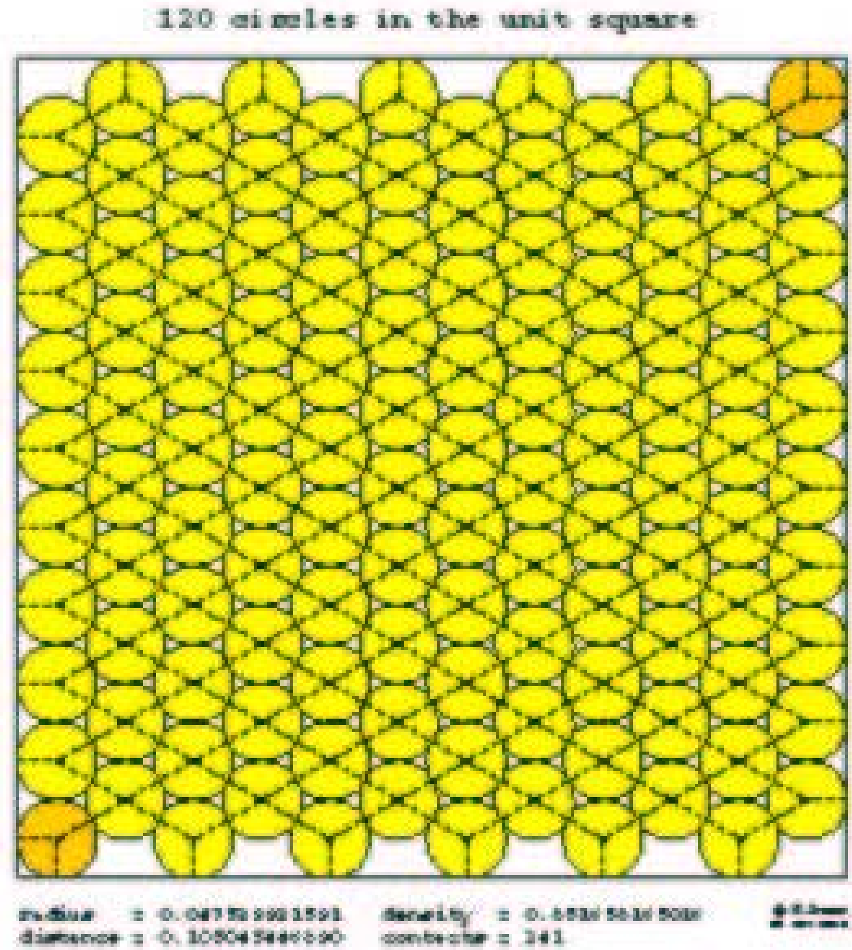
Conjecture: (Nurmela, Östergård -- 1999)

For these values of N, the "near-hexagonal" packing of N discs is optimal.

N = 12

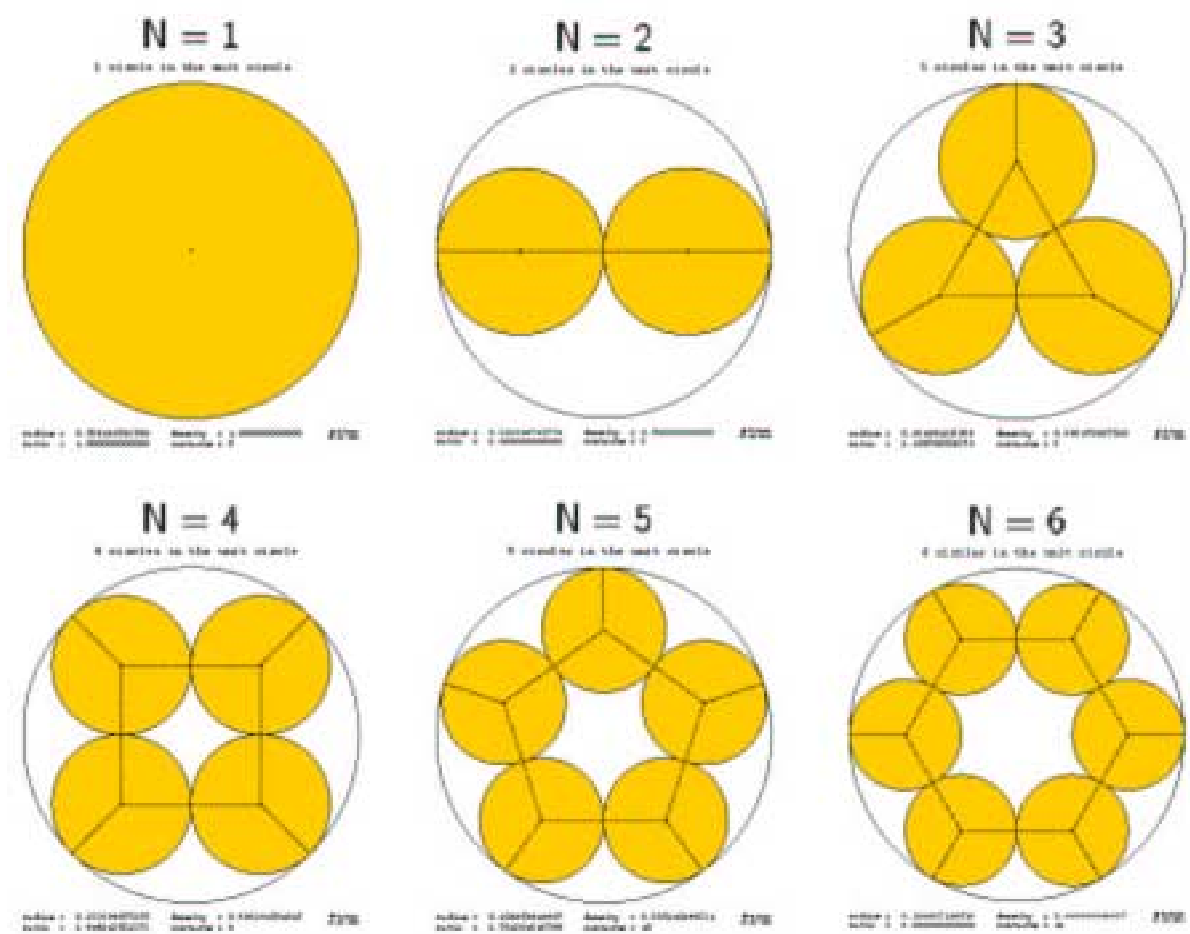


N = 120



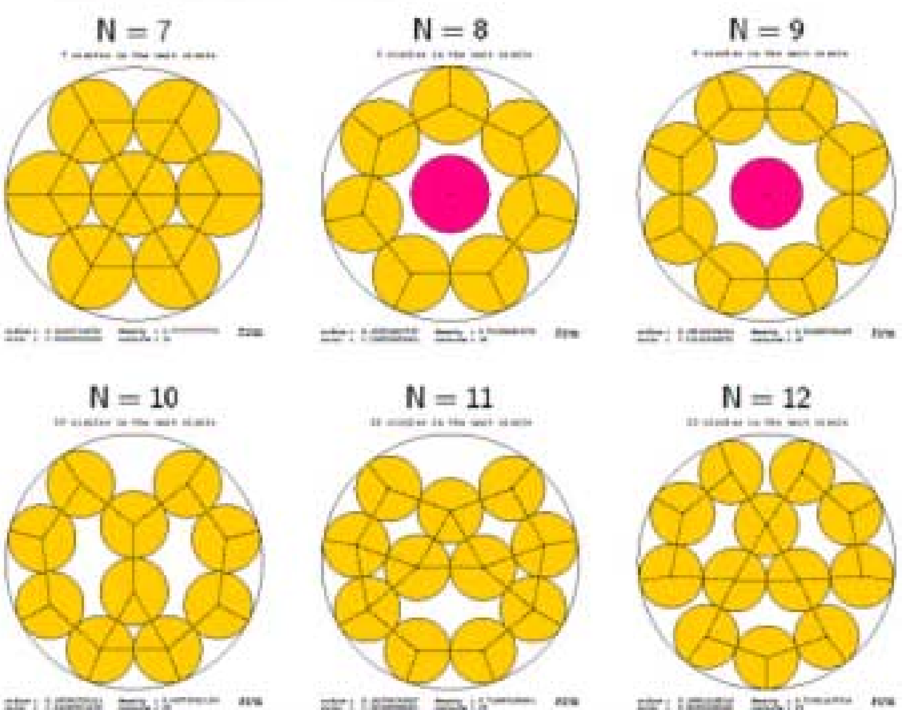
<u>How about packing discs in **circles**?</u>

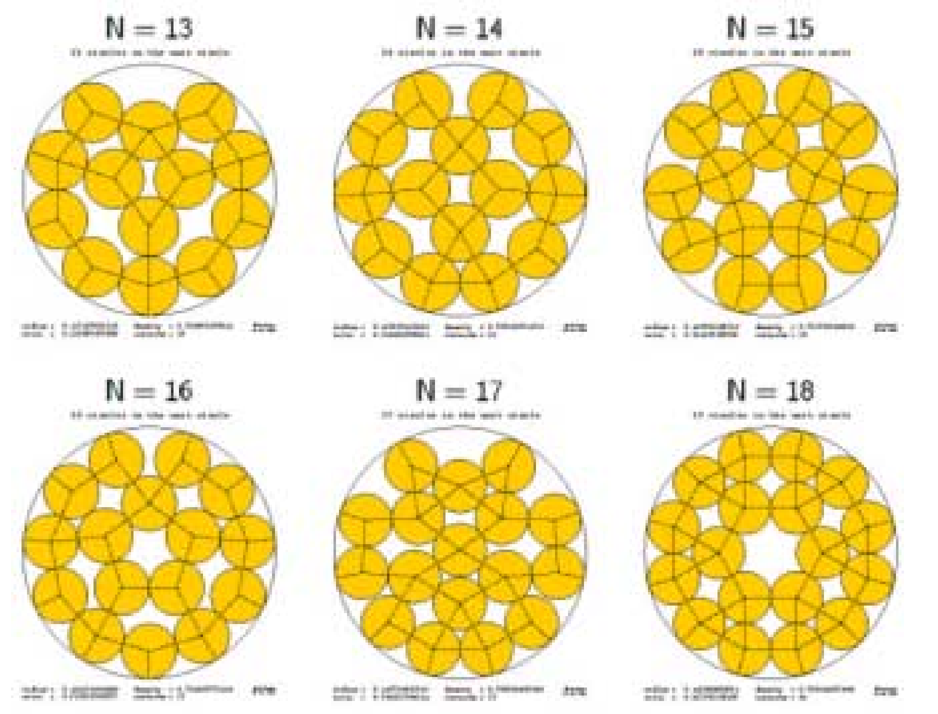
Of course, people have been doing that, too, for a long time (with even less success!)



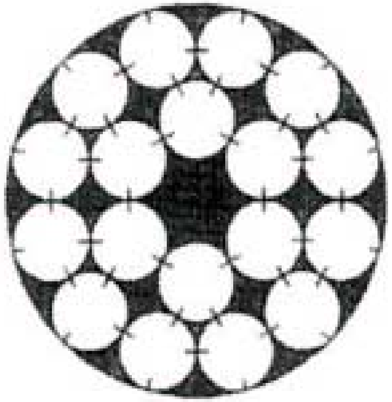
2. Earlier results

Kravitz [10] was, to our knowledge, the first to consider the problem of packing n congruent circles in a circle. In [10] packings of up to 19 circles are given without any optimality proofs.[3] Graham [6] and Pirl [17] independently proved optimality of packings of up to 7 and 10 circles, respectively. Pirl also presented good packings of up to 19 circles; some of these packings (for $n = 14, 16, 17$) were later improved by Goldberg [5], who also gave a packing of 20 circles. Goldberg's packing of 17 circles was further improved by Reis [18], who extended the range of n to 25. The packing of $n = 25$ is improved in this paper. Recently, Melissen [13] proved the optimality for the case $n = 11$.

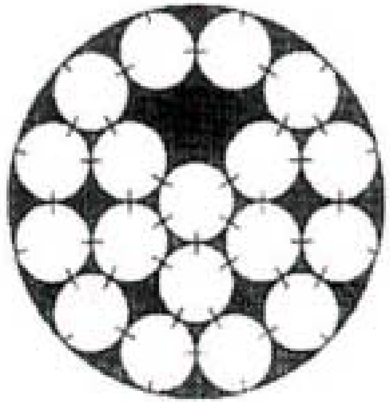




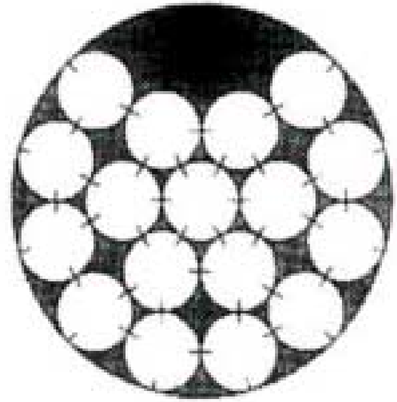
A peculiarity of the 18-circle case is that the best known packings of 18 circles have the same r as the best known packing of 19 circles. Three different, equally dense packings of 18 circles can be obtained by removing a circle in the packing of 19 circles in Fig. 2; see packings 18(a)–18(c) in Fig. 3. (A packing obtained by a congruence transformation, that is, by rotation or reflection, from another is considered the same.) In addition to these three packings, which apparently were the only ones known before, there are at least 7 more equally good packings. We suspect that there is no 11th equally good packing. At least, if one circle is removed from any of those 10 presumed best and then put back in the packing without overlaps with other circles, then one of these 10 packings is obtained. Furthermore, starting from any of these packings, all the others can be obtained with a series of such transformations.



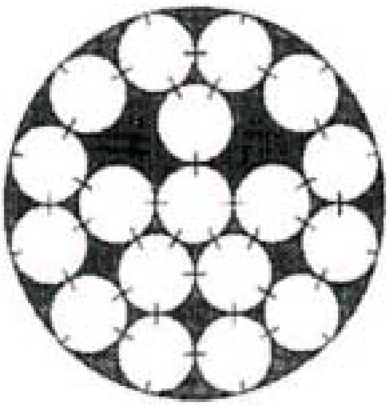
$n = 18$ (a)



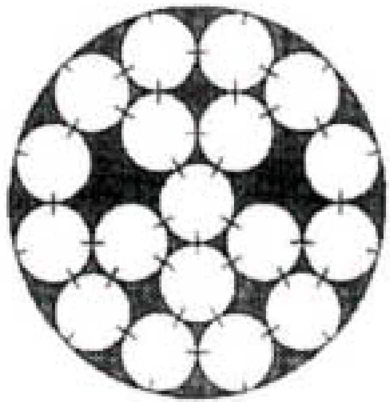
$n = 18$ (b)



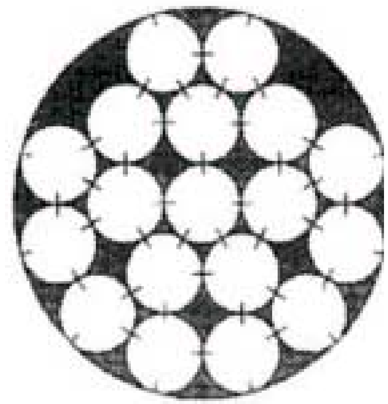
$n = 18$ (c)



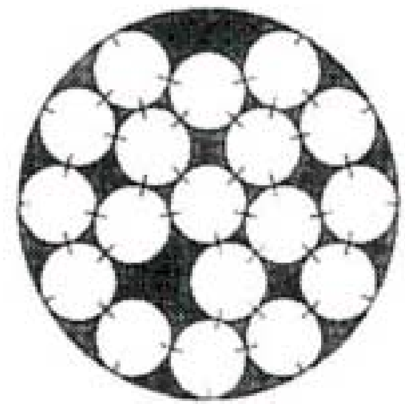
$n = 18$ (d)



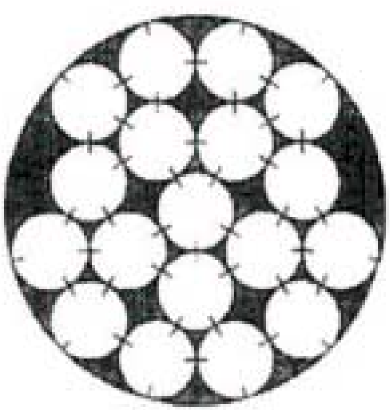
$n = 18$ (e)

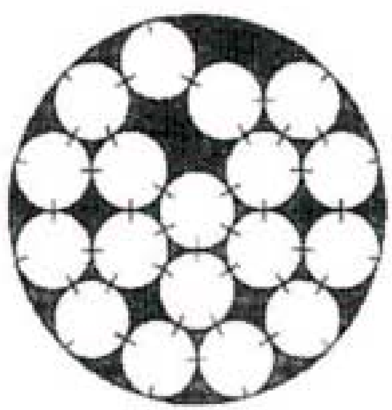


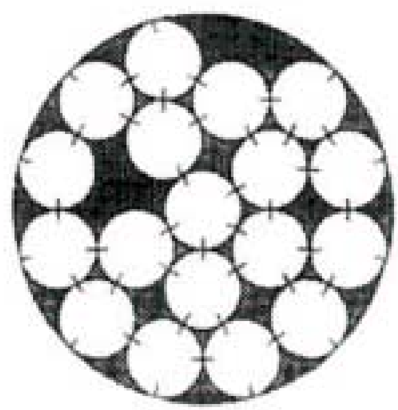
$n = 18$ (f)



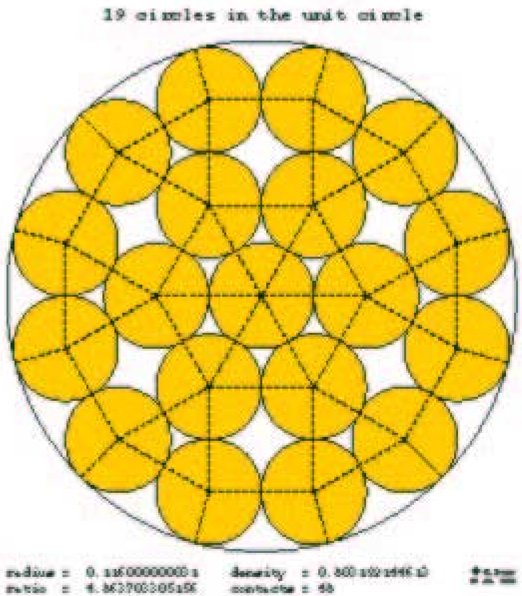
$n = 18$ (j)



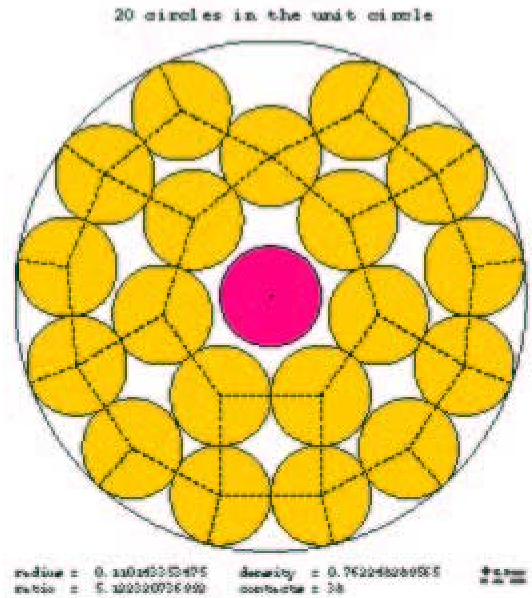




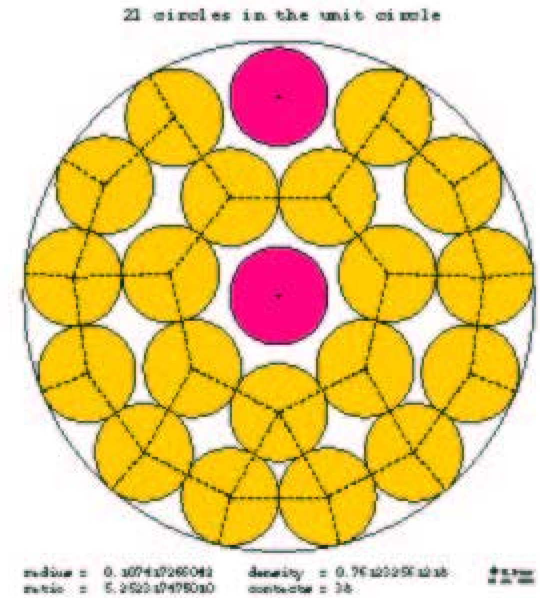
N = 19



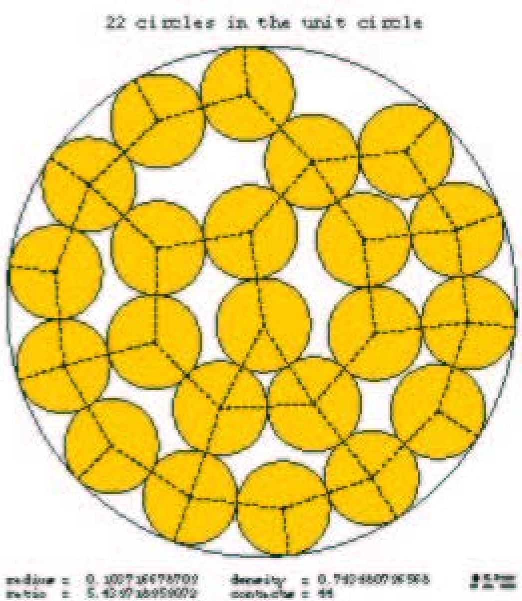
N = 20



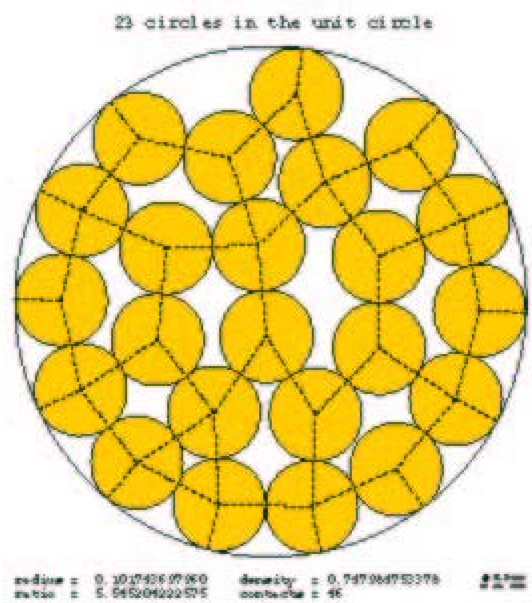
N = 21



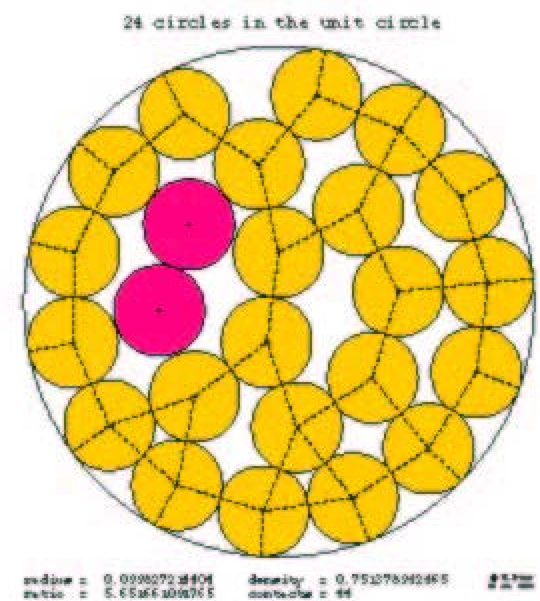
N = 22



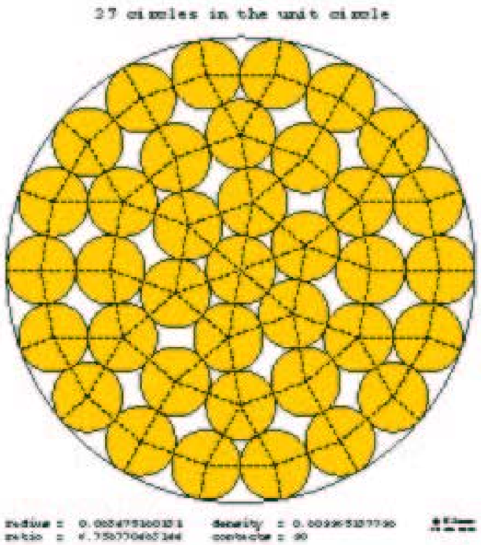
N = 23



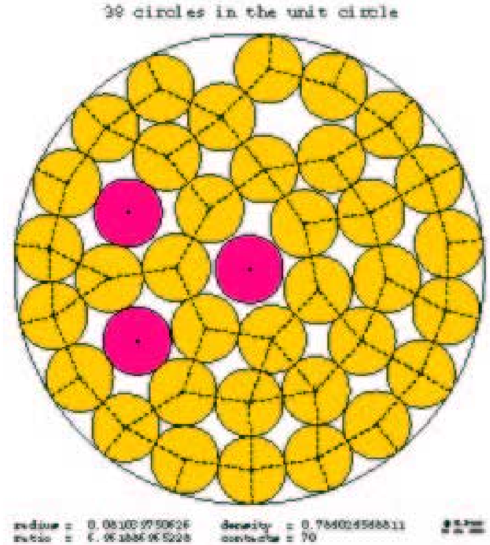
N = 24



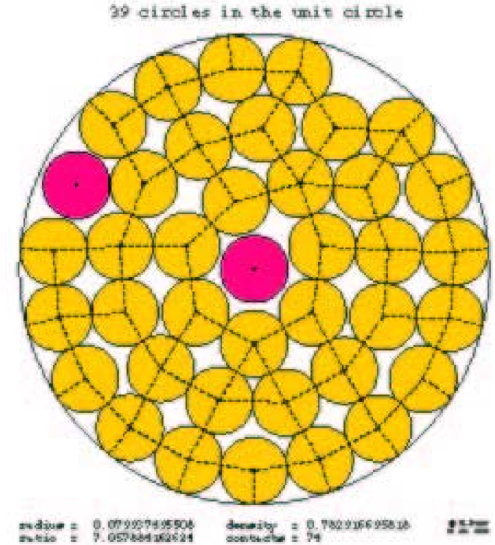
N = 37



N = 38



N = 39



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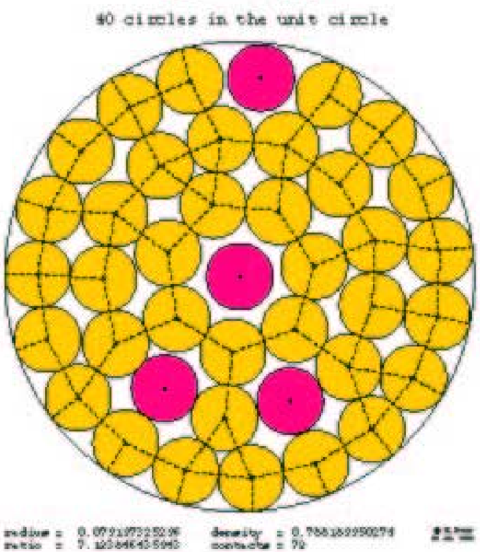

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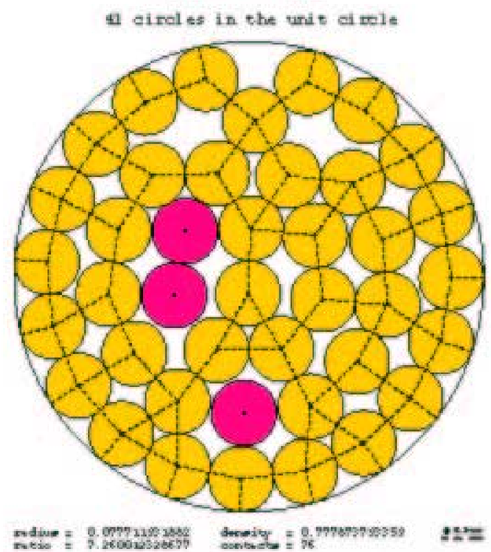

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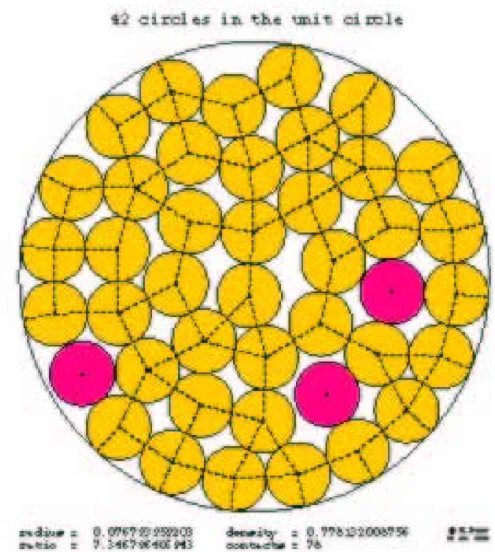
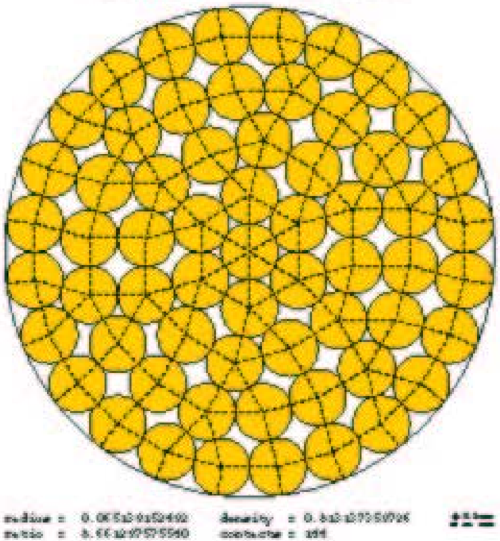


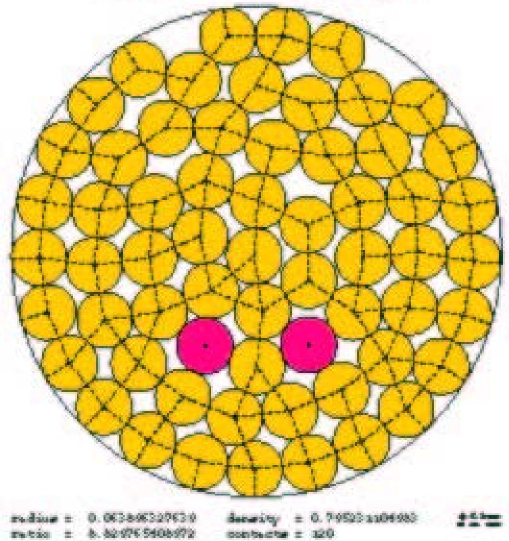
N = 61

61 circles in the unit circle



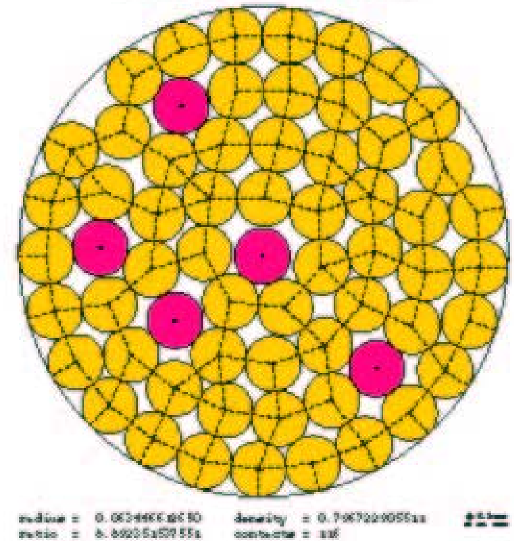
N = 62

62 circles in the unit circle



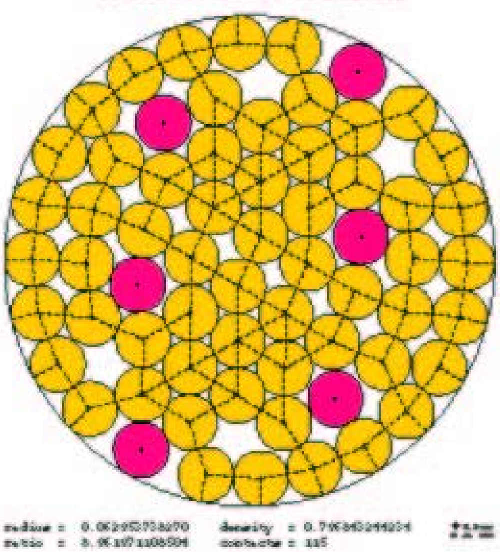
N = 63

63 circles in the unit circle



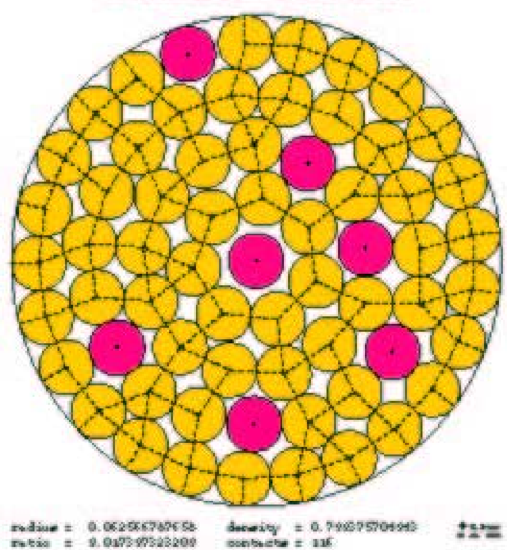
N = 64

64 circles in the unit circle



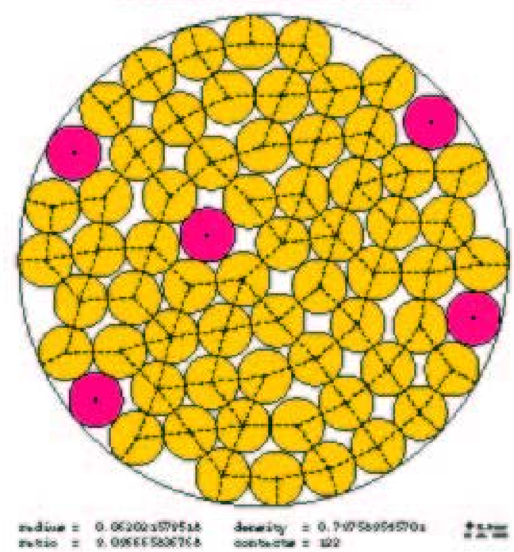
N = 65

65 circles in the unit circle



N = 66

66 circles in the unit circle



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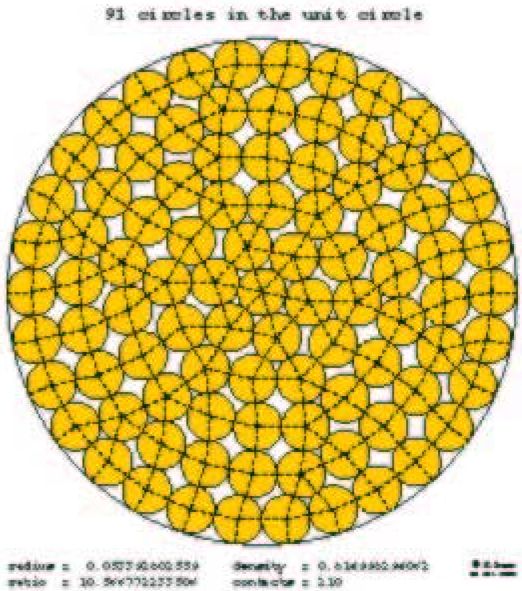

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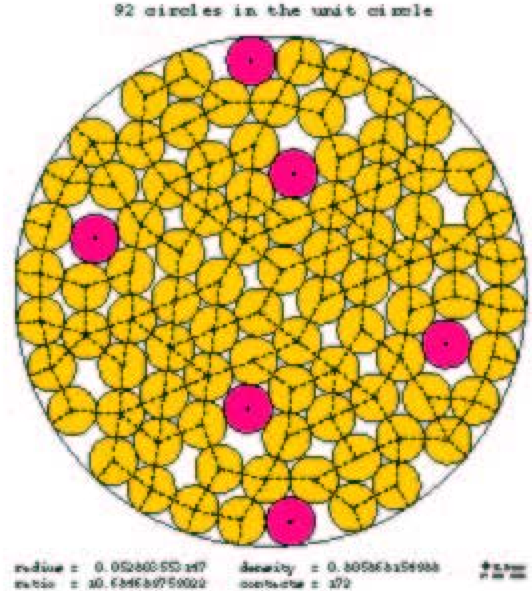

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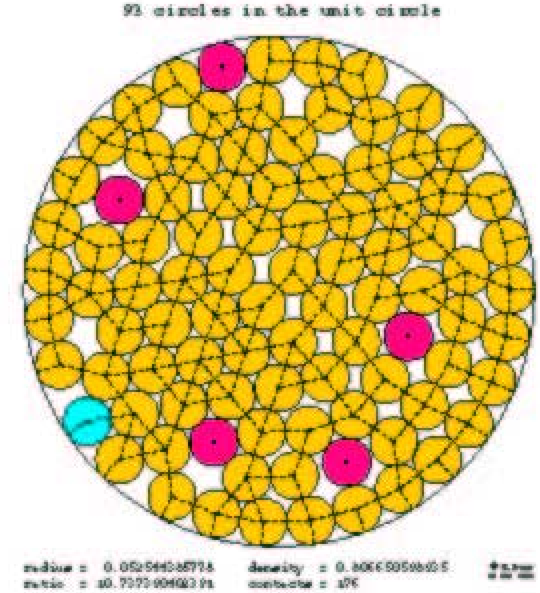

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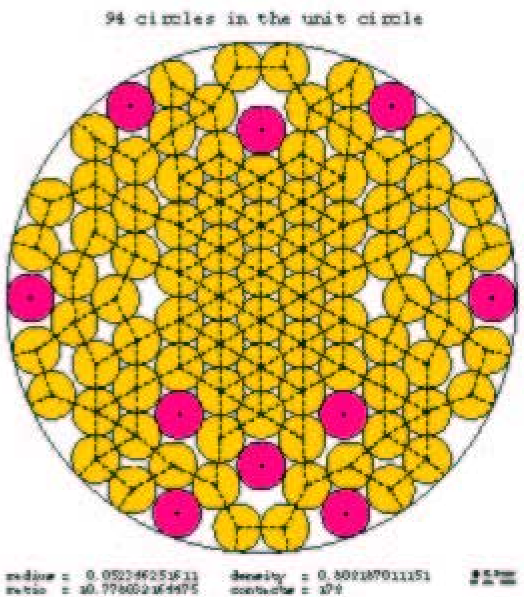

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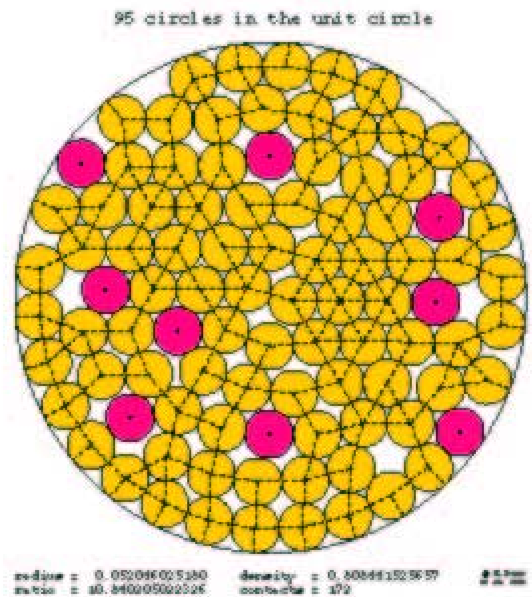

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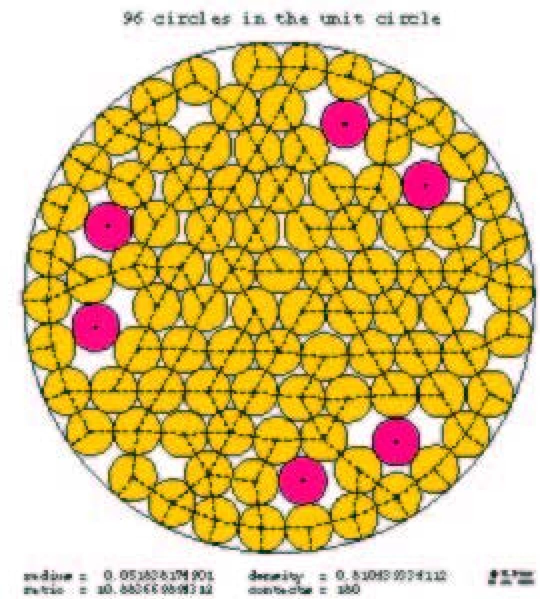
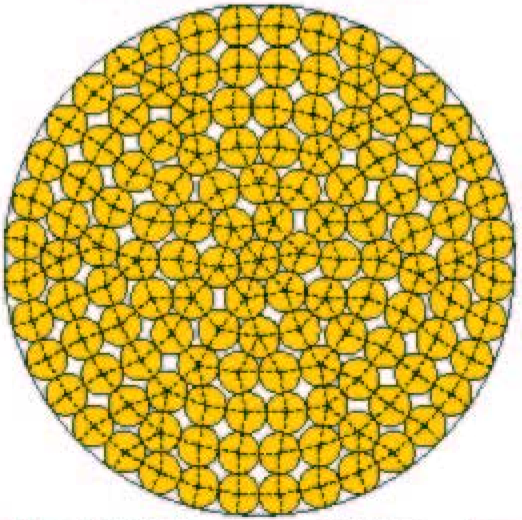


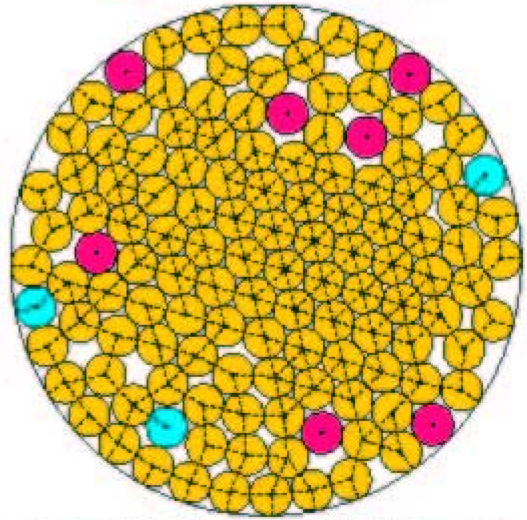
N = 127

127 circles in the unit circle



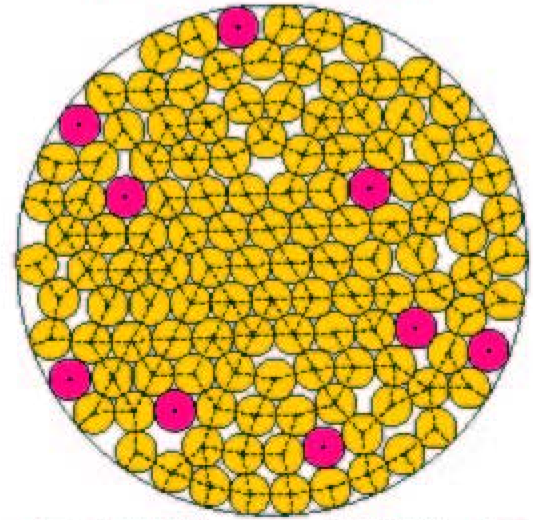
N = 128

128 circles in the unit circle



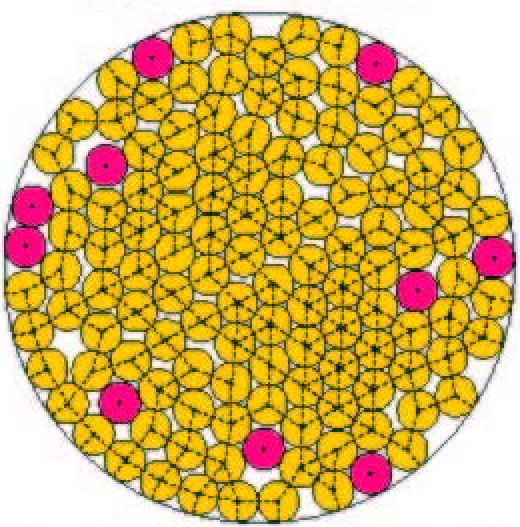
N = 129

129 circles in the unit circle



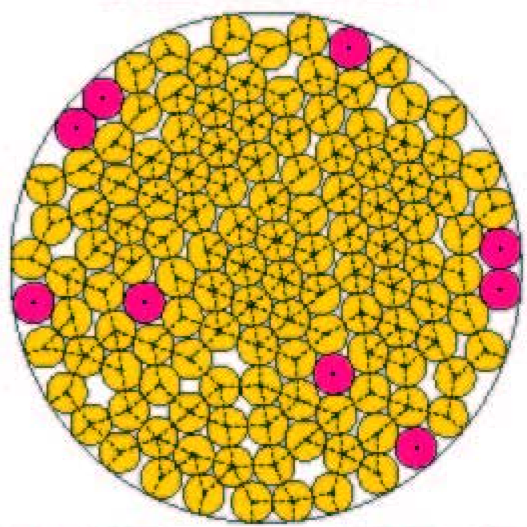
N = 130

130 circles in the unit circle



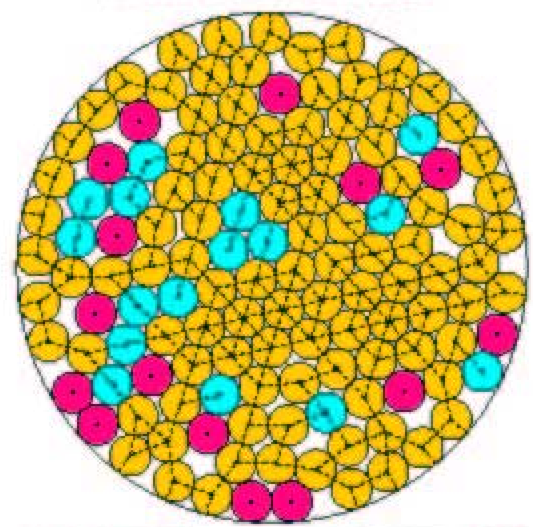
N = 131

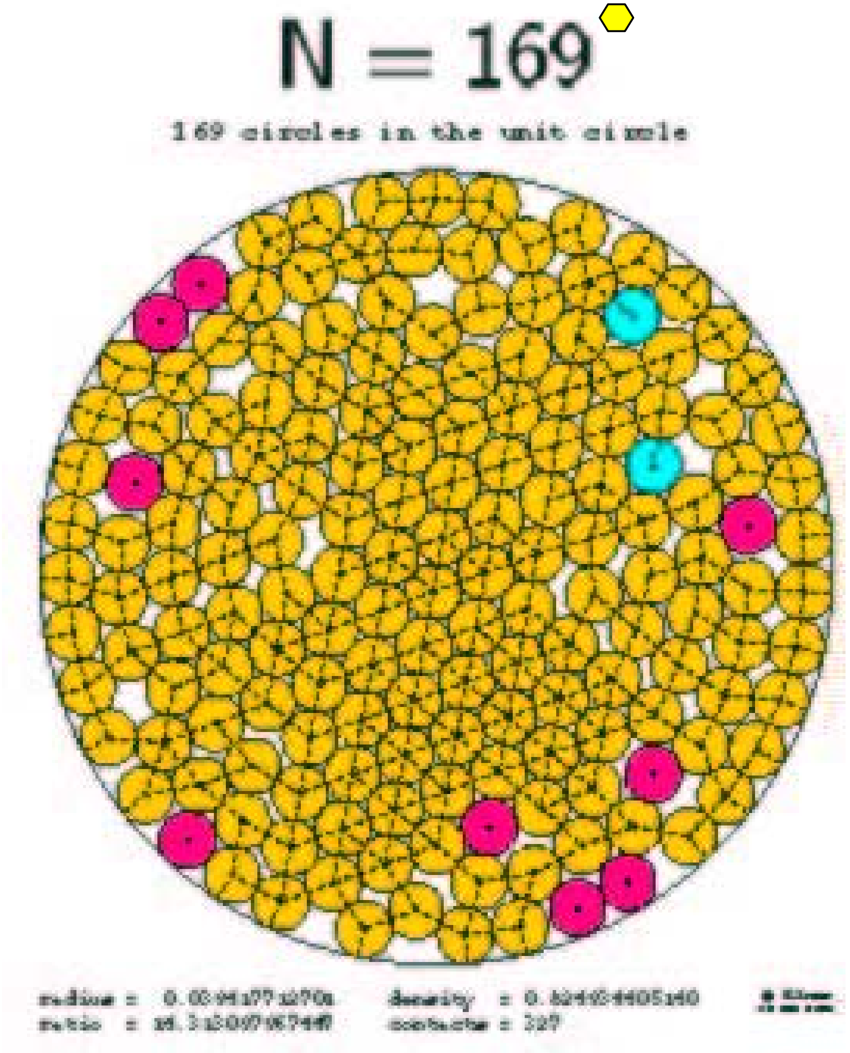
131 circles in the unit circle



N = 132

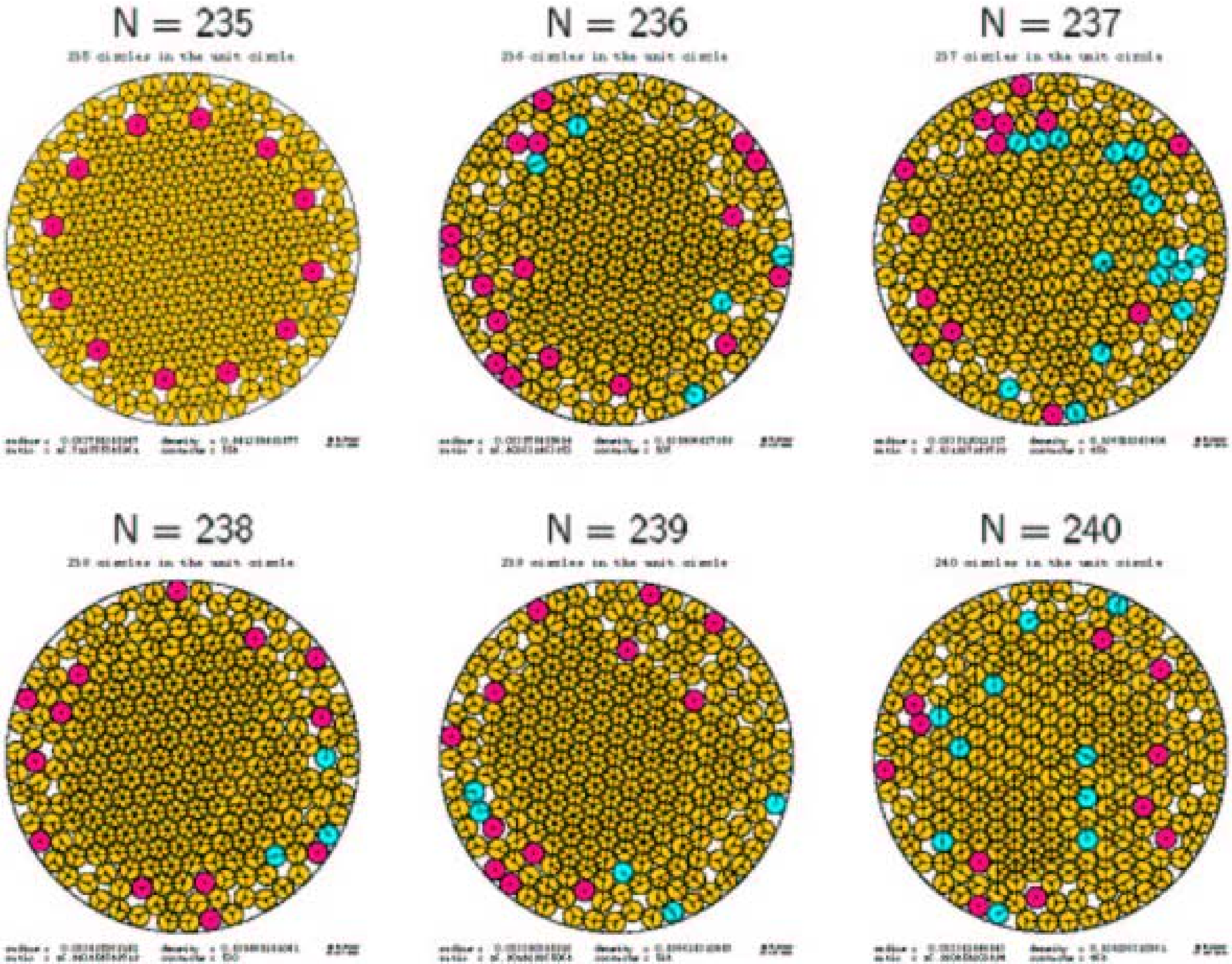
132 circles in the unit circle





$169 = 3 \cdot 7 \cdot 8 + 1 =$ ⬡

Too bad!



Any conjectures??

Dense packings of **2 sizes** of discs in the plane

(A. Heppes – 2003)

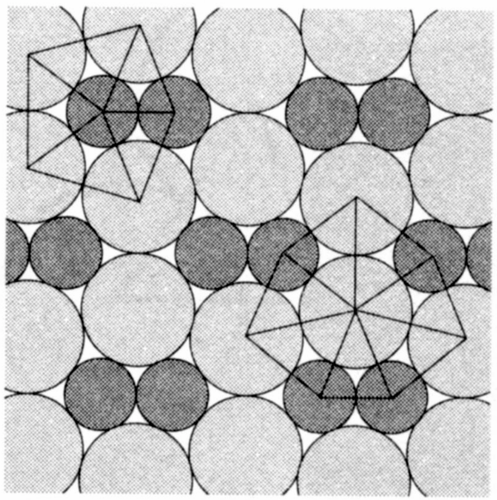


Fig. 1. $\mathcal{P}_1$.

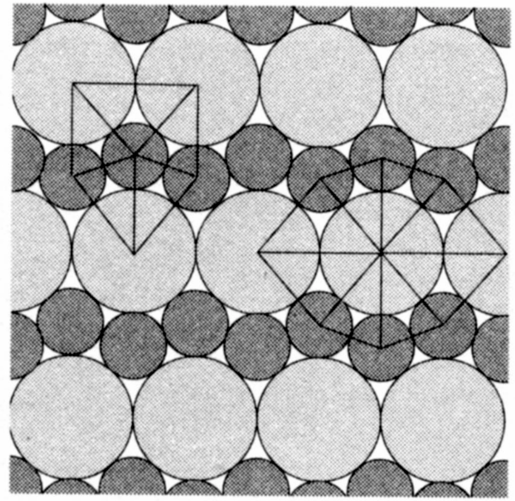


Fig. 2. $\mathcal{P}_2$.

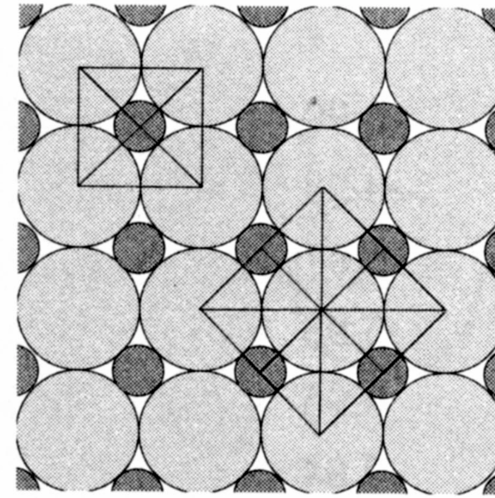


Fig. 3. $\mathcal{P}_3$.

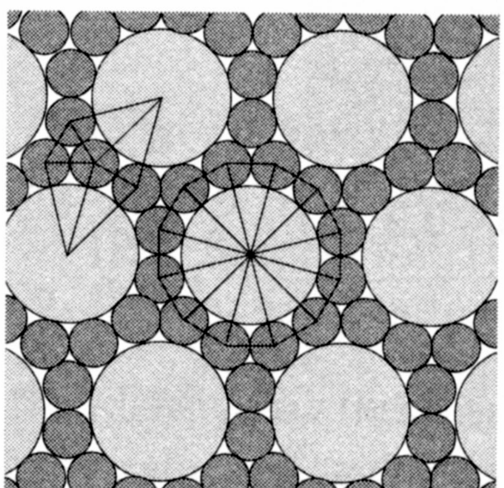


Fig. 4. $\mathcal{P}_4$.

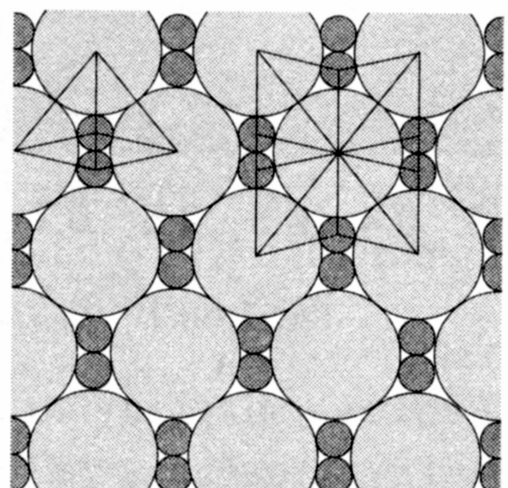


Fig. 5. $\mathcal{P}_5$.

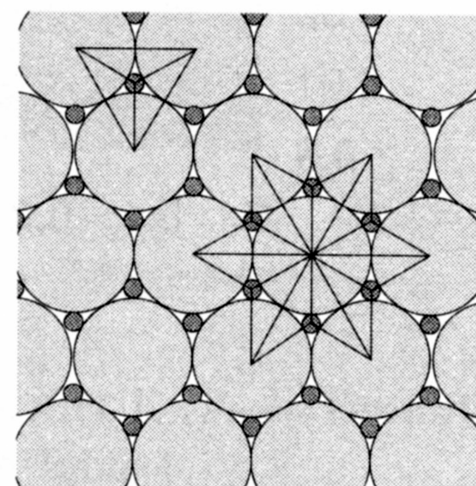
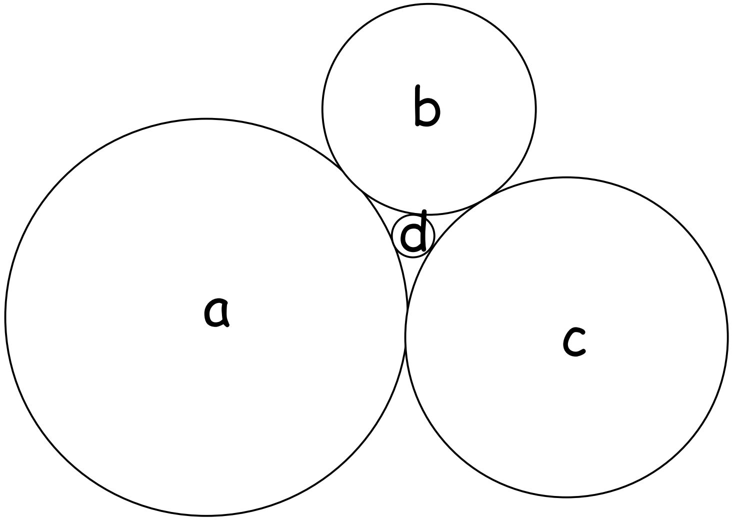


Fig. 6. $\mathcal{P}_6$.

Dense packings of discs of **many sizes** in the plane:

Apollonian circle packings

Apollonian circle packings arise by repeatedly filling the interstices between mutually tangent circles with further tnagent circles. It is possible for every circle in such a packing to have integer curvature. Such packings are called **integral Apollonian circle packings.**



a,b,c and d are reciprocals of the radii of the circles (also called the "bends" of the circles).

Descartes Theorem

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

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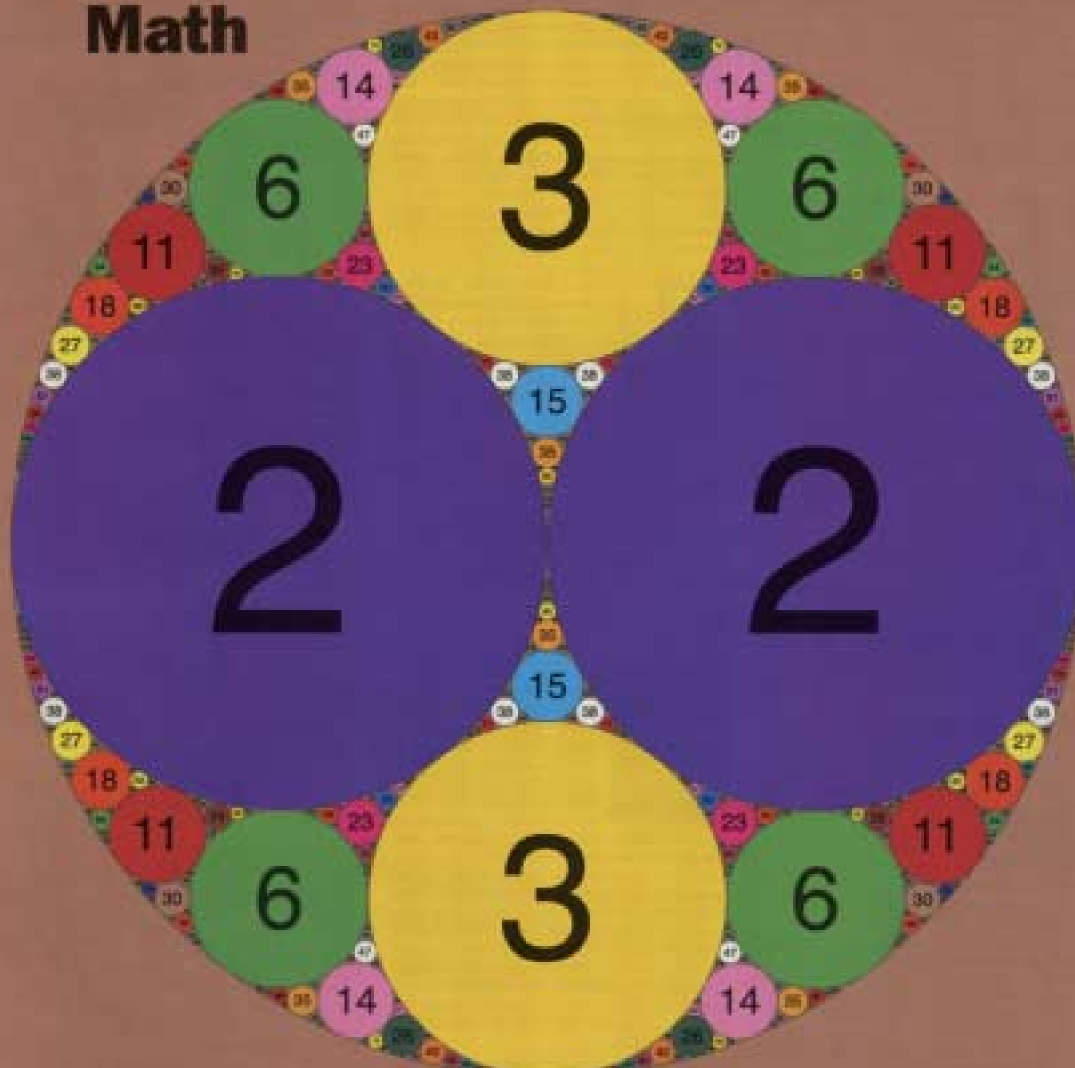
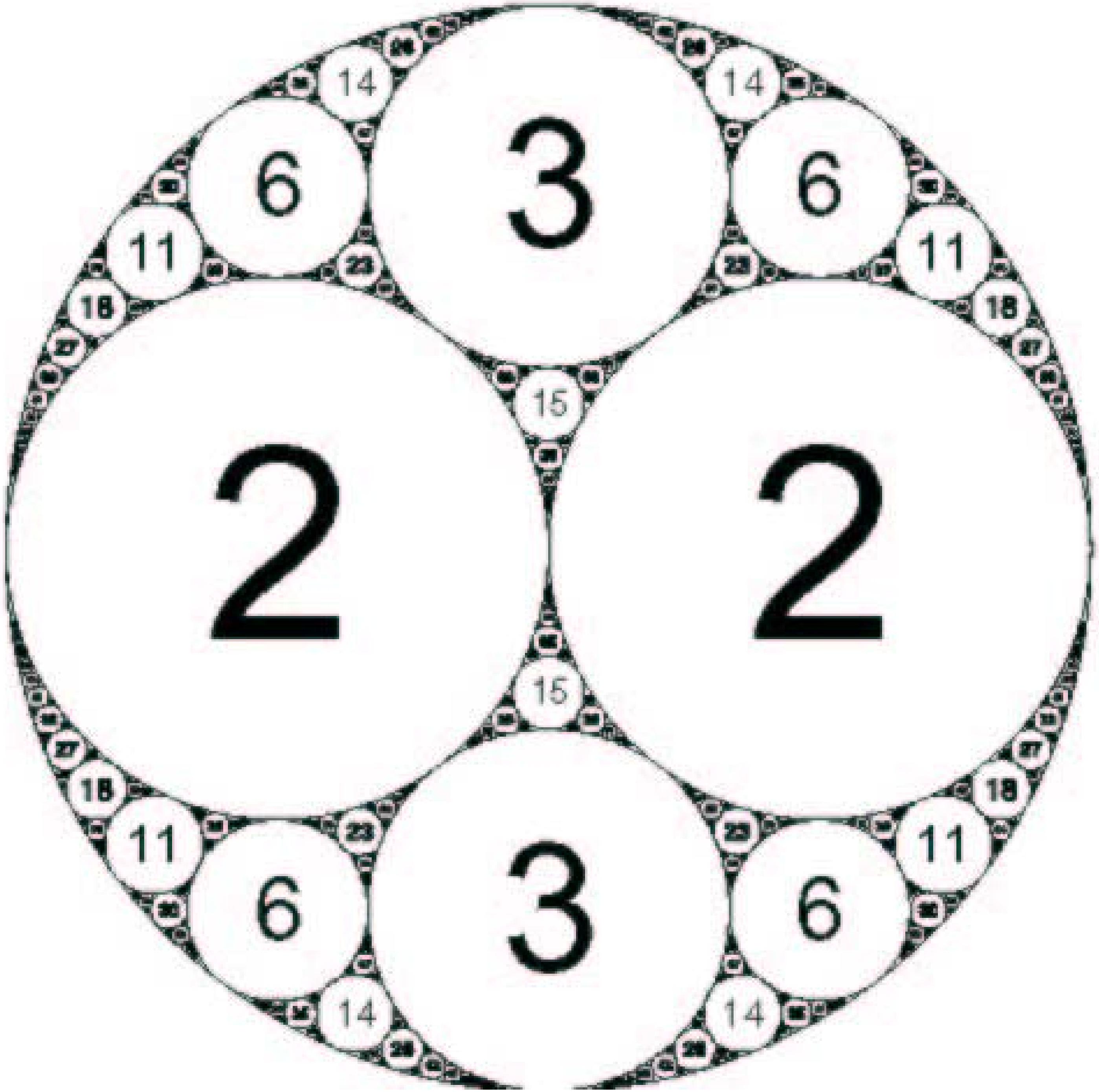
Circle Math

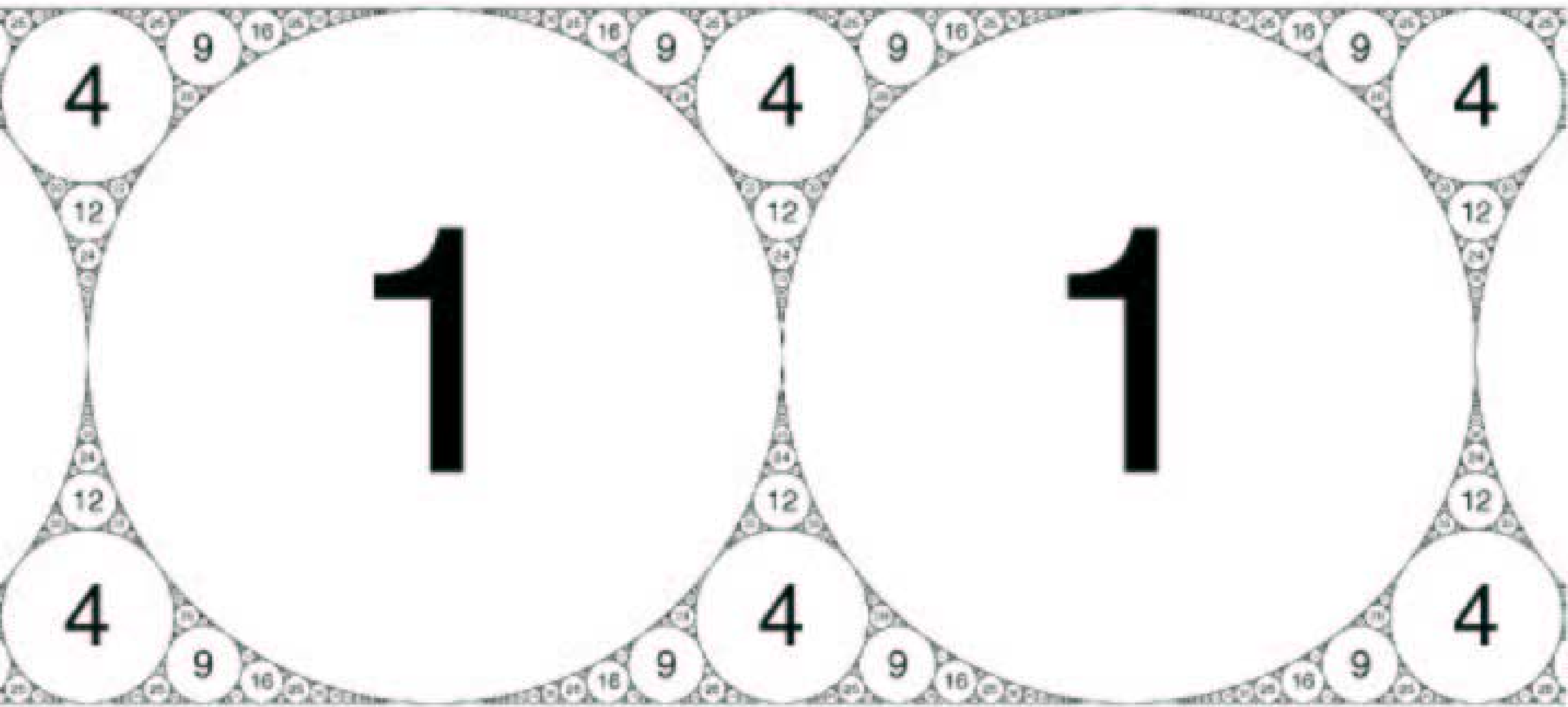


The integral Apollonian circle packing (-1,2,2,3)



Fact: All bends in the packing (-1,2,2,3) must be congruent to 2,3 6 or 11 (mod 12).

Conjecture ($500) All sufficiently large numbers satisfying these congruence conditions occur as bends in the packing (-1,2,2,3)



The Apollonian circle packing (0,0,1,1)

Fact: All bends occurring in (0,0,1,1) are congruent to 0,1,4,9,12 or 16 (mod 24)

Conjecture ($500) All sufficiently large numbers satisfying these congruence conditions occur as bends in the packing (0,0,1,1)

Fact: For any m with g.c.d (m,30) = 1, every congruence class modulo m occurs infinitely often as a bend in every integral Apollonian circle packing..

Conjecture: The above statement is true for all m with g.c.d. (m,6) = 1.

Conjecture: All congruential restrictions on bends in integral Apollonian circle packings can be expressed modulo 24.

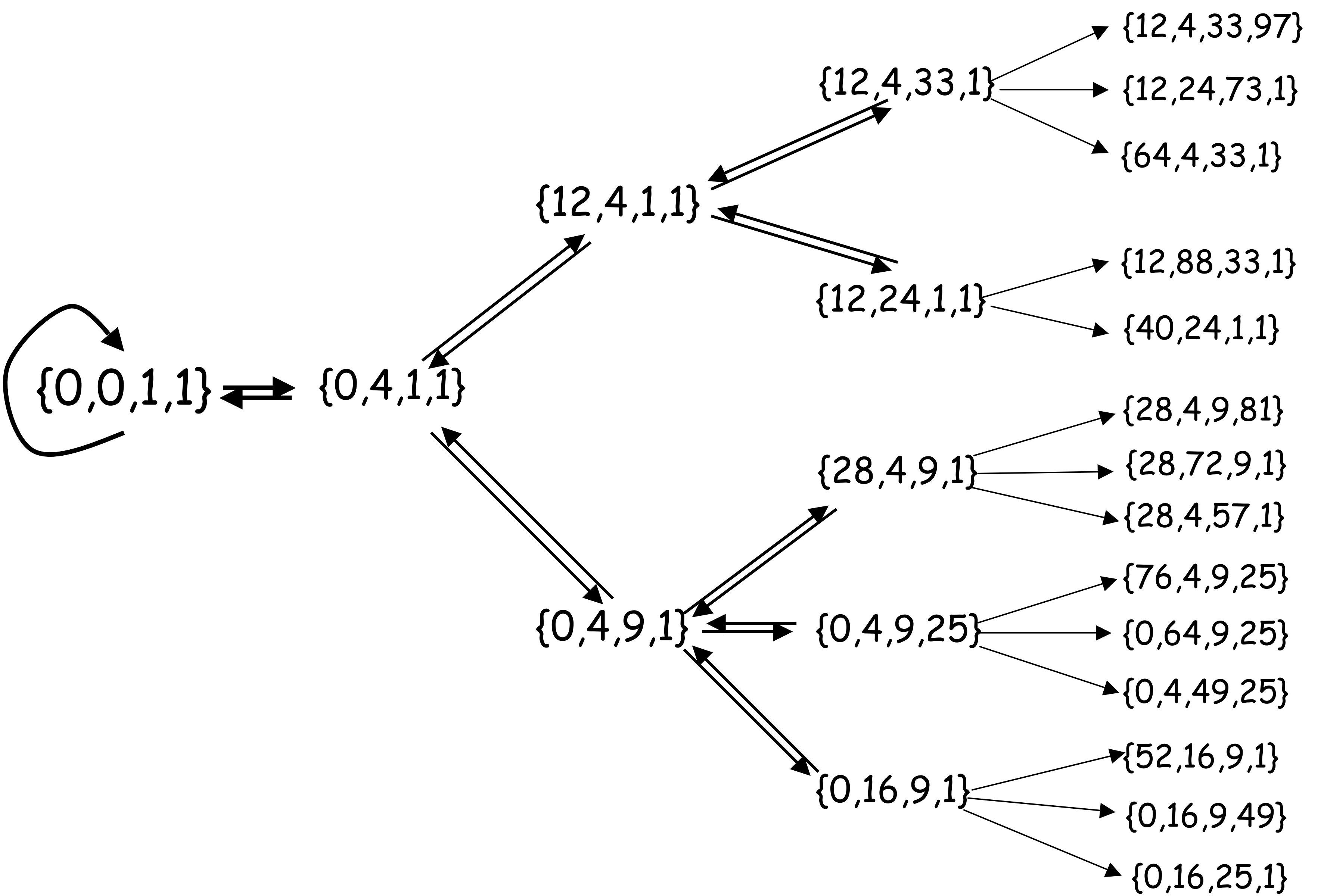
An equivalent number theory problem

Starting with some multiset $S = \{a,b,c,d\}$ of integers, **repeatedly** perform the following transformation:

Replace any element, say d, in S, by $d' = 2(a+b+c) - d$, forming $S' = \{a,b,c,d'\}$

Question: Which integers can ever be generated by this process?

For example, if we start with $\{0,0,1,1\}$, is it true that all sufficiently large integers congruent to 0,1,4,9,12 and 16 (mod 24) occur?


{0,0,1,1}
{0,4,1,1}
{12,4,1,1}
{0,4,9,1}
{12,4,33,1}
{12,24,1,1}
{28,4,9,1}
{0,4,9,25}
{0,16,9,1}
{12,4,33,97}
{12,24,73,1}
{64,4,33,1}
{12,88,33,1}
{40,24,1,1}
{28,4,9,81}
{28,72,9,1}
{28,4,57,1}
{76,4,9,25}
{0,64,9,25}
{0,4,49,25}
{52,16,9,1}
{0,16,9,49}
{0,16,25,1}

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Journal of Number Theory 100 (2003) 1–45

JOURNAL OF Number Theory

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Apollonian circle packings: number theory

Ronald L. Graham,[a,*,1] Jeffrey C. Lagarias,[a,2] Colin L. Mallows,[a,3] Allan R. Wilks,[a] and Catherine H. Yan[b]

Now for something completely different......

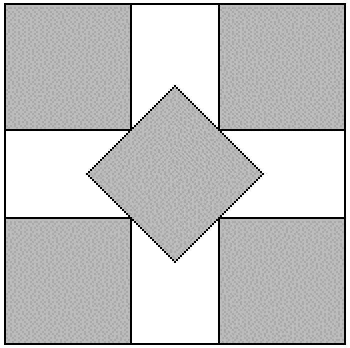
Packing **squares** in squares.

Let s(n) denote the side length of the **smallest** square into which n non-overlapping unit squares can be packed.

Of course, $s(m^2) = m$, for any integer m.

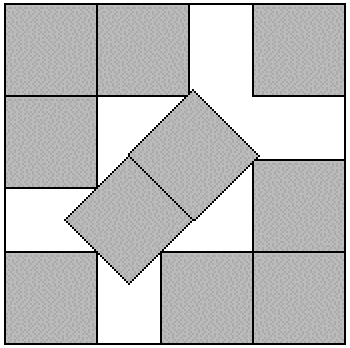
n	2	3	5	6	7	8	14	15	24	35
s(n)	2	2	$2+\frac{1}{\sqrt{2}}$	3	3	3	4	4	5	6

All currently known optimal values of s(n) for $n \neq \square$



$S(5) = 2+\frac{1}{\sqrt{2}}$

Some other currently best known packings



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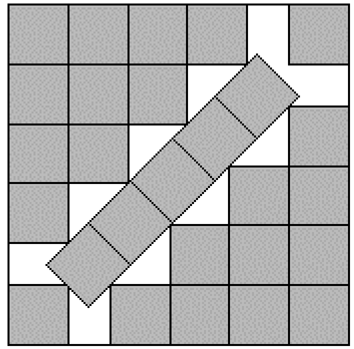

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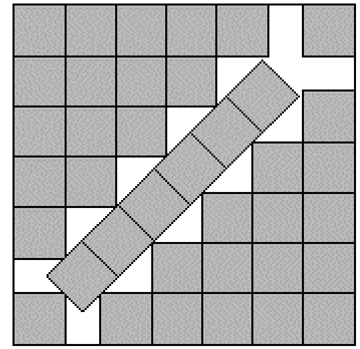

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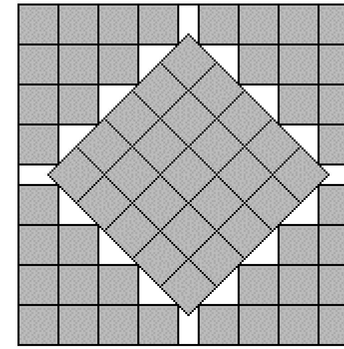

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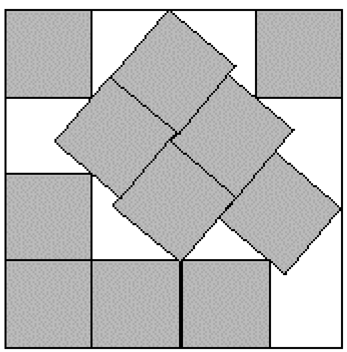

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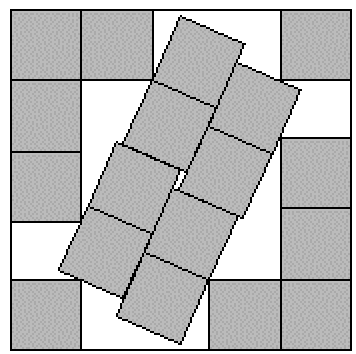

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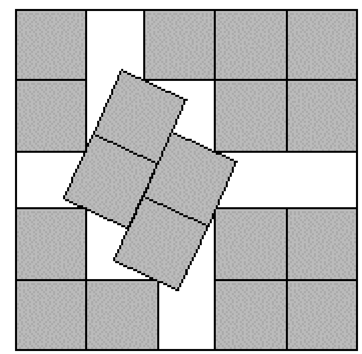

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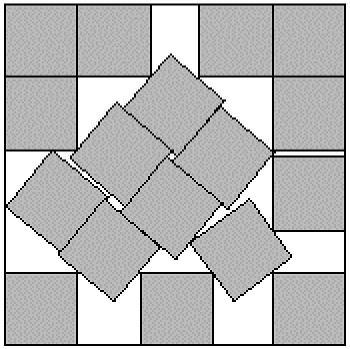
It starts getting harder…



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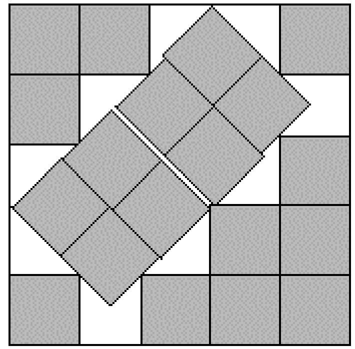

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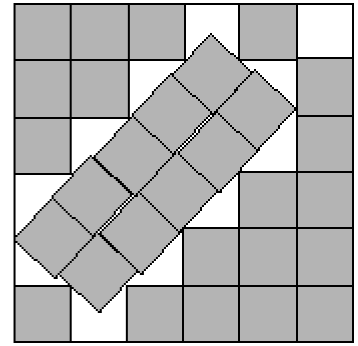

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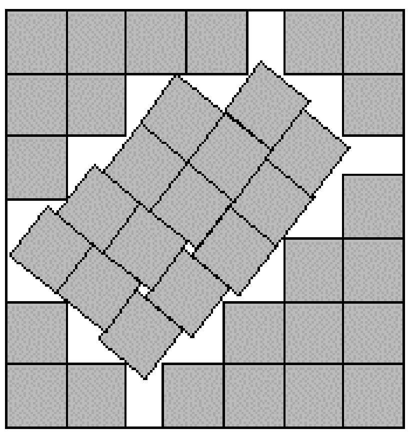

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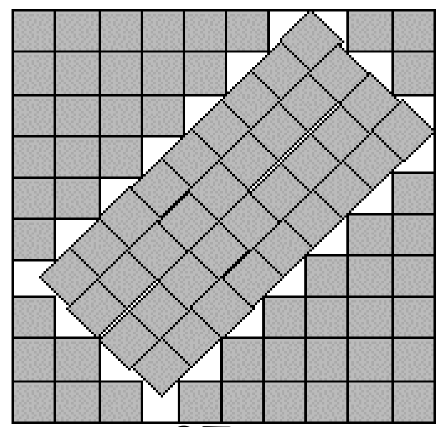

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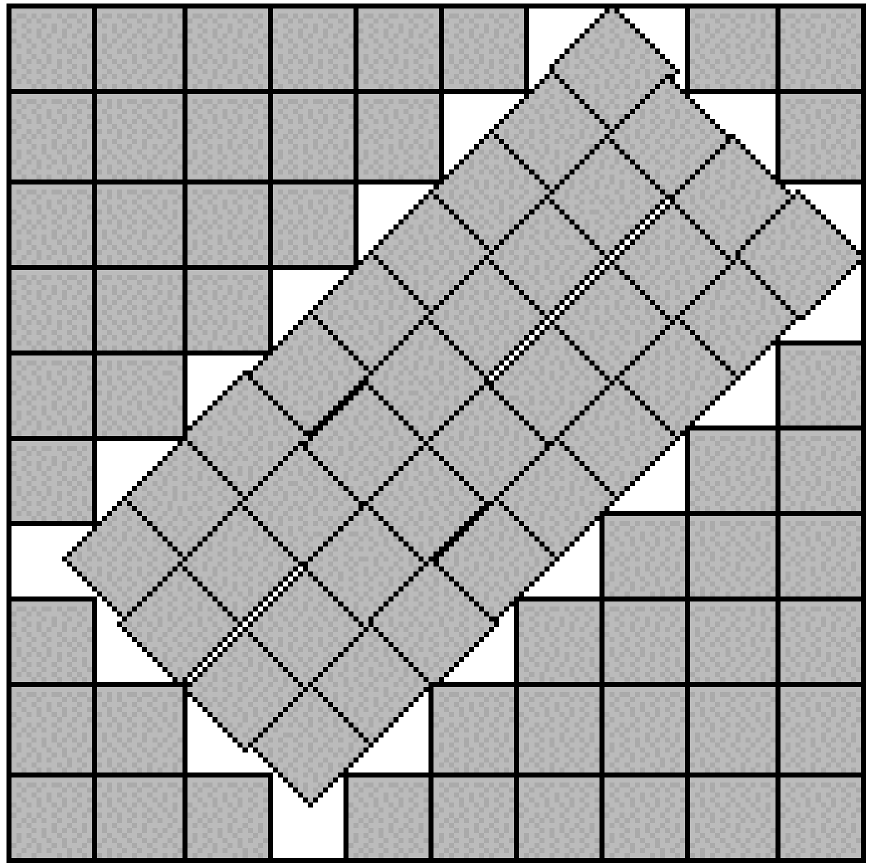

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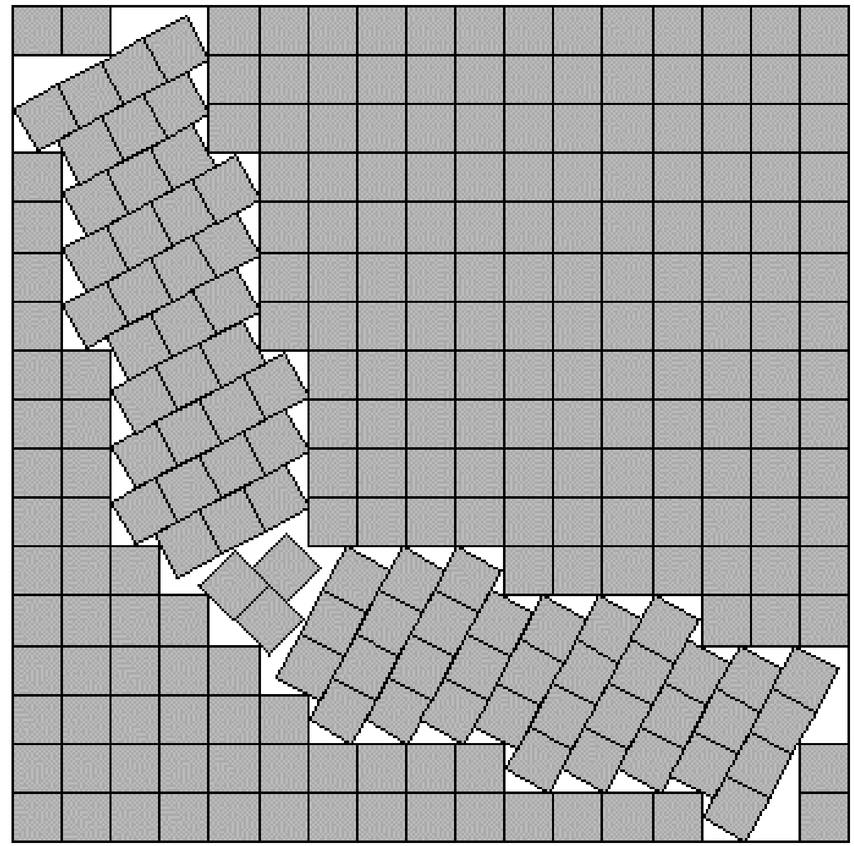
Could this really be “the truth” ?

Old conjecture: $s(n^2 - n) = n$

(New) counterexample:



$s(17^2 - 17) < 17$ (L. Cleemann)

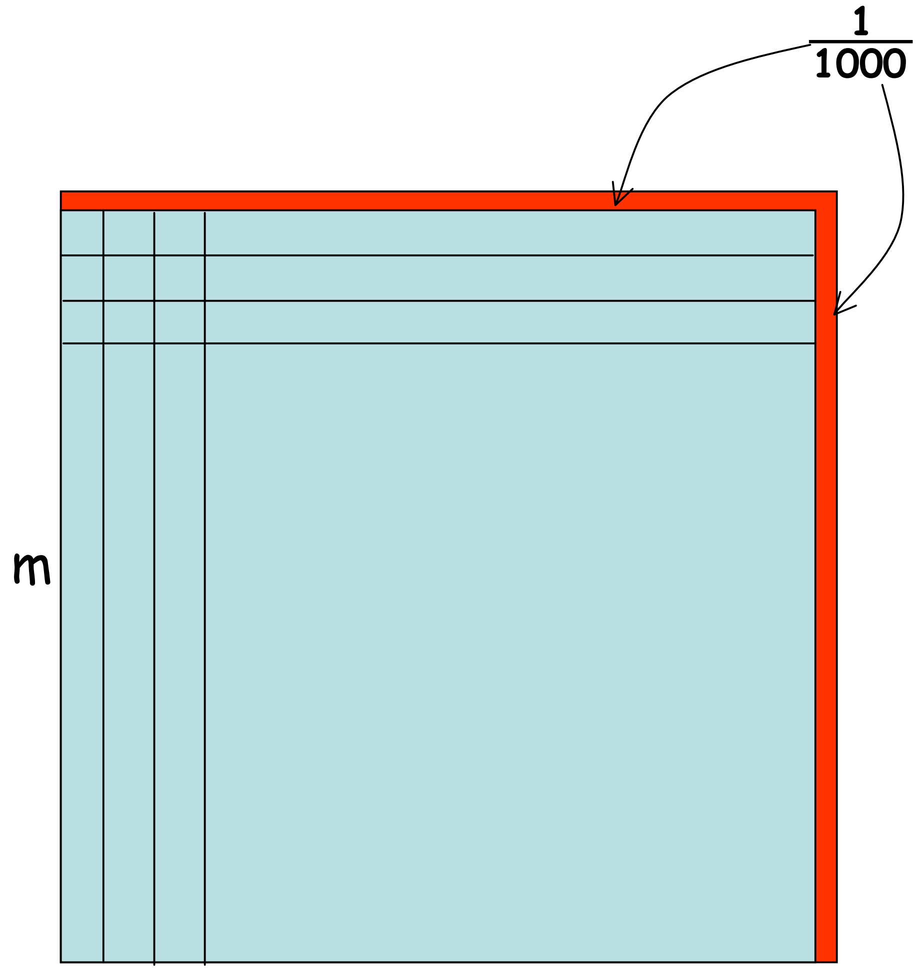
Define the wasted space W(s) in a packing of an sxs square by:

$$W(s) := s^2 - \max(n : s(n) \leq s)$$

$W(m) = 0$ if m is an integer.

What is $W(m + \frac{1}{1000})$?

$$W(m + \frac{1}{1000}) \approx \frac{m}{500}?$$



Define the wasted space W(s) in a packing of an sxs square by:

$$W(s) := s^2 - \max(n : s(n) \leq s)$$

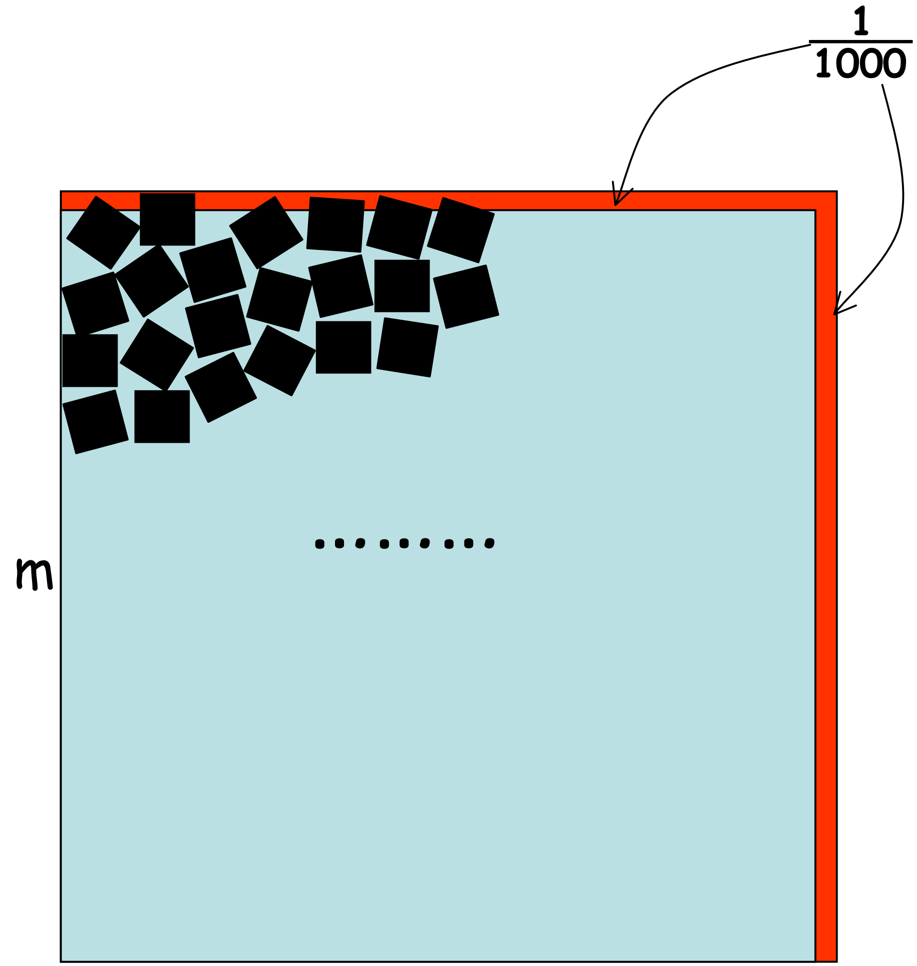
W(m) = 0 if m is an integer.

What is $W(m + \frac{1}{1000})$?

A non-obvious packing!

Theorem (Erdős-RLG – 1975)

$$W(s) = O(s^{\frac{7}{11}})$$



Theorem: (H. Montgomery)

For any $\varepsilon > 0$,

$$W(s) = O(s^{\frac{3-\sqrt{3}}{2}+\varepsilon})$$

Note: $\frac{3-\sqrt{3}}{2} = 0.63397... < 0.63636... = \frac{7}{11}$

What about a lower bound?

Theorem: (K. F. Roth-R. C. Vaughan – 1978)

Suppose $s(s-\lfloor s \rfloor) > \frac{1}{6}$.

Then

$$W(s) > 10^{-100} \sqrt{s \, | \, s - \lfloor s + \tfrac{1}{2} \rfloor |}$$

Thus, $W(s)$? $s^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$, (for s bounded away from integers)

Conjecture: ($1000)

For some $\varepsilon > 0$, $W(s) > s^{\frac{1}{2}+\varepsilon}$ (for s bounded away from integers)

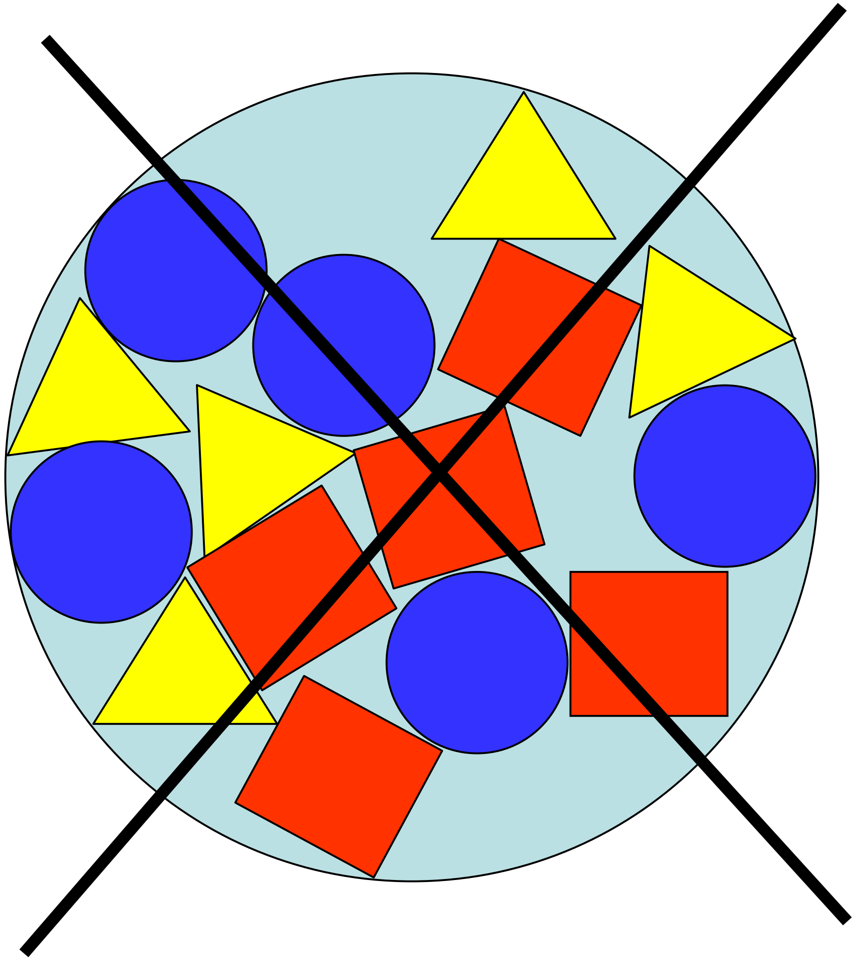
E. Friedman, Packing unit squares in squares: a survey and new results,

Electronic J. Combinatorics 7 (2000) DS #7

P. Erdös and R. L. Graham, On packing squares with equal squares,
J. Combin. Theory Ser. A **19** (1975) 119-123

K. F. Roth and R. C. Vaughan, Inefficiency in packing squares
with unit squares, *J. Combin. Theory Ser. A* **24** (1978) 170-186

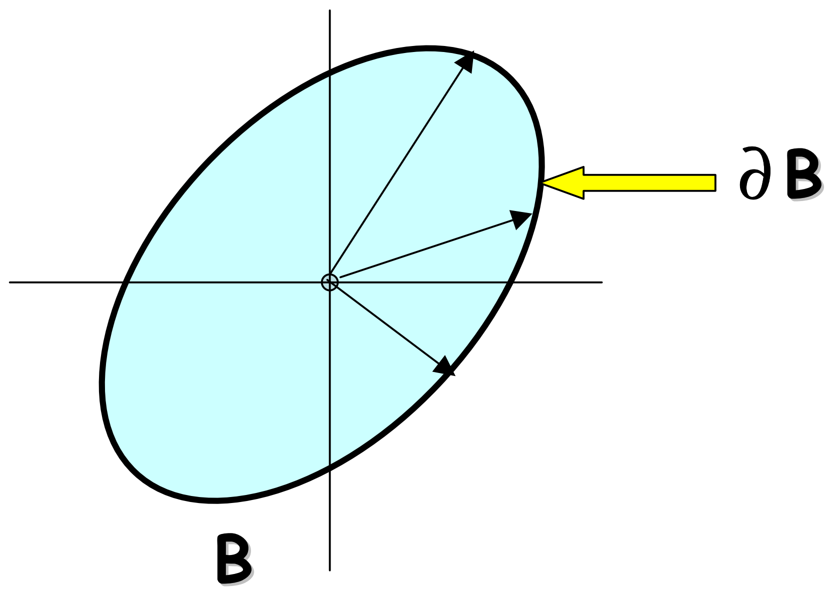
What next?



What next?

A different metric

The Minkowski plane — unit ball determined by a compact convex centrally symmetric domain B.



Theorem (Folkman, RLG—1969)

If K is a simplicial complex in the plane, and $d(x,y) \geq \mathbf{1}$ for $x \neq y$ in K,

then
$$a_0(K) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K)$$

Theorem (RLG, H. Witsenhausen, H. Zassenhaus – 1972)

For any Minkowski plane, and any finite simplicial complex K in the plane, we have

$$a_0(K) \leq \frac{1}{2\Delta^*} A(K) + \frac{1}{2} P(K) + \chi(K)$$

where Δ^* is the minimum (by compactness) over all areas of triangles with unit side lengths.

Instead of **packings**, we consider the same questions for coverings.

In general, these seem to be more difficult.

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Finally, how about all of these questions in three (or more) dimensions !

Even the first question we started with, namely determining the densest packing of Euclidean 3-space with unit balls still seems to be rather challenging!

(Kepler conjecture, Hales/Ferguson, Hsiang, ...)

To be continued......