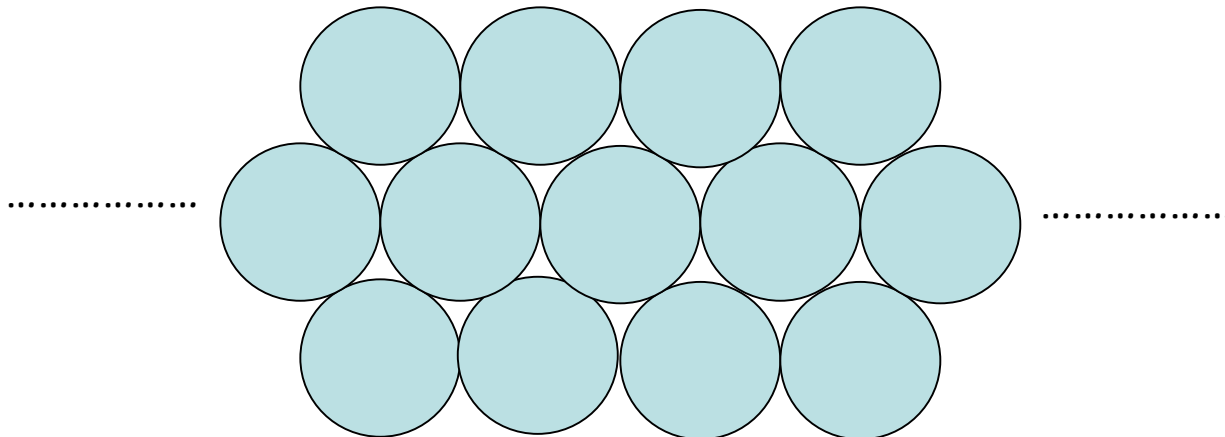


Packing Equal Discs in the Plane

Object: To maximize the density ρ_0 of the covered region

Theorem: (Thue - 1892)

Hexagonal is optimal, with $\rho_0 = \frac{\pi}{2\sqrt{3}}$

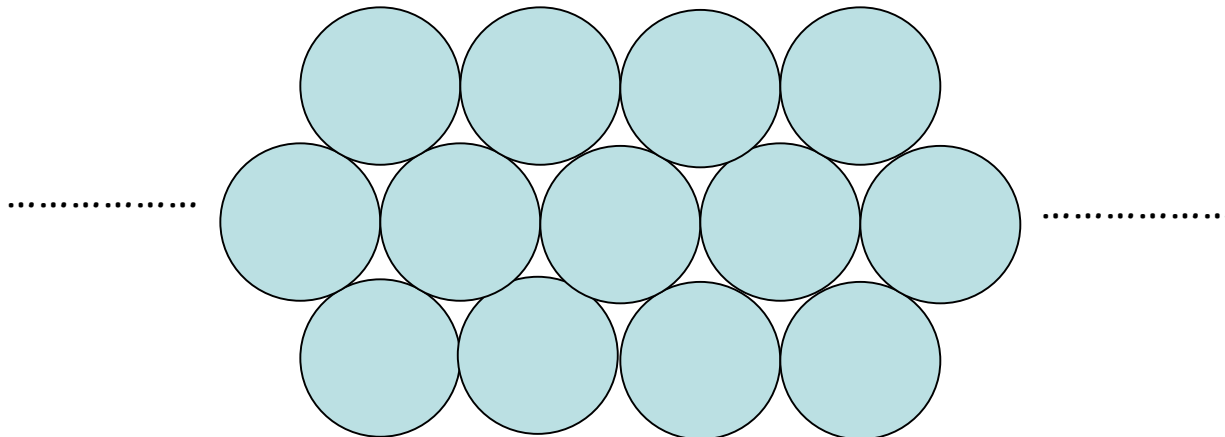


Packing Equal Discs in the Plane

Object: To maximize the density ρ_0 of the covered region

Theorem: (Thue - 1892, 1910) (Fejes Tóth - 1940)

Hexagonal is optimal, with $\rho_0 = \frac{\pi}{2\sqrt{3}}$ (Lagrange - 1773)

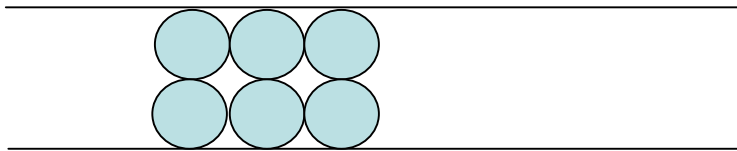


The next simplest case

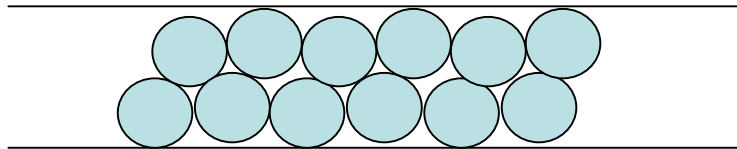
Packing an **infinite strip of width w** with equal discs.

(We will always assume that our discs have diameter 1.)

For example, take $w = 2$:

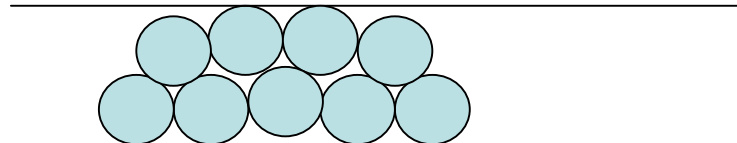


trivial --- **center density** = $1/2$



Fejes Tóth (1971)---**center density** =

$$\frac{1}{4}(\sqrt{2} + \frac{1}{\sqrt{3}}) = 0.49789\dots$$

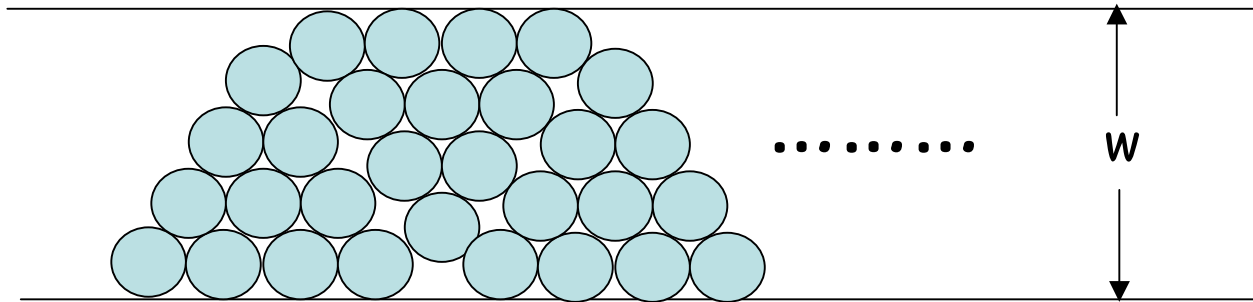


RLG (1971)---**center density** =

$$\frac{1}{6}(1 + \sqrt{4\sqrt{3} - 3}) = 0.49699\dots$$

Conjecture:(J. Molnár -- 1970's)

The "alternating triangles" packing of discs is optimal for **every** value of w .



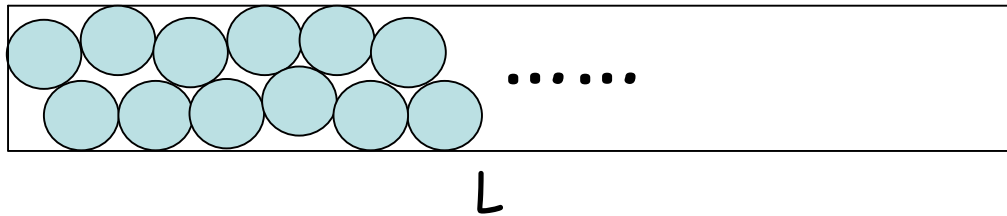
Theorem: (Füredi—1992)

Conjecture is true for $w \leq 1 + \sqrt{3}$

and also for $w = 1 + k\sqrt{3}$ for any positive integer k .

What if the strip is finite (but very long)?

For example, what is the length L_2 of the shortest rectangle of width 2 for which a **non-trivial** packing of units discs is optimal?



If $L = 164.992765\dots$, then 332 units discs can be non-trivially packed into a 2-by- L rectangle. The trivial packing requires a 2-by-165 rectangle.

What is the correct value of L_2 ??

How far into the interior do the **irregularities** penetrate in on optimal packing?

$N = 49$

49 circles in the 10x1 unit rectangle



radius = 0.069177492110 density = 0.715530035067 © E. Searc
distance = 0.142499429277 contacts = 110 20-09-2002

$N = 50$

50 circles in the 10x1 unit rectangle



radius = 0.067466165729 density = 0.714976775101 © E. Searc
distance = 0.140946423785 contacts = 110 20-09-2002

$N = 51$

51 circles in the 10x1 unit rectangle



radius = 0.066792996000 density = 0.714793503551 © E. Searc
distance = 0.139477929761 contacts = 112 01-02-2002

$N = 52$

52 circles in the 10x1 unit rectangle



radius = 0.066176054501 density = 0.715409918949 © E. Searc
distance = 0.139133465924 contacts = 117 20-09-2002

$N = 53$

53 circles in the 10x1 unit rectangle



radius = 0.065602609207 density = 0.716595420917 © E. Searc
distance = 0.136994663452 contacts = 117 20-09-2002

$N = 54$

54 circles in the 10x1 unit rectangle



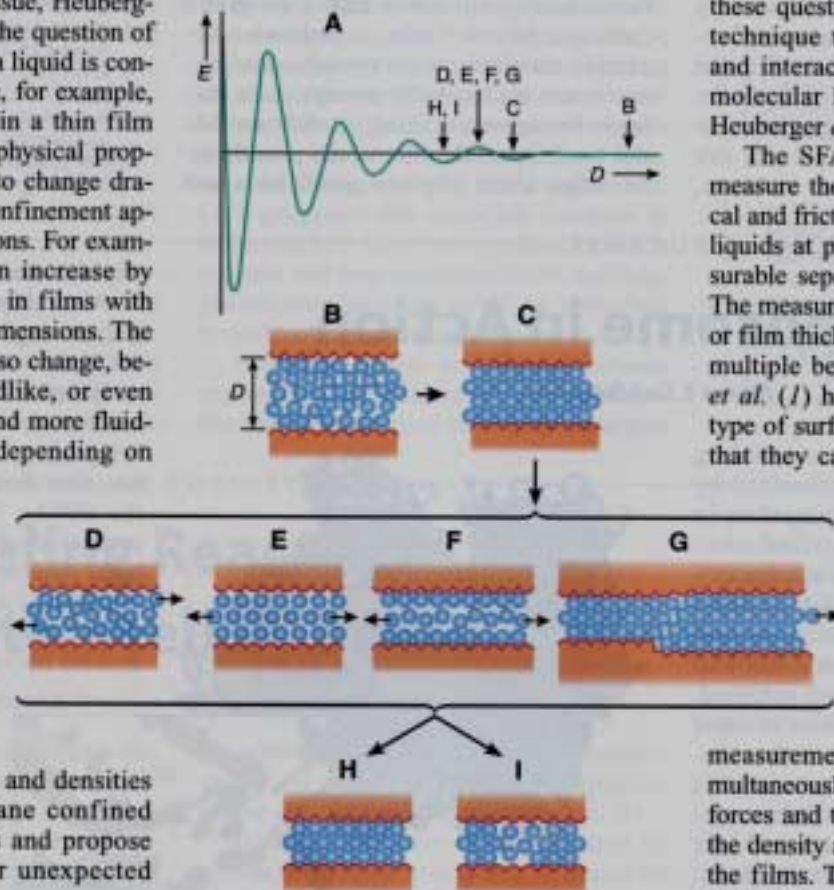
radius = 0.065042745209 density = 0.717697375990 © E. Searc
distance = 0.135666349710 contacts = 119 01-02-2002

Putting Liquids Under Molecular-Scale Confinement

Jacob Israelachvili and Delphine Gourdon

On page 905 of this issue, Heuberger *et al.* (1) address the question of what happens when a liquid is confined within a small volume, for example, in an ultrathin capillary or in a thin film between two surfaces. The physical properties of liquids are known to change dramatically as the degree of confinement approaches molecular dimensions. For example, a liquid's viscosity can increase by several orders of magnitude in films with molecular or "nanoscale" dimensions. The "structure" of a liquid can also change, becoming more ordered, solidlike, or even crystalline or less ordered and more fluidlike than the bulk liquid, depending on how the microscopic shape and atomic structure of the confining walls match that of the liquid molecules (2). Many aspects of these changes remain poorly characterized and understood. Heuberger *et al.* (1) report unprecedentedly detailed measurements of the forces and densities of thin films of cyclohexane confined between two mica surfaces and propose new explanations for their unexpected observations.

The properties of confined liquids (and solids) are of great interest and importance in areas as diverse as materials science, microfabrication, adhesion and lubrication, biology, geology, and the bud-



How does the confined liquid respond? The short-range oscillatory "solvation" force (also known as the potential of mean force) between two surfaces in a liquid varies between attraction and repulsion with a periodicity

liquid film undergo a succession of liquid-to-solid-to-liquid phase transitions ($B \rightarrow C \rightarrow D \rightarrow H$), does the film collapse in an ordered fashion ($C \rightarrow E \rightarrow H$), or do individual layers get forced out through dislocations ($C \rightarrow G \rightarrow H$)? And are there both out-of-plane and in-plane (lateral) heterogeneities, such as two-dimensional domains, in the films (F or I)? To answer these questions experimentally requires a technique that can probe both structure and interactions in real time at the sub-molecular level (<0.1 nm). This is what Heuberger *et al.* (1) have achieved.

The SFA (6) is traditionally used to measure the normal and lateral (rheological and friction) forces between surfaces in liquids at precisely controllable and measurable separations at the angstrom level. The measurement of the surface separation or film thickness is achieved optically with multiple beam interferometry. Heuberger *et al.* (1) have designed and built a new type of surface force-measuring apparatus that they call an extended surface forces apparatus (eSFA). Using fast spectral correlation spectroscopy to record the interference fringes, they were able to measure surface separation D and film refractive index n at least 10 times more accurately than in conventional SFA measurements. This enabled them to simultaneously measure both the interaction forces and the refractive index (and hence the density and, indirectly, the structure) of the films. This allows for the first time a direct correlation between these two intimately related factors.

Using the eSFA, Heuberger *et al.* measured the oscillatory force profile between two molecularly smooth surfaces of mica across liquid cyclohexane, C_6H_{12} , a rough-

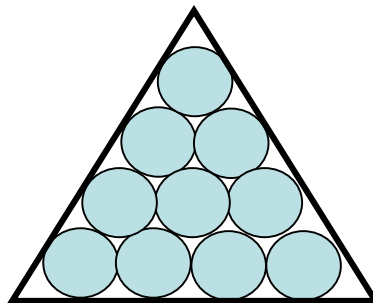
Packing bounded domains

Let $T(s)$ denote an **equilateral triangle** of side s .

What does the **densest packing** of n unit discs in $T(s)$ look like?

If $n = \binom{m}{2}$, the answer is "obvious".

For example, for $n = 10$:



Conjecture (Zassenhaus) This is always optimal for $n = \binom{m}{2}$

Theorem (Oler--1961) This conjecture is true.

One of the simplest proofs is based on the following result.

Let K denote a simplicial complex in the plane, and let $\alpha_i(K)$ denote the number of i -simplices of K , $i = 0, 1, 2$.

As usual, let $\chi(K) = \alpha_0(K) - \alpha_1(K) + \alpha_2(K)$

denote the Euler characteristic of K

Let $A(K)$ and $P(K)$ denote the **area** and **perimeter** of K , respectively

Theorem (Folkman, RLG—1969)

If K is a simplicial complex in the plane, and $d(x, y) \geq 1$ for $x \neq y$ in K ,

then

$$\alpha_0(K) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K)$$

Let X be a compact convex subset of the plane.

By a **packing** of X , we mean a subset S of X such that

$$x, y \in S \Rightarrow d(x, y) \geq 1$$

The **packing number** $\rho(X)$ is defined to be $\max \{|S|: S \text{ is a packing of } X\}$

Corollary (Oler - 1961)

$$\rho(X) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + 1$$

Example: $X = T(n)$, an equilateral triangle of side n .

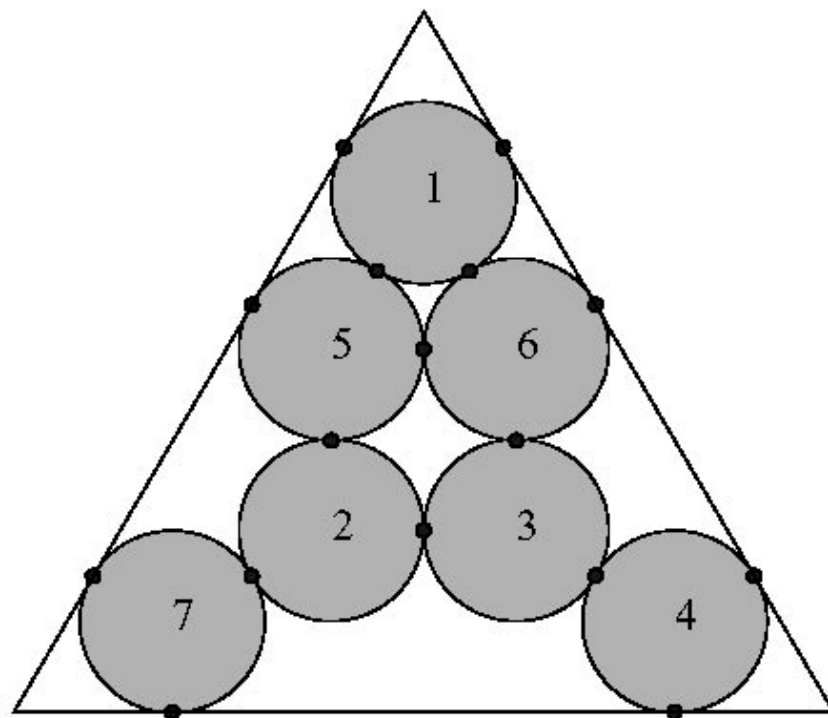
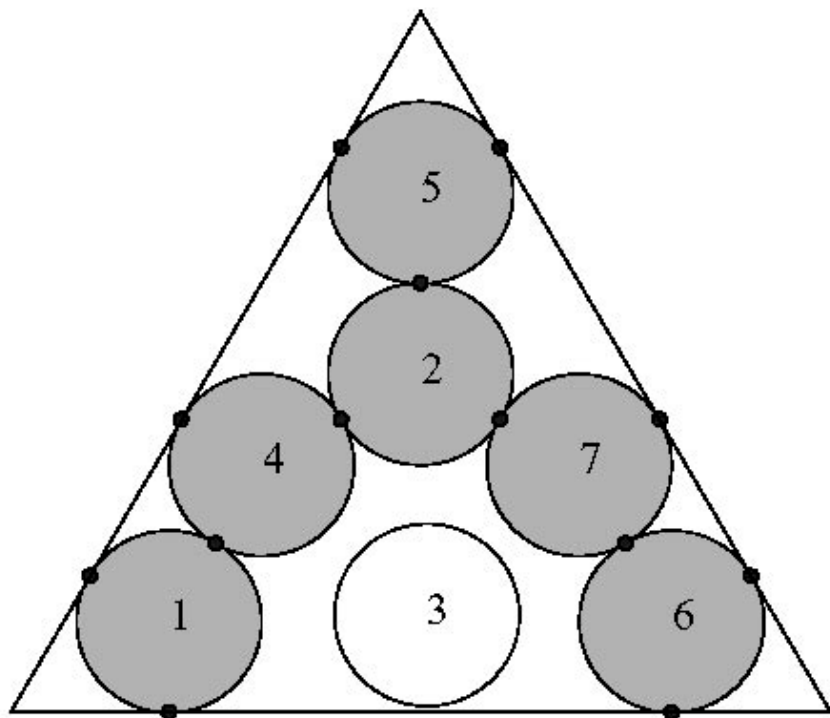
Then, $A(T(n)) = \frac{\sqrt{3}}{4} n^2$, and $P(T(n)) = 3n$

$$\text{Thus, } \rho(T(n)) \leq \frac{2}{\sqrt{3}} \frac{\sqrt{3}}{4} n^2 + \frac{3}{2} n + 1 = \frac{1}{2} (n^2 + 3n + 2) = \binom{n+2}{2}$$

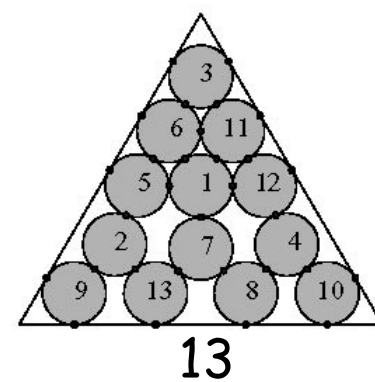
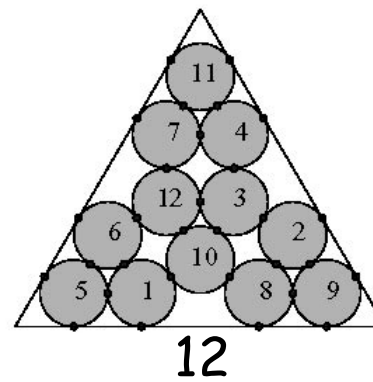
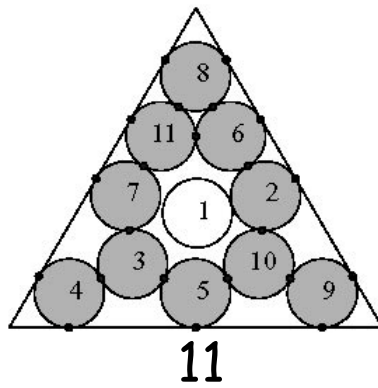
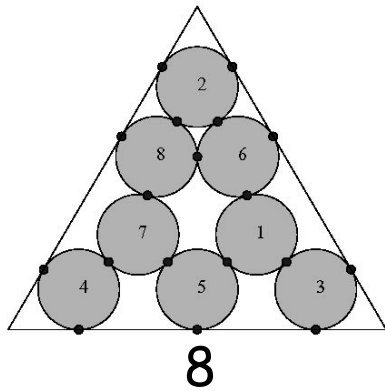
Conjecture (D. J. Newman) - \$100

The smallest equilateral triangle into which $\binom{n+2}{2} - 1$ points can be packed is (still) n !

This is no longer true for $\binom{n+2}{2} - 2$ points.



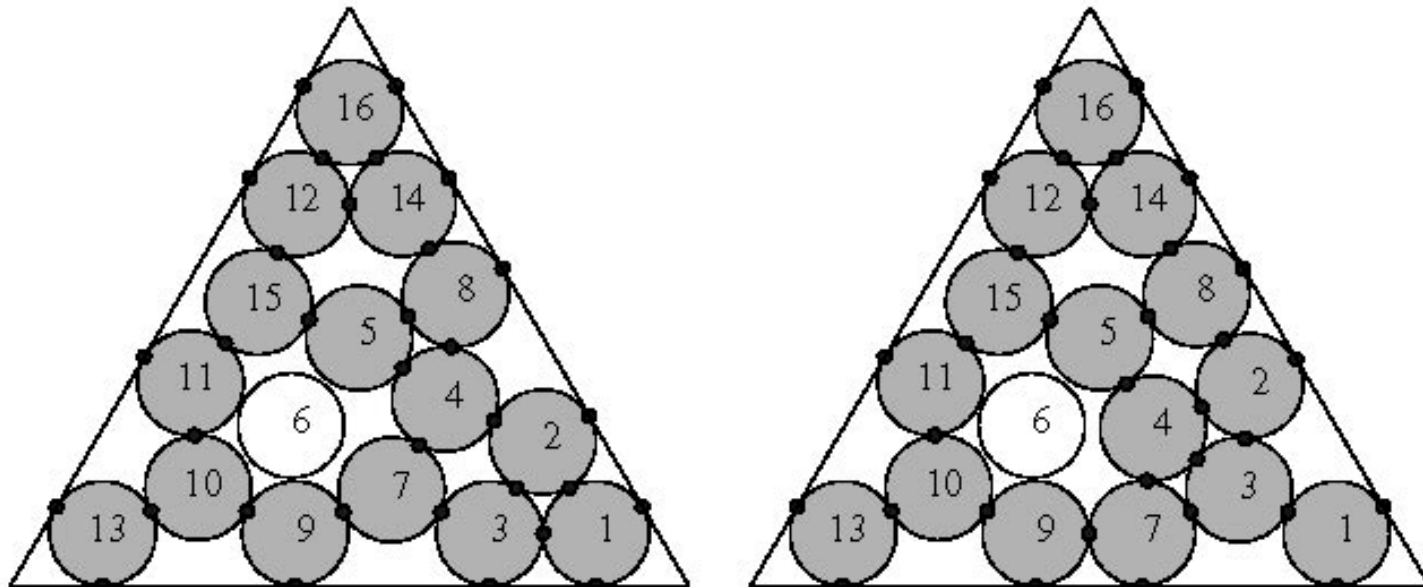
Two different optimal packings of 7 discs in an equilateral triangle



Optimal packings for $n = 8, 11,$ and 12 (H. Melissen - 1995)

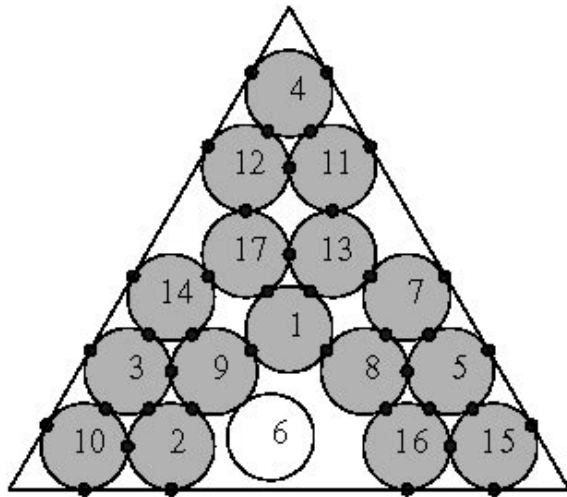
Packing for $n = 13$ is conjectured to be optimal.

$n = 12$ is the last known optimal value when $n \neq \Delta$

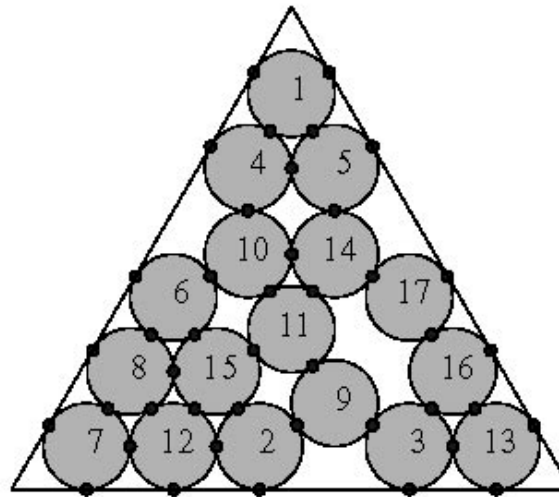


Two equally good packings for $n = 16$, conjectured to be optimal.

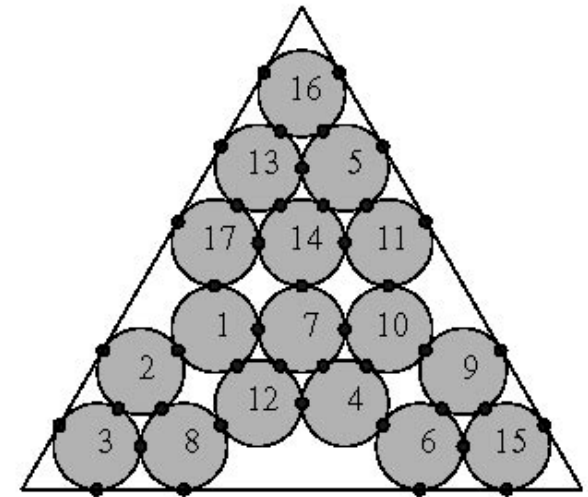
Especially hard cases seem to be when $n = \Delta + 1$.



40 "bonds"



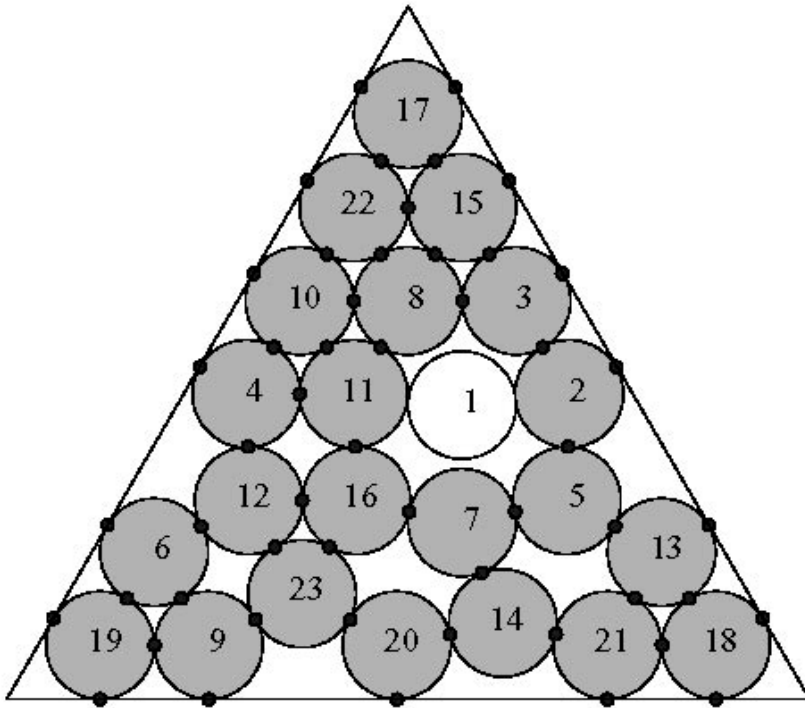
42 "bonds"



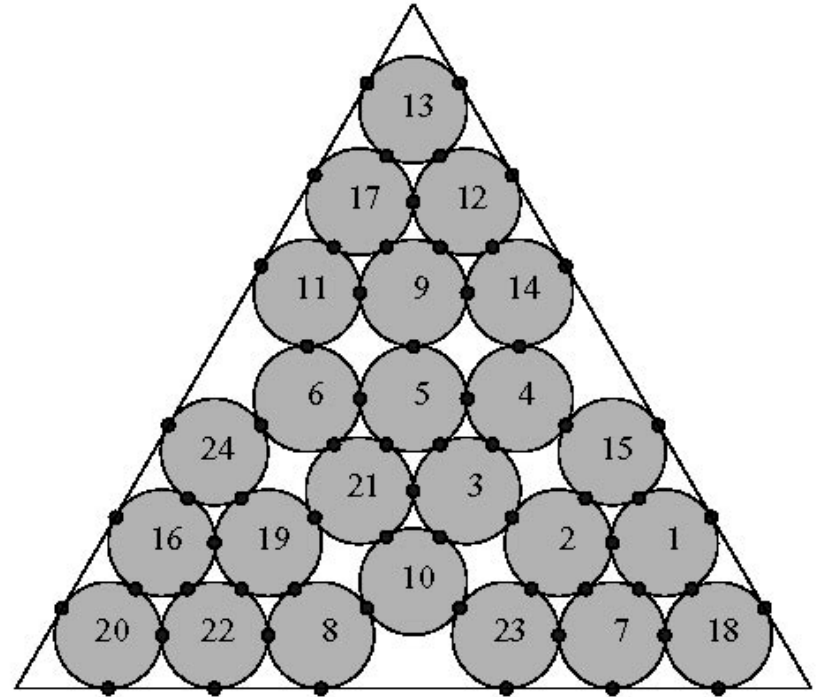
43 "bonds"

Three equally good conjectured optimal packings of 17 discs

.Note the "rattler" in the first packing



23

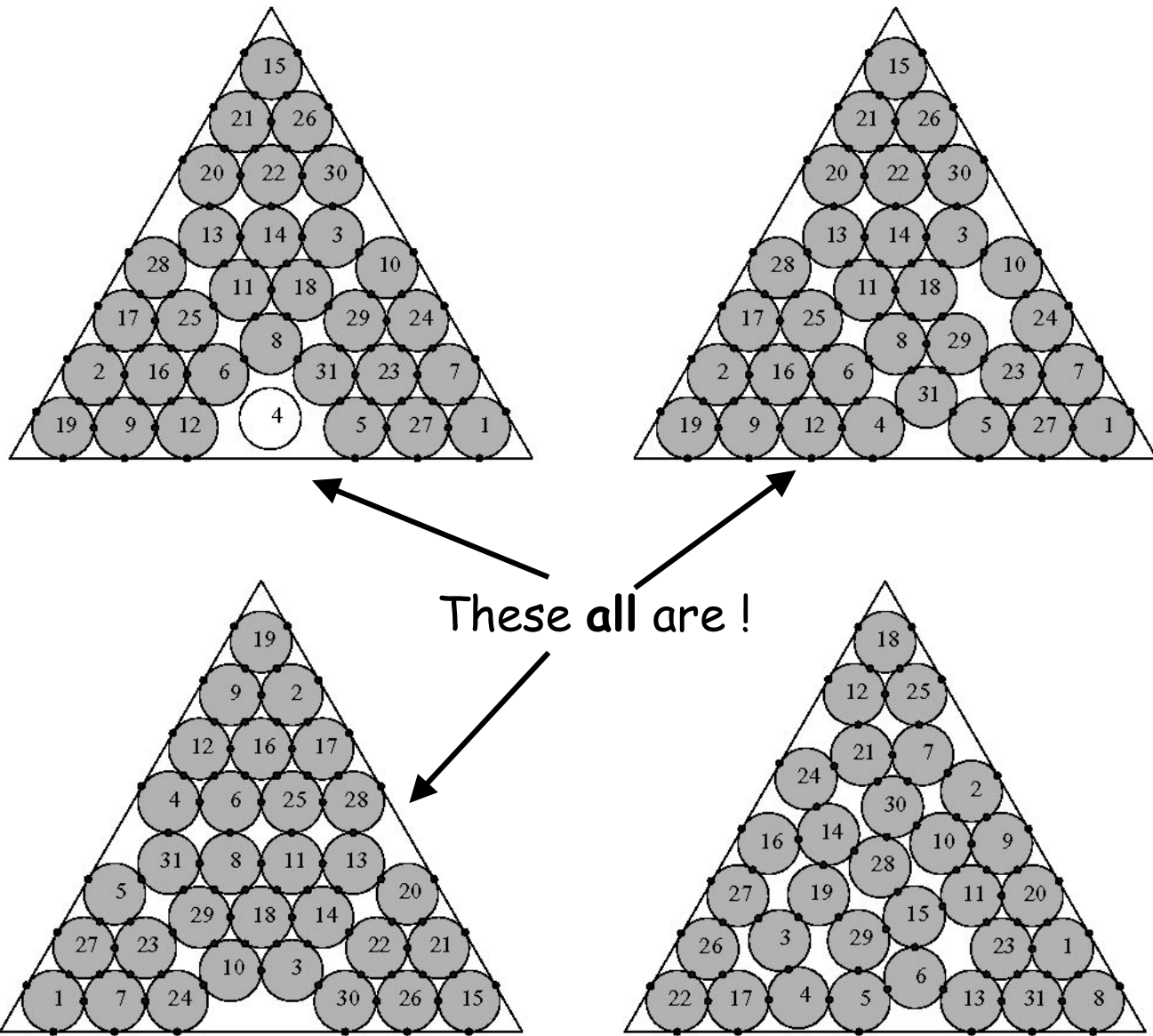


24

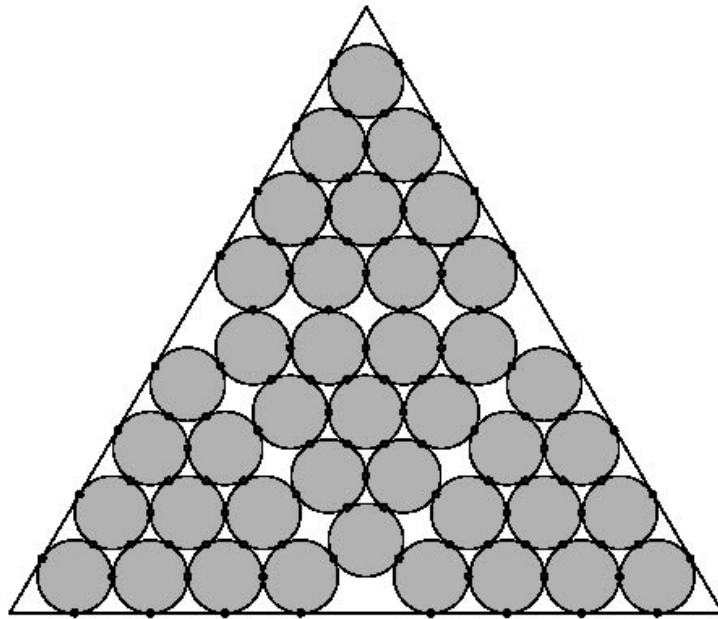
Conjectured optimal packings of 23 and 24 discs.

Which is nicer?

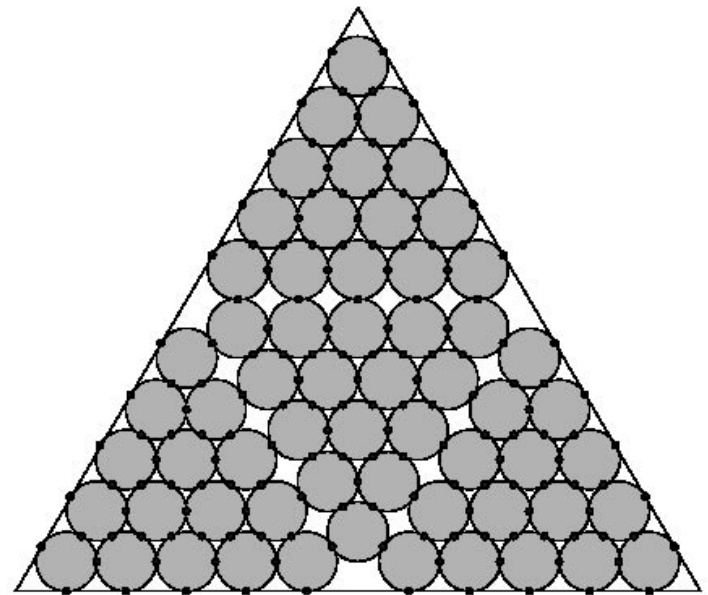
Any conjectures??



Four packings of 31 discs. Which is the best?

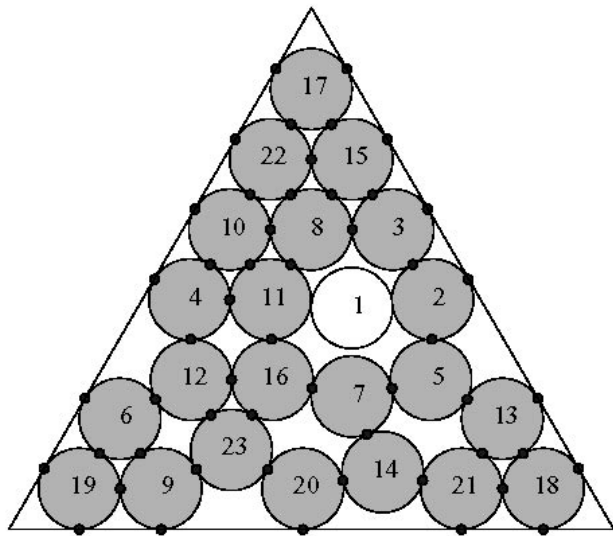


40

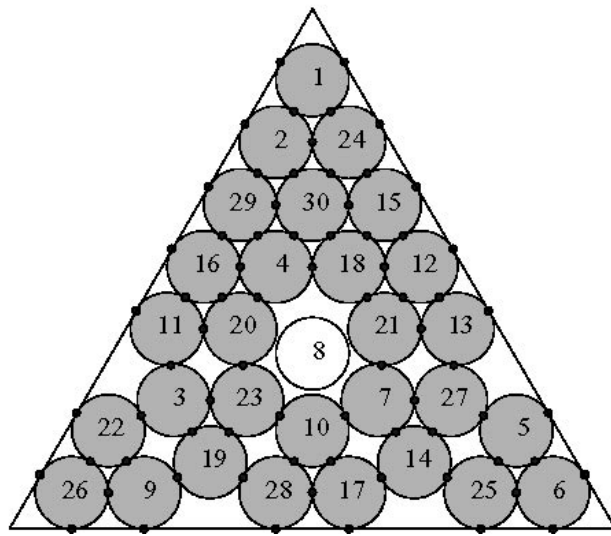


60

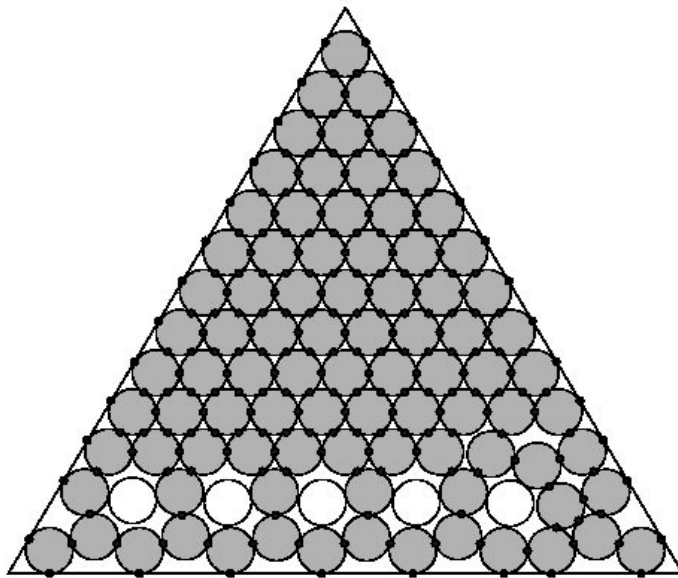
Conjectured optimal packings when $n = 4\Delta$



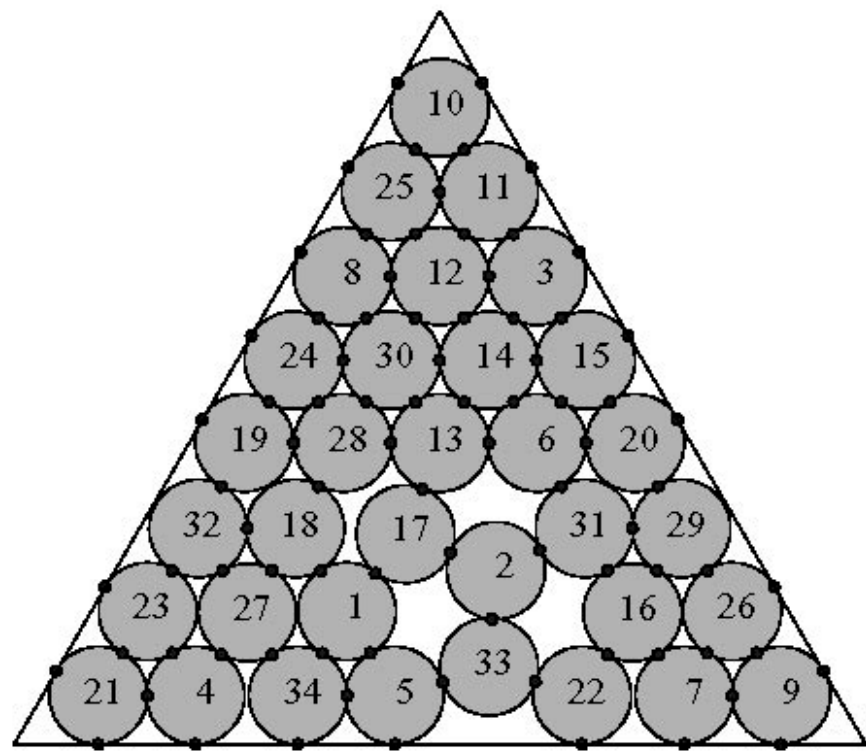
23



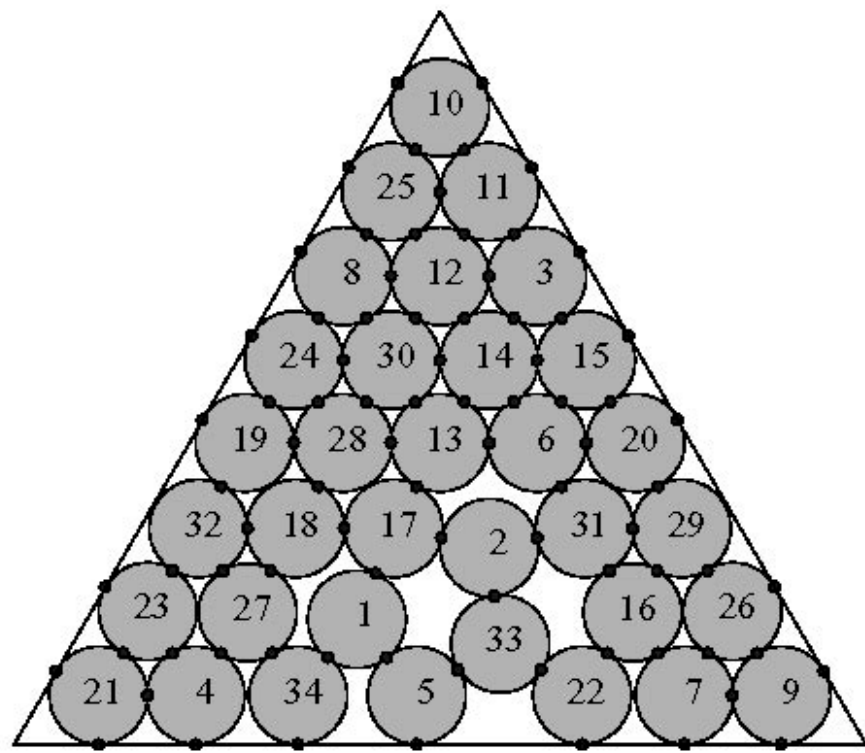
30



92



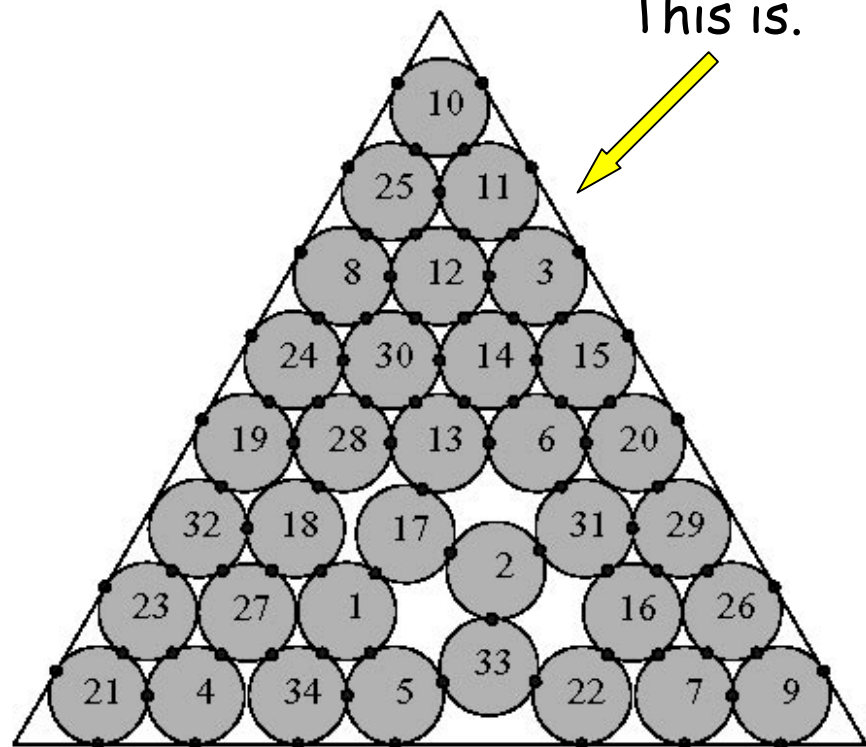
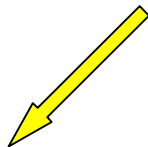
34



34

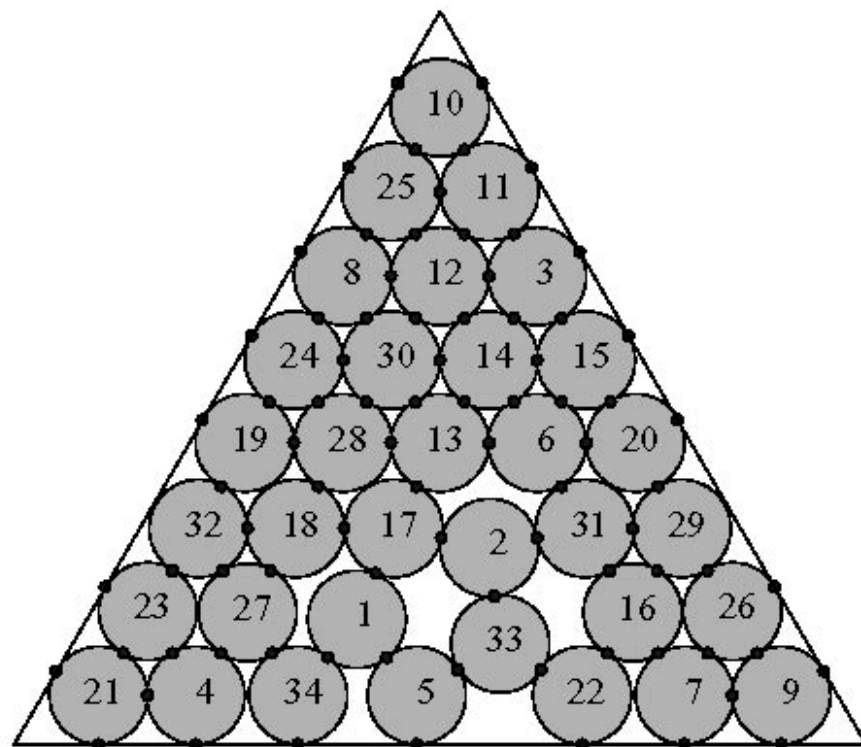
Which is better ??

This is.



34

0.142869646754496 84 bonds



34

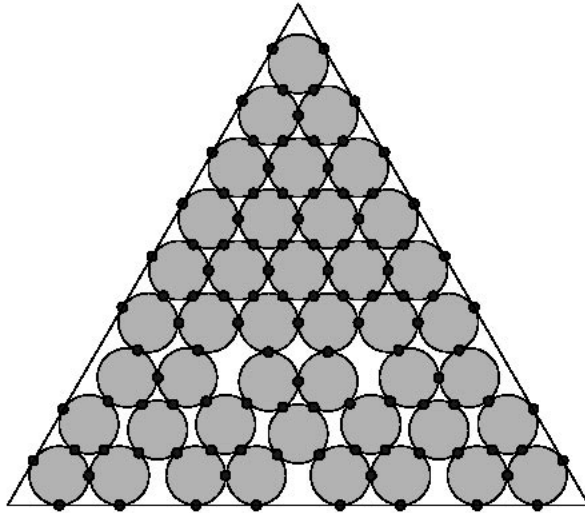
0.142867647681844 83 bonds



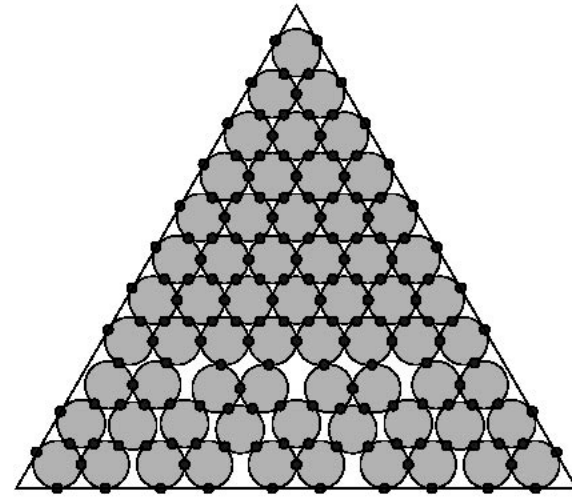
Which is better ??

(radius of discs if reduced triangle has side 1)

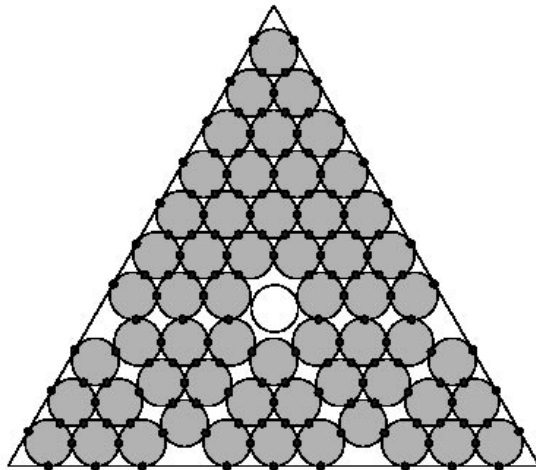
Patterns ?



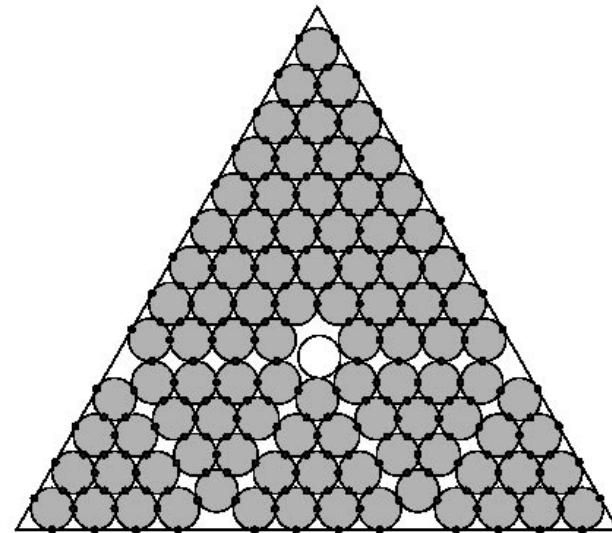
42



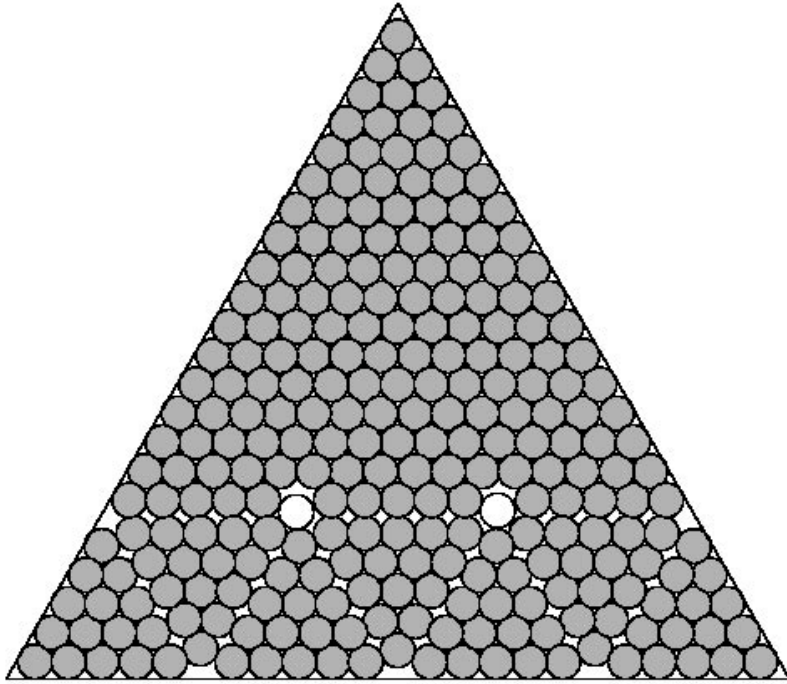
63



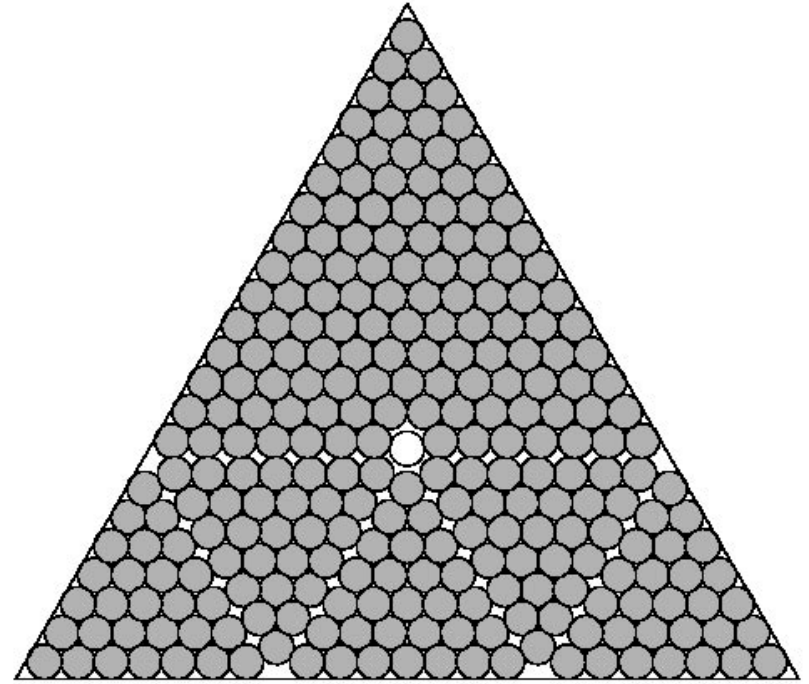
58



95



$n = 258$



$n = 260$

Part of general patterns?

$T(n)$:= minimum side length of equilateral triangle into which n unit discs can be packed

$$\delta(n) := T(n) - \frac{1}{2}(-3 + \sqrt{8n + 1})$$

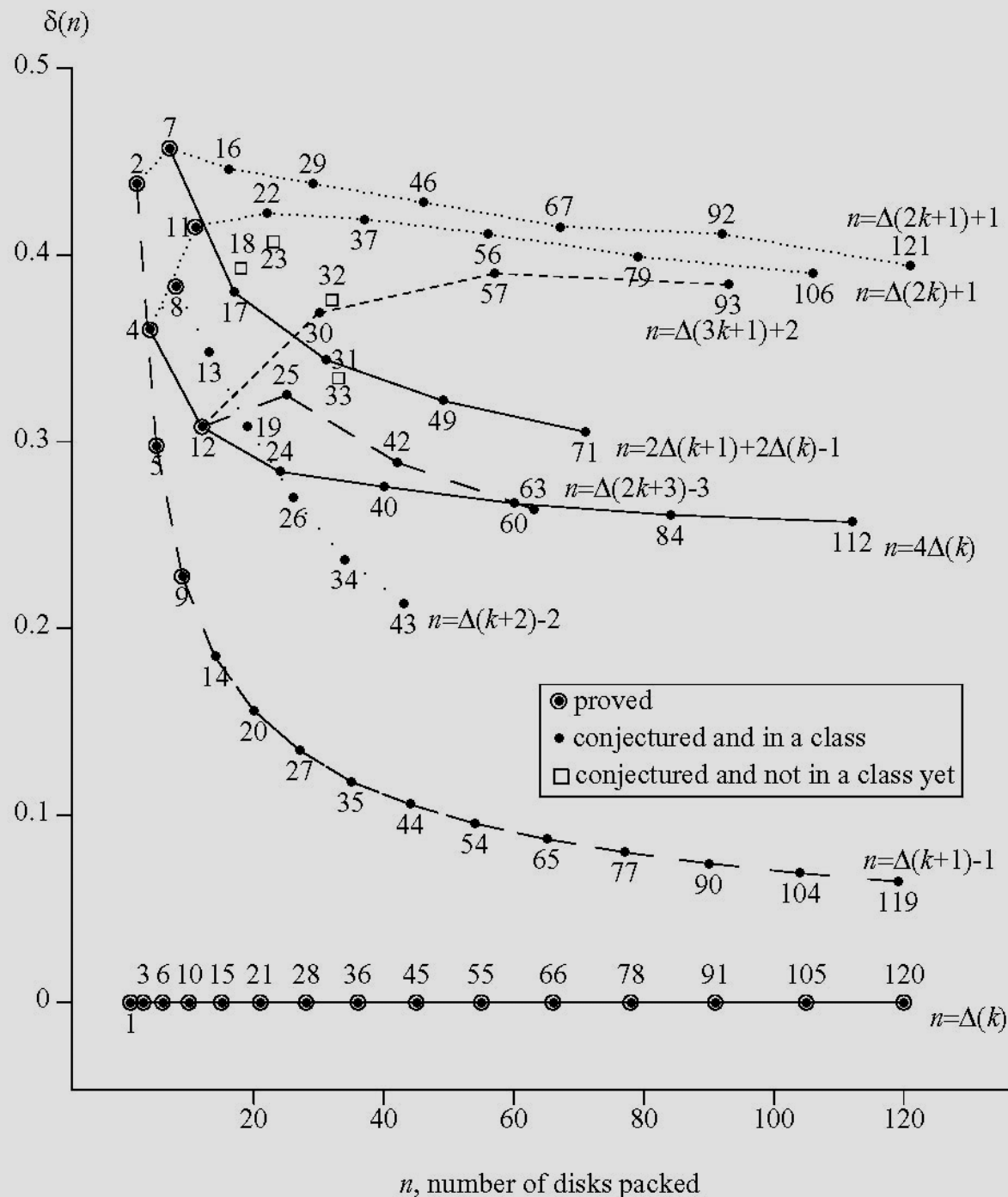
$\delta(n)$

$$\Delta(n) := \frac{1}{2}n(n + 1)$$

Conjecture:

$$T(\Delta(n) + 1) - T(\Delta(n)) > \varepsilon$$

for some $\varepsilon > 0$.

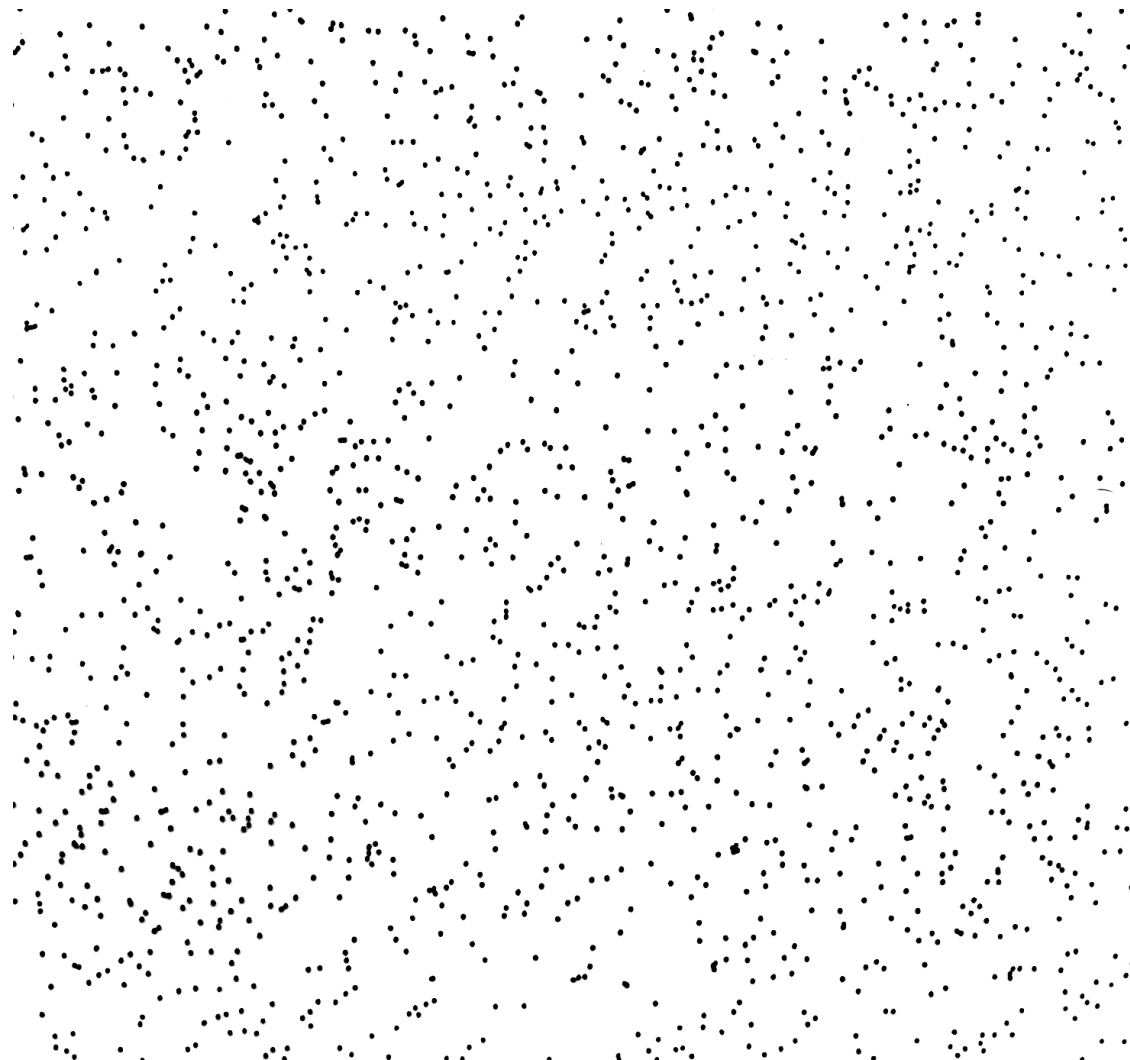


Packings were generated by an "event driven" billiards simulation algorithm written by Boris Lubachevsky (formerly of Bell Labs). They were designed in part to understand crystal growth in the presence of irregularities.

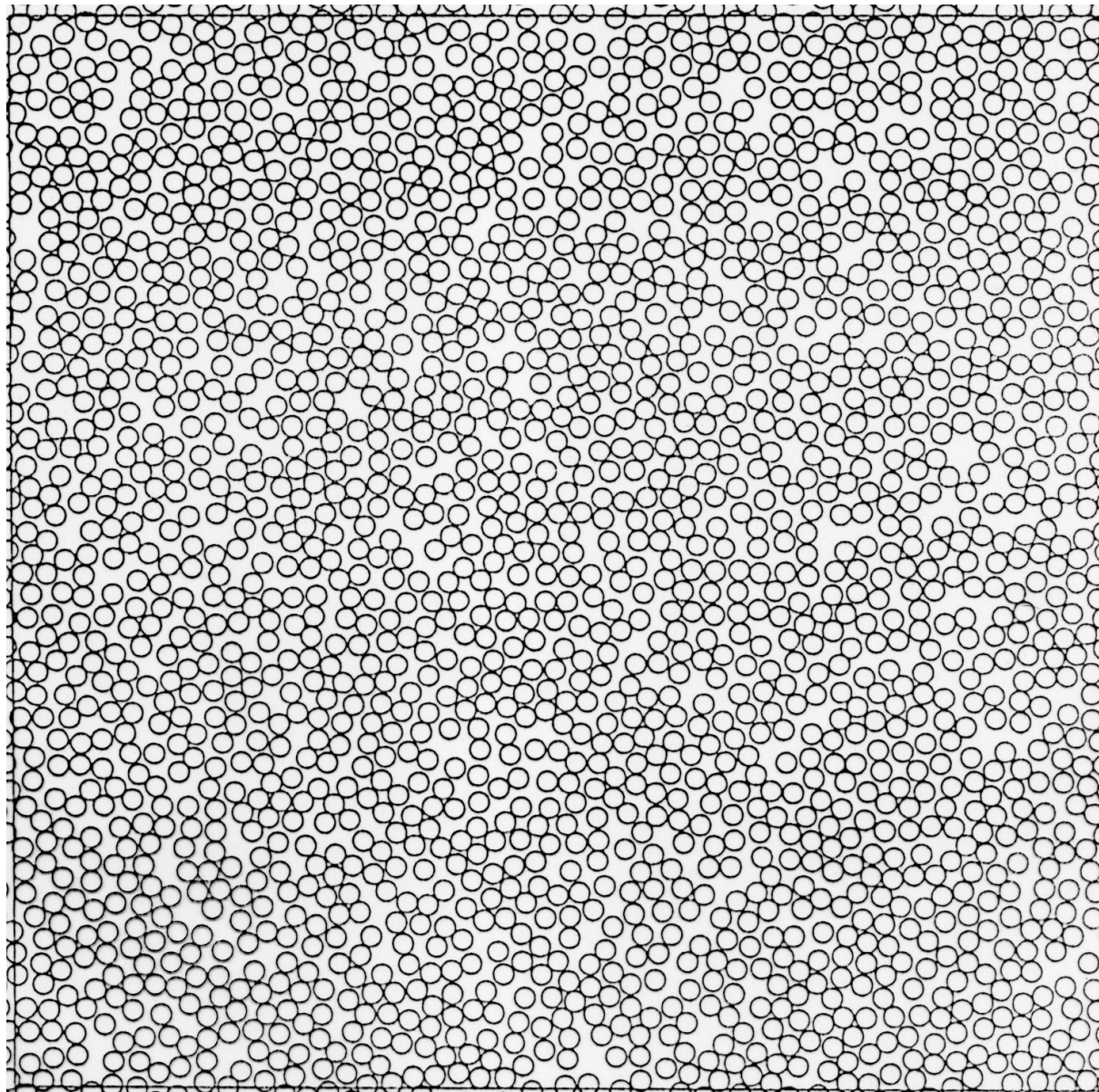
We start with very small discs with random positions and velocities and then let bounce around elastically while slowly increasing in size, until after many millions of bounces, they become "stuck".

You then repeat this process thousand of times!

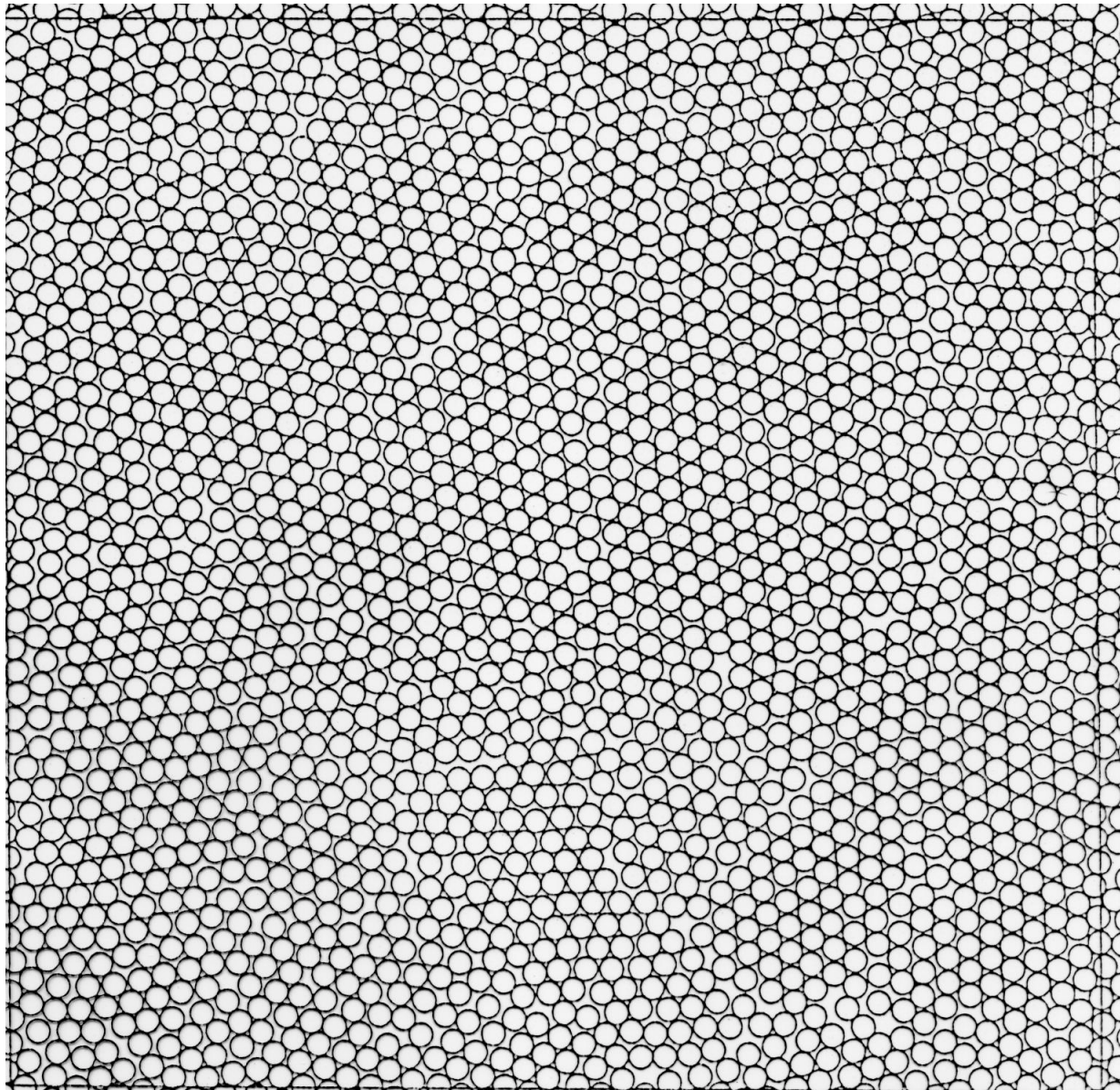
Stillinger, Frank H.; Lubachevsky, Boris D. Crystalline-amorphous interface packings for disks and spheres. *J. Statist. Phys.* 73 (1993), no. 3-4, 497--514



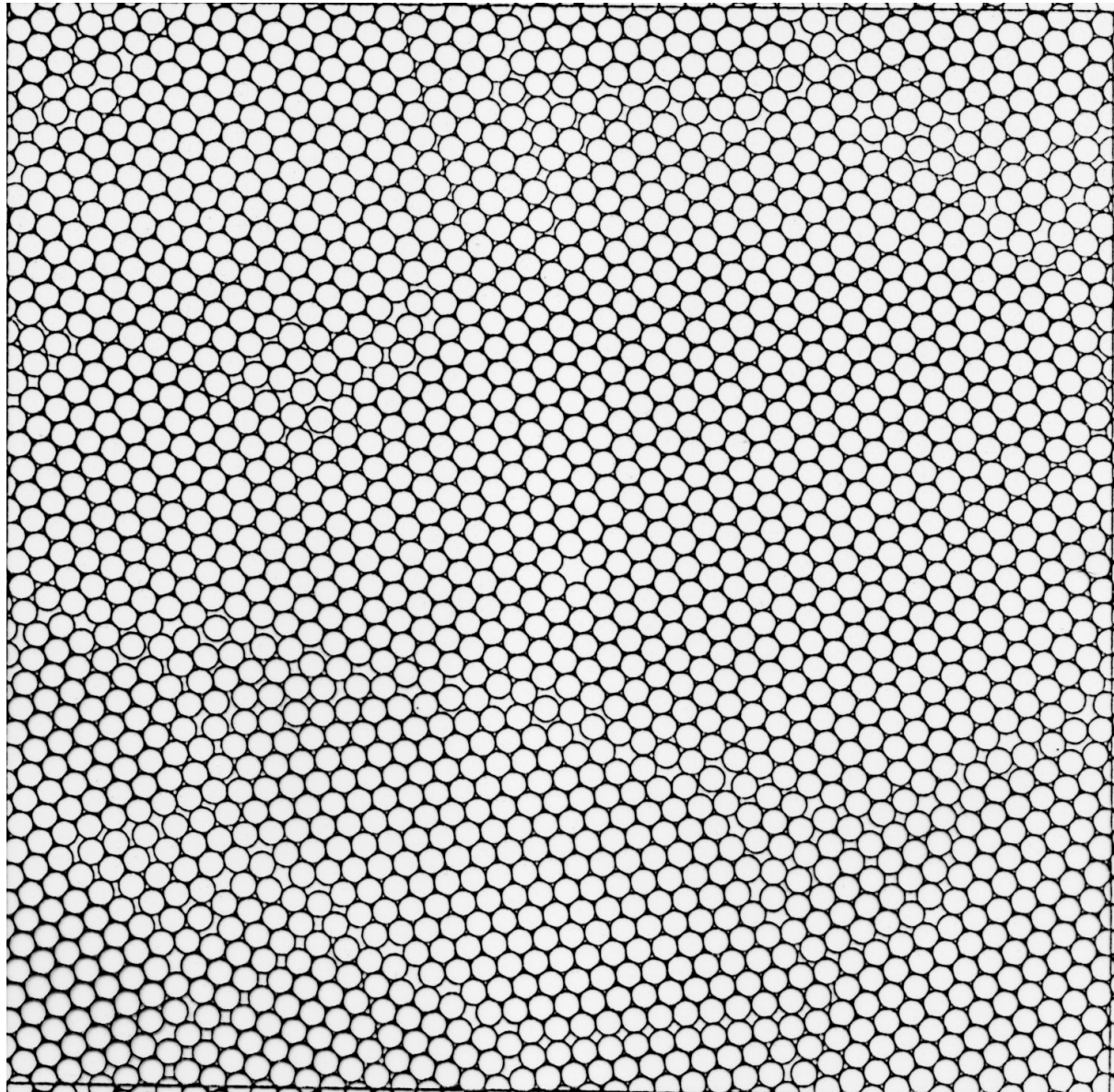
2000 random points



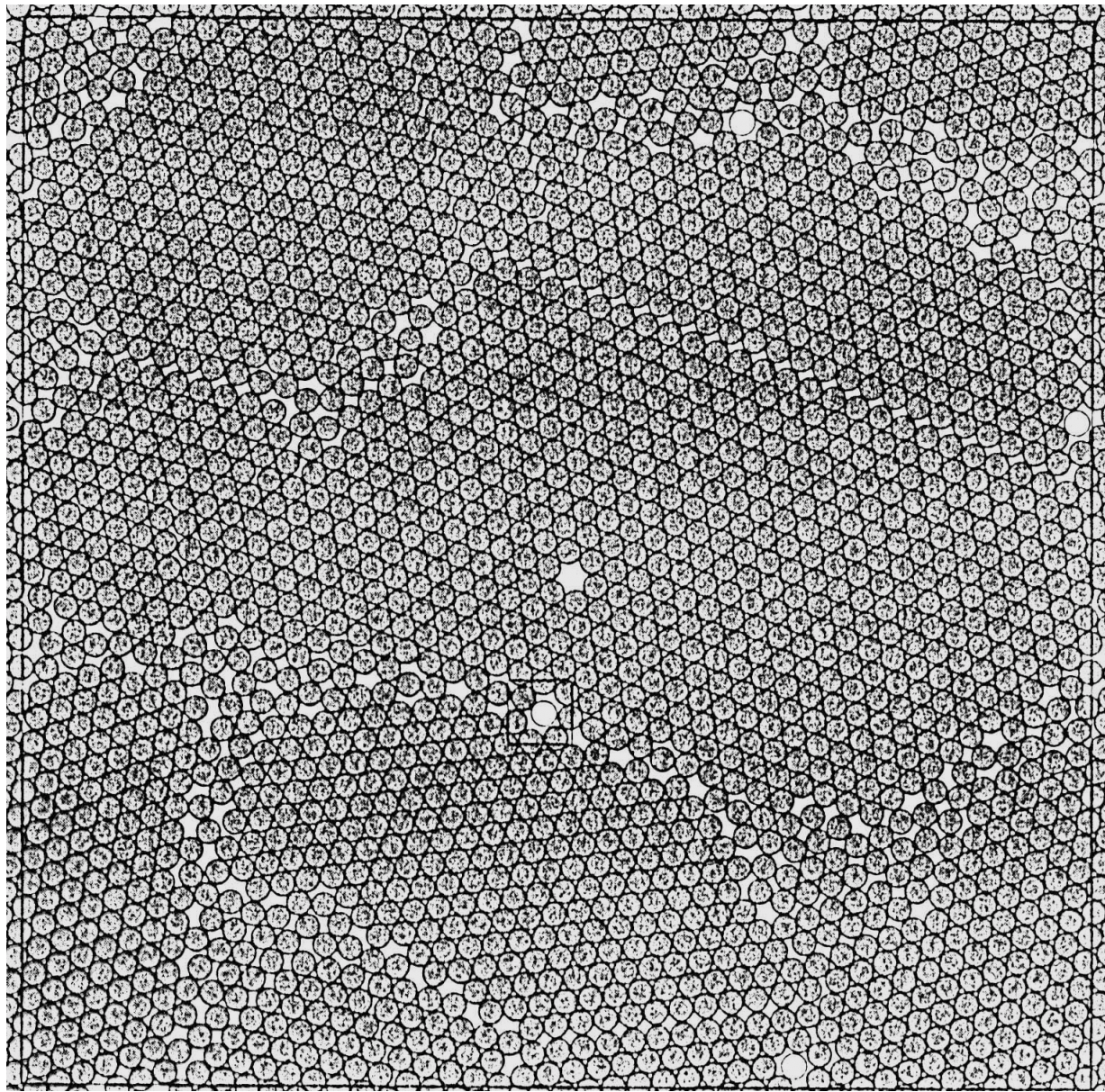
After 4×10^4 collisions



After 4×10^5 collisions



After 4×10^6 collisions



After 4×10^7 collisions

H. Melissen, Densest packings of congruent circles in an equilateral triangle, *Amer. Math. Monthly* 100 (1993), 916-925

B. D. Lubachevsky and RLG, Dense packings of equal discs in an equilateral triangle: from 22 to 34 and beyond, *Electronic J. Combinatorics* 2 (1995), #A1

H. Melissen, Packing and covering with circles, Ph.D. dissertation, Utrecht University, 1997, viii + 180 pp.

Packing discs into squares

This is one of the classic disc packing problems.

2. Old Results

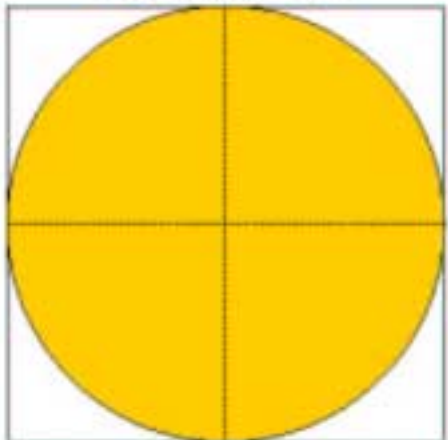
The problem of packing circles into different geometrical shapes has received much attention since the seminal work of Fejes Tóth [7]. A recent survey of results and problems still open can be found in [3]. One of the most natural and most studied of these problems is that of packing circles in a square.

This problem was solved for up to nine circles in the 1960s by Graham, Meir and Schaer; the proofs of these cases have been reported in [12], [18], [20], and [22]. The proofs for $n \leq 5$ are easy, whereas the cases $6 \leq n \leq 9$ require more elaborate mathematical tools. For example, for $n = 5$ we can divide the square into four subsquares as indicated in Fig. 1. Now at least one square must contain two points due to the pigeon-hole principle, so the length of the diagonals in the subsquares ($\sqrt{2}/2$) upper-bounds d_5 . This is also a lower bound, since in the solution in Fig. 1 (which is the only possible optimal solution), this is the smallest distance between two points. Thereby $d_5 = \sqrt{2}/2$.

For $n \geq 10$, only the optimal packings of 14 [26], 16 [24], 25 [25], and 36 [10] circles have been proved by hand.

$N = 1$ *

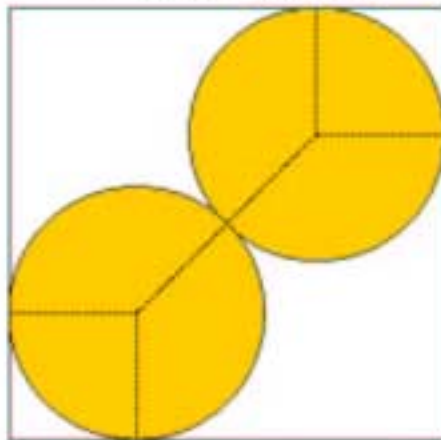
1 circle in the unit square



radius = 0.5000000000 Area = 0.7853981634
Perimeter = 3.1415926536 Perimeter = 3

$N = 2$ *

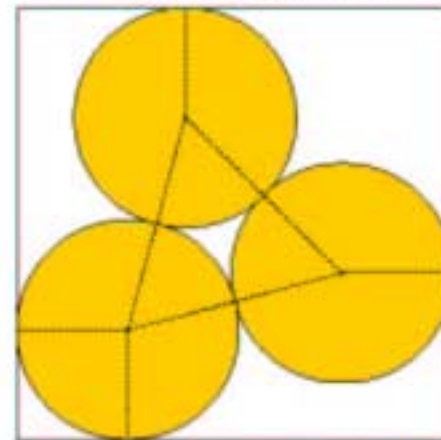
2 circles in the unit square



radius = 0.3660254037 Area = 0.4241183637
Perimeter = 2.3430804417 Perimeter = 2

$N = 3$ *

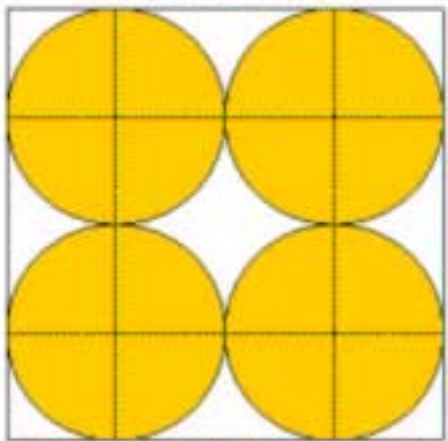
3 circles in the unit square



radius = 0.2886751345 Area = 0.2454925766
Perimeter = 1.8853981634 Perimeter = 3

$N = 4$ *

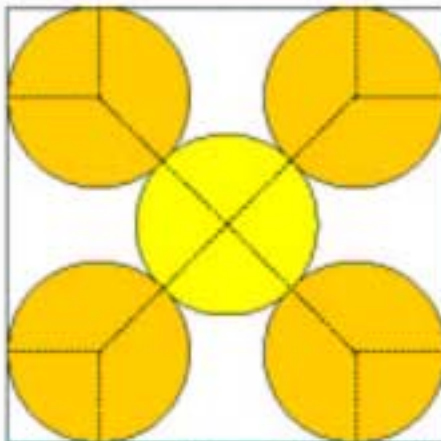
4 circles in the unit square



radius = 0.2500000000 Area = 0.1963495408
Perimeter = 1.5707963268 Perimeter = 4

$N = 5$ *

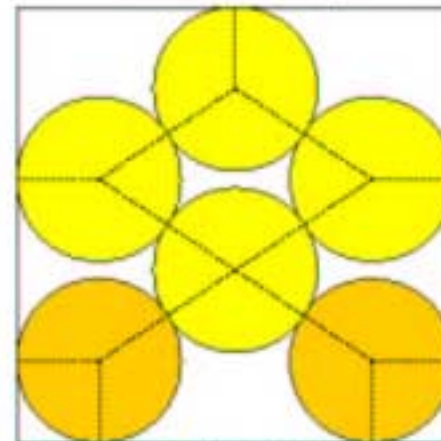
5 circles in the unit square



radius = 0.1877459462 Area = 0.1017614699
Perimeter = 1.1771277112 Perimeter = 5

$N = 6$ *

6 circles in the unit square

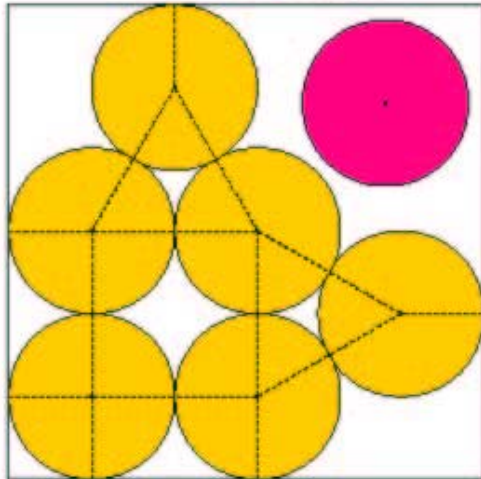


radius = 0.1644853627 Area = 0.0844444444
Perimeter = 1.0296215177 Perimeter = 6

* \Rightarrow optimal

$N = 7^*$

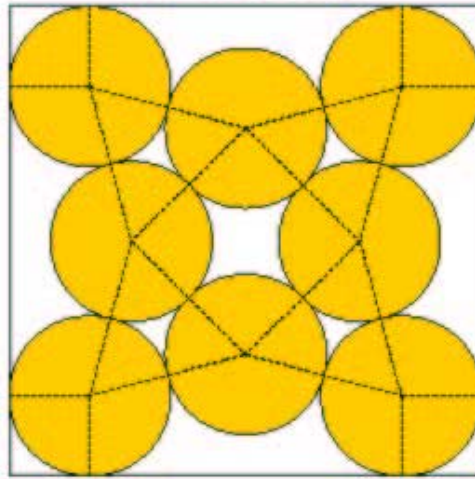
7 circles in the unit square



radius = 0.17483763137 density = 0.4423082541 # of contacts = 12
 distance = 0.5233834462 contacts = 12

$N = 8^*$

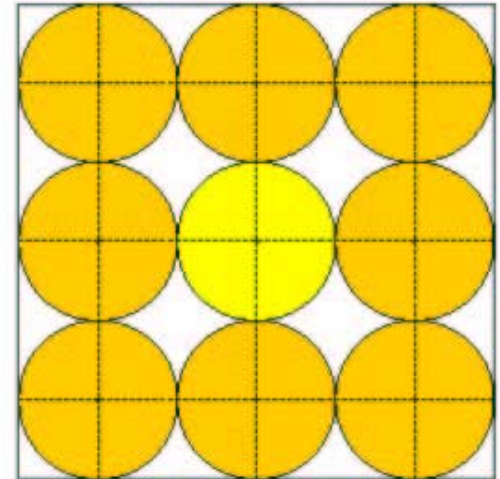
8 circles in the unit square



radius = 0.17054096701 density = 0.7506733234 # of contacts = 18
 distance = 0.53707699205 contacts = 18

$N = 9^*$

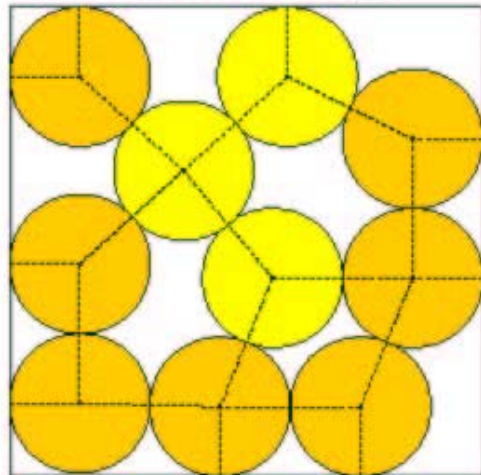
9 circles in the unit square



radius = 0.16666666667 density = 0.7832637297 # of contacts = 24
 distance = 0.50000000000 contacts = 24

$N = 10^*$

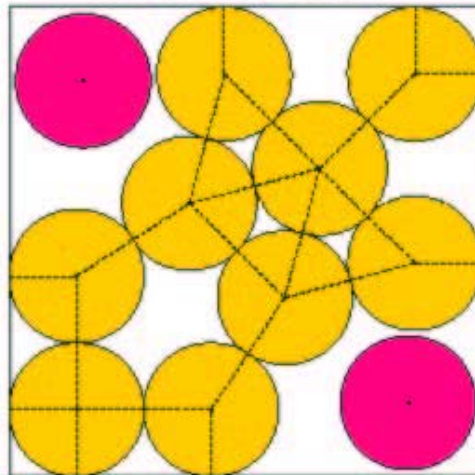
10 circles in the unit square



radius = 0.16024052385 density = 0.4900294534 # of contacts = 18
 distance = 0.62323267364 contacts = 18

$N = 11^*$

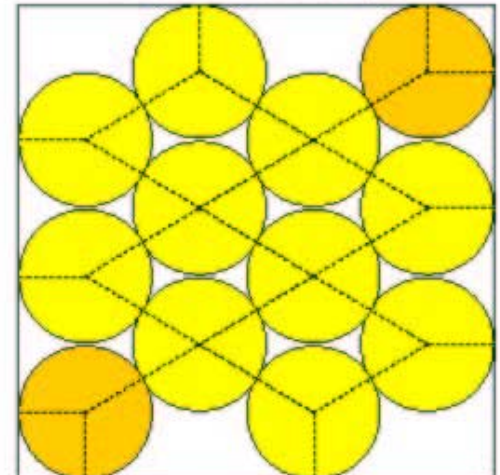
11 circles in the unit square



radius = 0.14223427374 density = 0.7007187774 # of contacts = 20
 distance = 0.70007733037 contacts = 20

$N = 12^*$

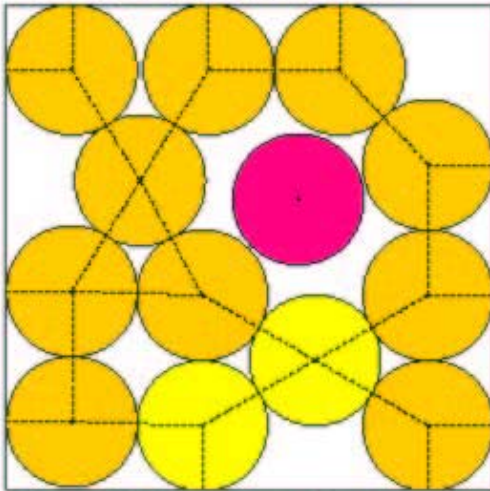
12 circles in the unit square



radius = 0.12995864081 density = 0.75846323884 # of contacts = 24
 distance = 0.76670263321 contacts = 24

$N = 13^*$

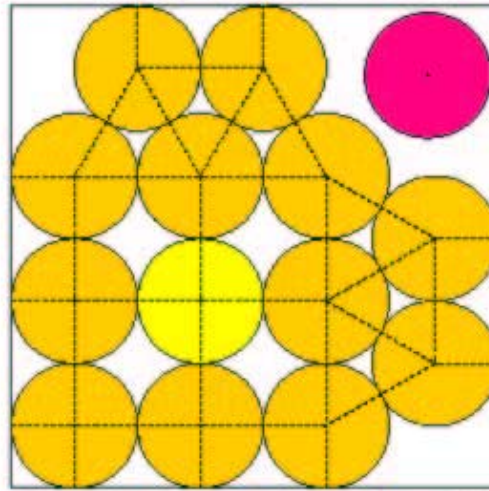
13 circles in the unit square



radius = 0.42599513499 density = 0.75226469608 # of contacts = 25
 distance = 0.36806009516

$N = 14^*$

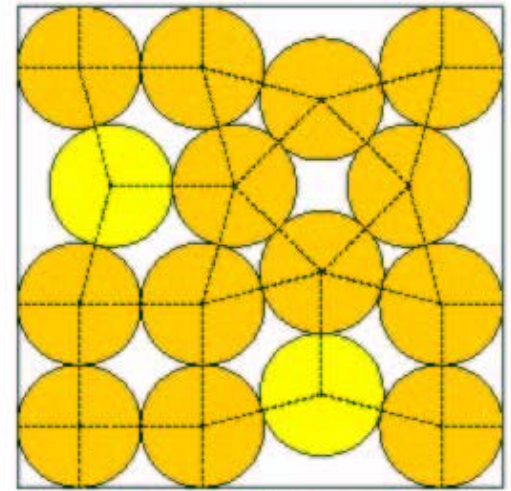
14 circles in the unit square



radius = 0.42871727120 density = 0.77397267947 # of contacts = 27
 distance = 0.34691504074

$N = 15^*$

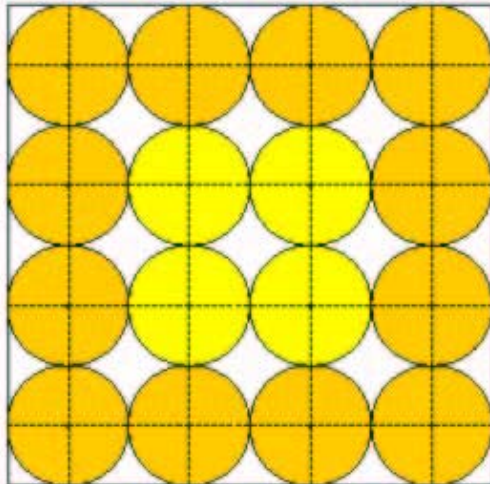
15 circles in the unit square



radius = 0.42720750712 density = 0.76209030027 # of contacts = 28
 distance = 0.343063277400

$N = 16^*$

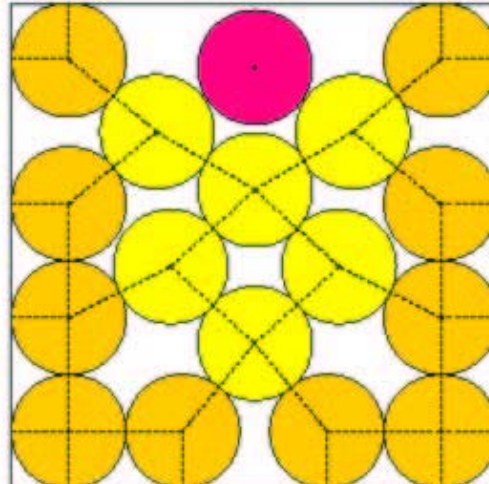
16 circles in the unit square



radius = 0.319000000000 density = 0.70599833397 # of contacts = 40
 distance = 0.333333333333

$N = 17^*$

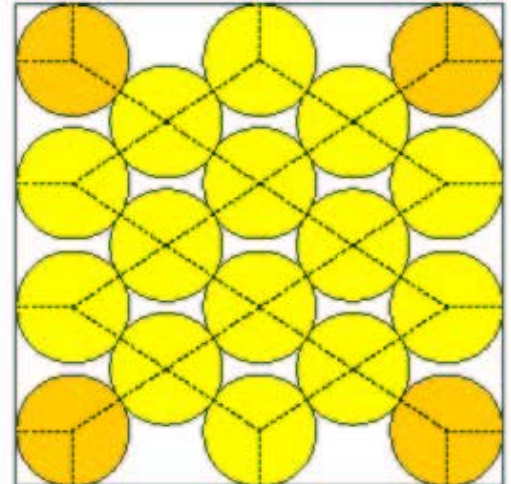
17 circles in the unit square



radius = 0.41719742783 density = 0.73358063308 # of contacts = 34
 distance = 0.30613745300

$N = 18^*$

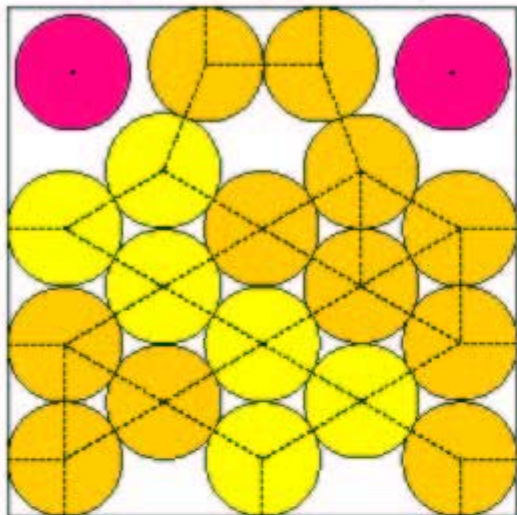
18 circles in the unit square



radius = 0.412521473444 density = 0.75462577474 # of contacts = 35
 distance = 0.300422402111

N = 19*

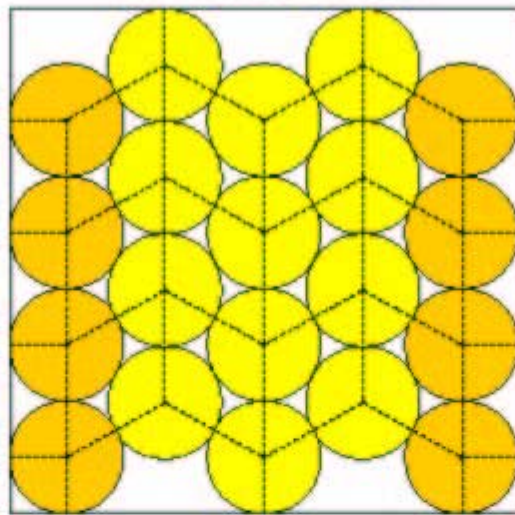
19 circles in the unit square



radius = 0.112065407571 density = 0.75200760742 #SPR
 distance = 0.289561991995 contacts = 37 # of SPR

N = 20*

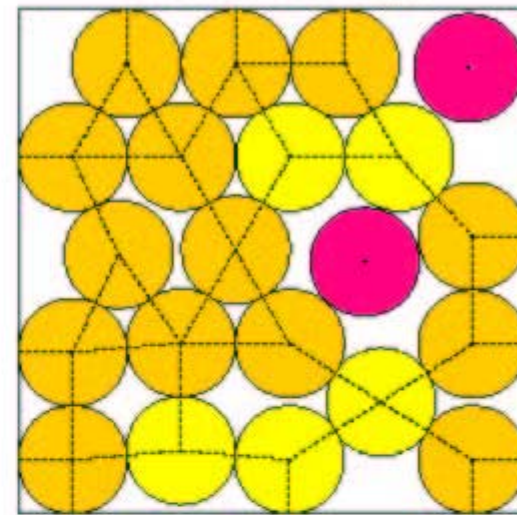
20 circles in the unit square



radius = 0.111365247512 density = 0.77049564665 #SPR
 distance = 0.286613852330 contacts = 44 # of SPR

N = 21*

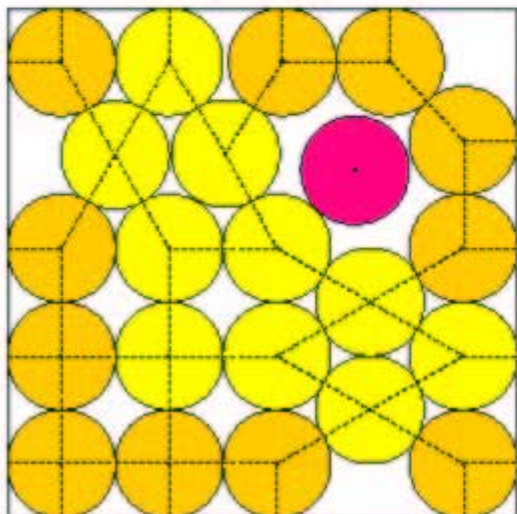
21 circles in the unit square



radius = 0.10666012755 density = 0.78275702640 #SPR
 distance = 0.27103229553 contacts = 53 # of SPR

N = 22*

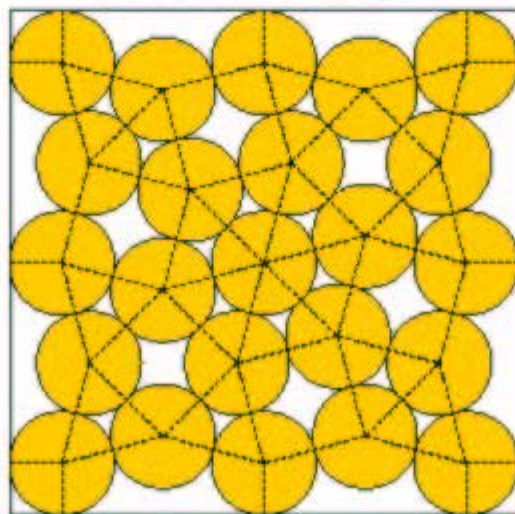
22 circles in the unit square



radius = 0.1056526757 density = 0.7743612096 #SPR
 distance = 0.26795261551 contacts = 41 # of SPR

N = 23*

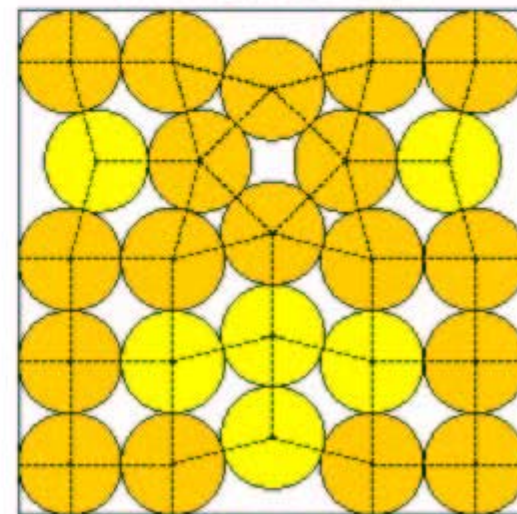
23 circles in the unit square



radius = 0.10506213760 density = 0.76742107114 #SPR
 distance = 0.256619045105 contacts = 56 # of SPR

N = 24*

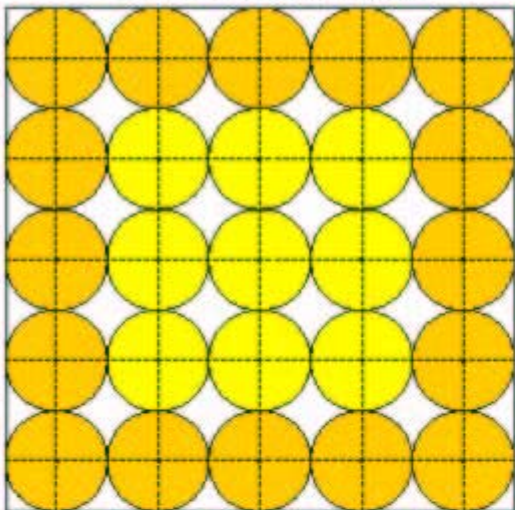
24 circles in the unit square



radius = 0.10126300451 density = 0.77467230756 #SPR
 distance = 0.25425509500 contacts = 56 # of SPR

N = 25 *

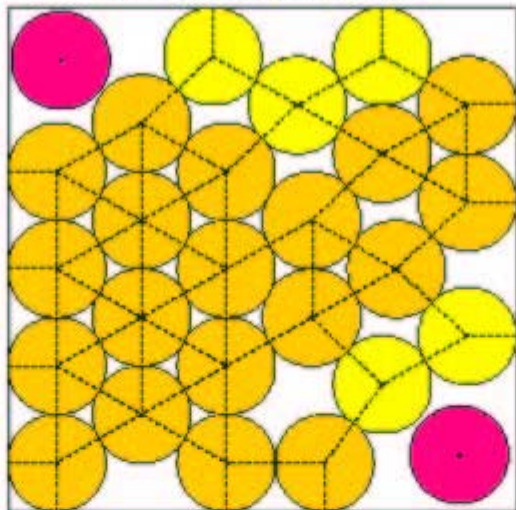
25 circles in the unit square



radius = 0.200000000000 density = 0.785398163397 # of
 distance = 0.250000000000 contacts = 00

N = 26 *

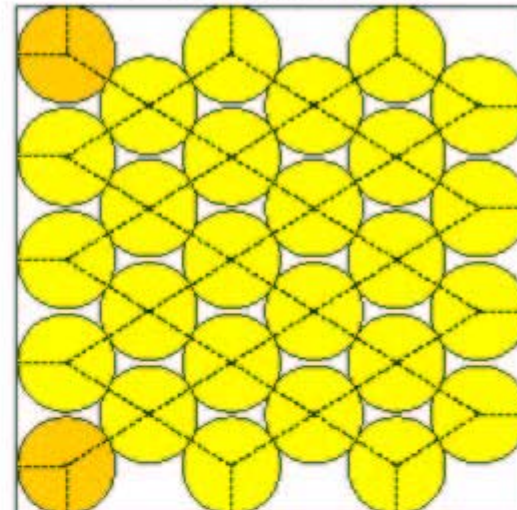
26 circles in the unit square



radius = 0.095702329030 density = 0.75062020434 # of
 distance = 0.250754792441 contacts = 00

N = 27 *

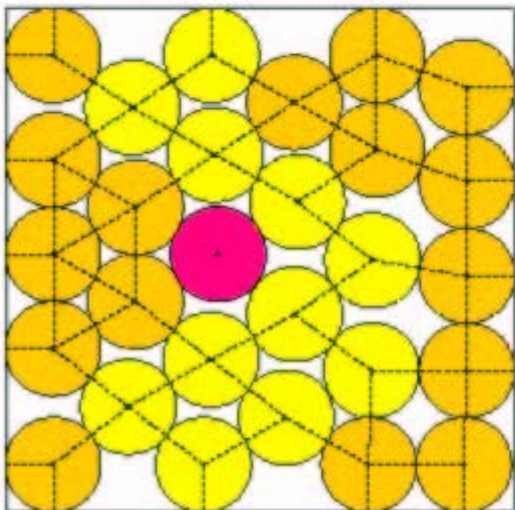
27 circles in the unit square



radius = 0.09540003144 density = 0.77021295467 # of
 distance = 0.225842983701 contacts = 00

N = 28

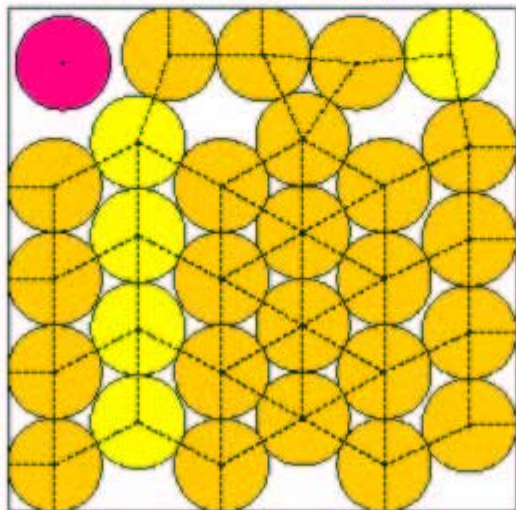
28 circles in the unit square



radius = 0.08672622025 density = 0.773054113404 # of
 distance = 0.250305420443 contacts = 01

N = 29

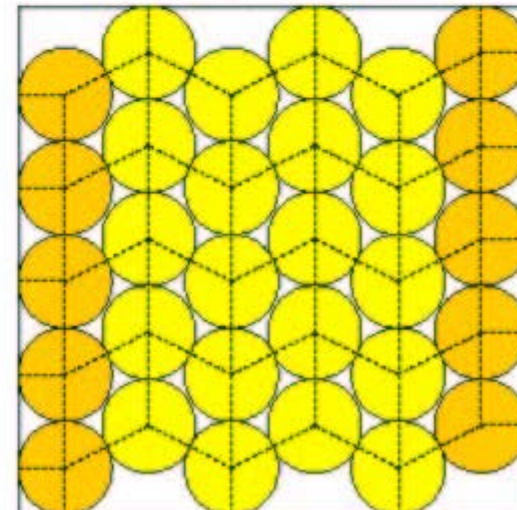
29 circles in the unit square



radius = 0.09240324040 density = 0.75620243740 # of
 distance = 0.22900200744 contacts = 01

N = 30

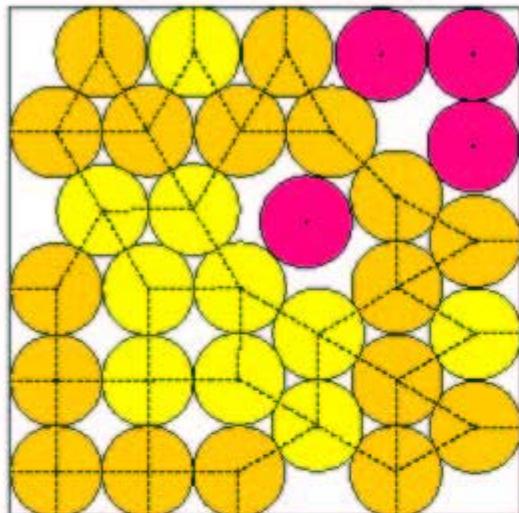
30 circles in the unit square



radius = 0.09167207986 density = 0.740019025461 # of
 distance = 0.224902264321 contacts = 01

N = 31

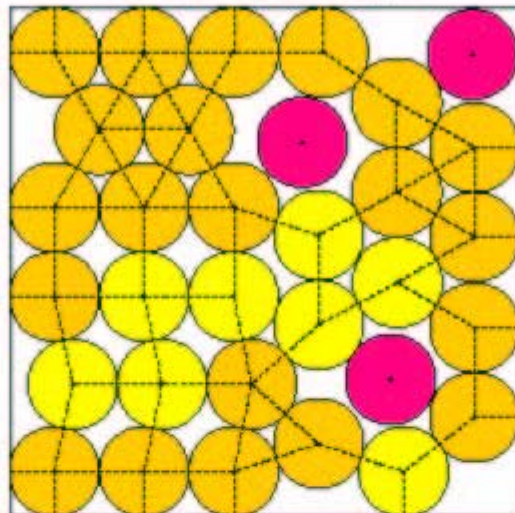
31 circles in the unit square



radius = 0.088233333331 density = 0.777237478723
 distance = 0.21049229818 contacts = 55

N = 32

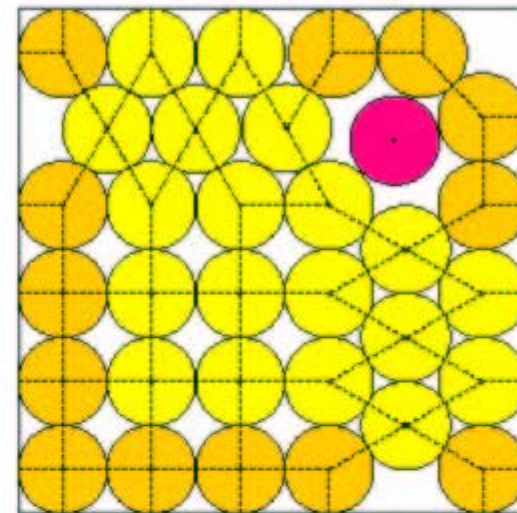
32 circles in the unit square



radius = 0.087652157066 density = 0.77600424405
 distance = 0.211174522590 contacts = 62

N = 33

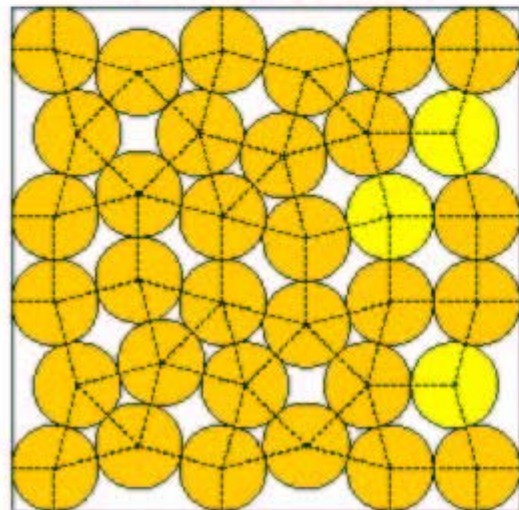
33 circles in the unit square



radius = 0.087250016124 density = 0.76665228254
 distance = 0.211820824385 contacts = 65

N = 34

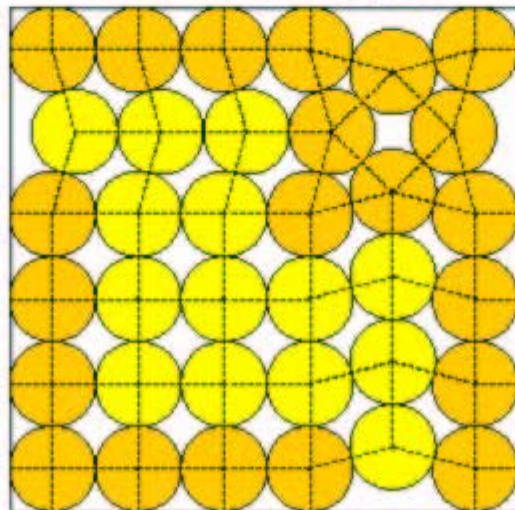
34 circles in the unit square



radius = 0.08570042051 density = 0.77164264552
 distance = 0.10396616760 contacts = 60

N = 35

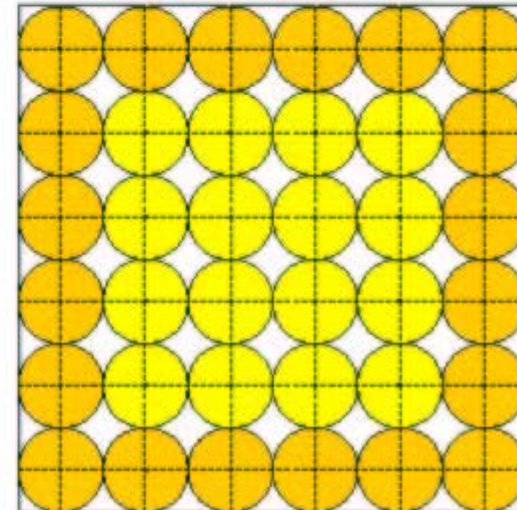
35 circles in the unit square



radius = 0.084290712123 density = 0.761227121299
 distance = 0.10274500067 contacts = 60

N = 36*

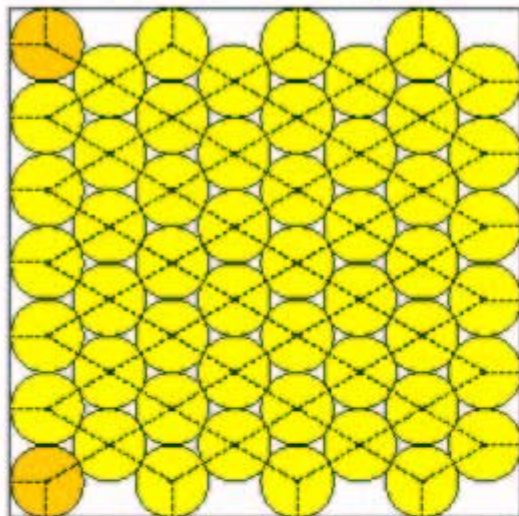
36 circles in the unit square



radius = 0.083333333333 density = 0.76074630328
 distance = 0.100000000000 contacts = 64

N = 52

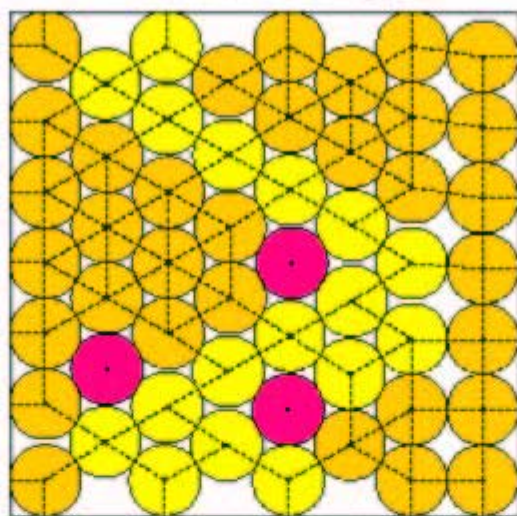
52 circles in the unit square



radius = 0.070857193073 density = 0.82950840063 # of contacts = 105

N = 53

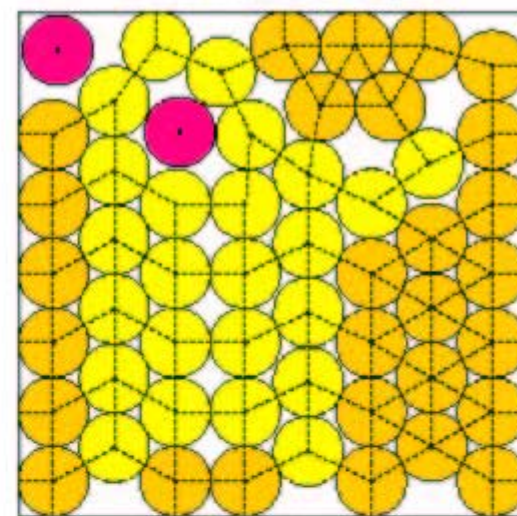
53 circles in the unit square



radius = 0.07094704447 density = 0.83042404208 # of contacts = 110

N = 54

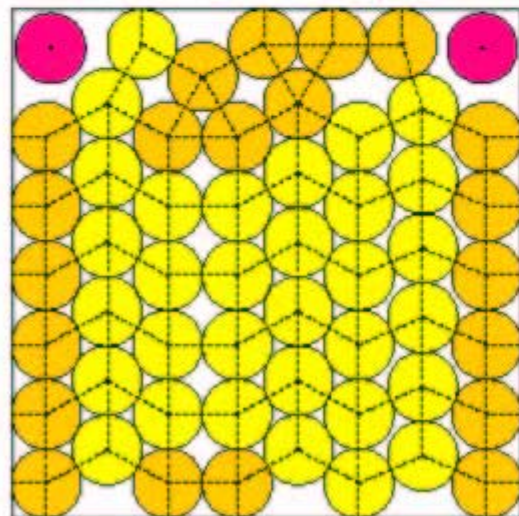
54 circles in the unit square



radius = 0.07045540303 density = 0.79460712064 # of contacts = 115

N = 55

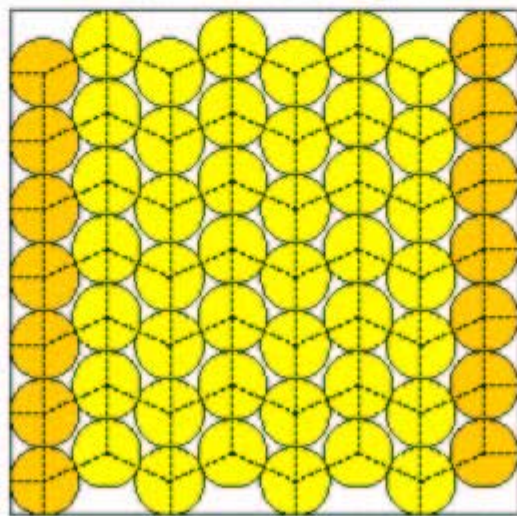
55 circles in the unit square



radius = 0.06865040559 density = 0.80007107045 # of contacts = 110

N = 56

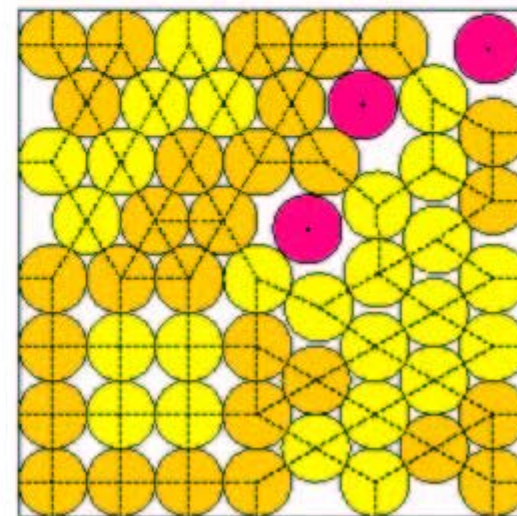
56 circles in the unit square



radius = 0.070702304322 density = 0.80291719903 # of contacts = 113

N = 57

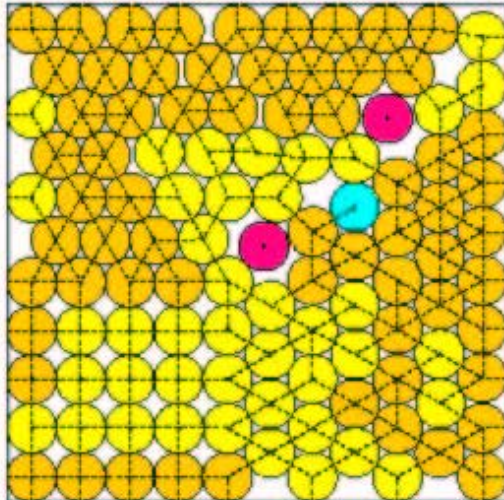
57 circles in the unit square



radius = 0.07004677304 density = 0.80247670441 # of contacts = 117

N = 109

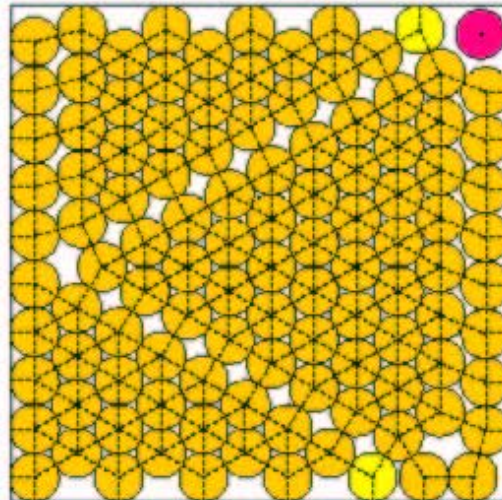
109 circles in the unit square



radius = 0.04305543086 density = 0.824045052234
distance = 0.300765030713 contacts = 226

N = 110

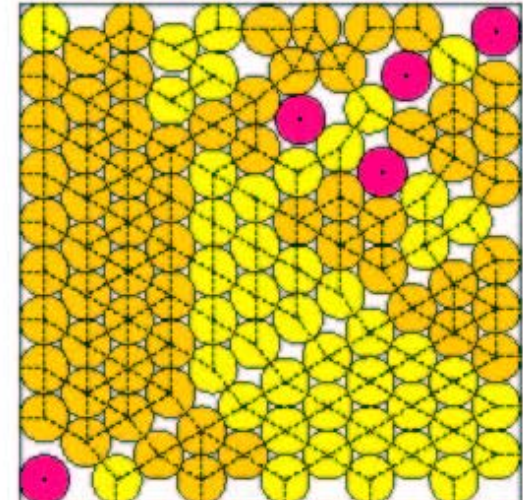
110 circles in the unit square



radius = 0.04277980531 density = 0.82274209044
distance = 0.300305906933 contacts = 235

N = 111

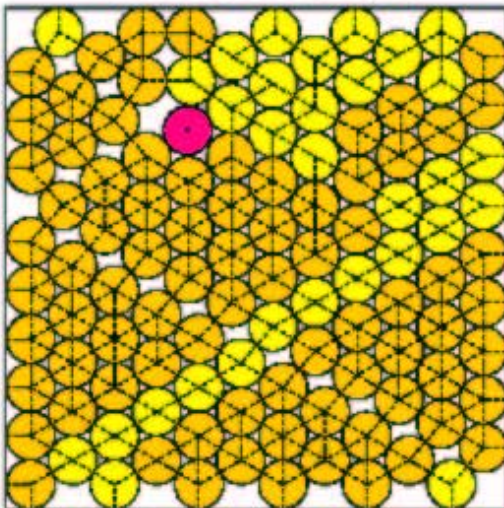
111 circles in the unit square



radius = 0.04264205941 density = 0.825066945510
distance = 0.307767230300 contacts = 226

N = 112

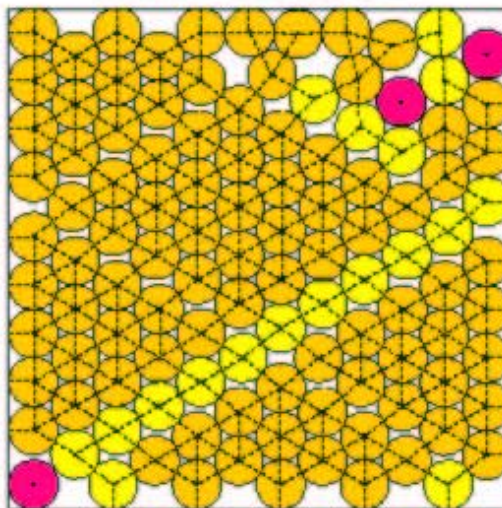
112 circles in the unit square



radius = 0.042645805903 density = 0.825796605810
distance = 0.307955822215 contacts = 251

N = 113

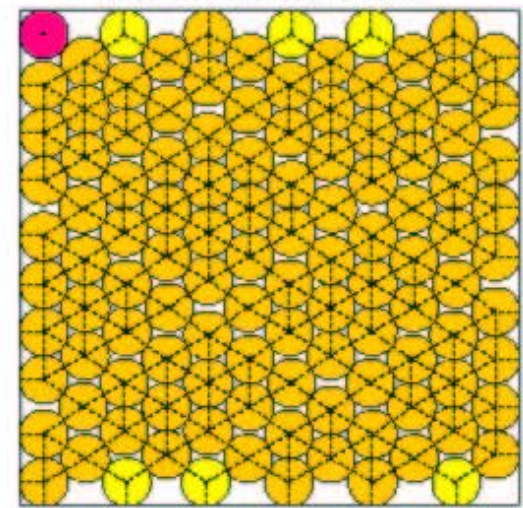
113 circles in the unit square



radius = 0.042257532089 density = 0.826720067986
distance = 0.306525535451 contacts = 257

N = 114

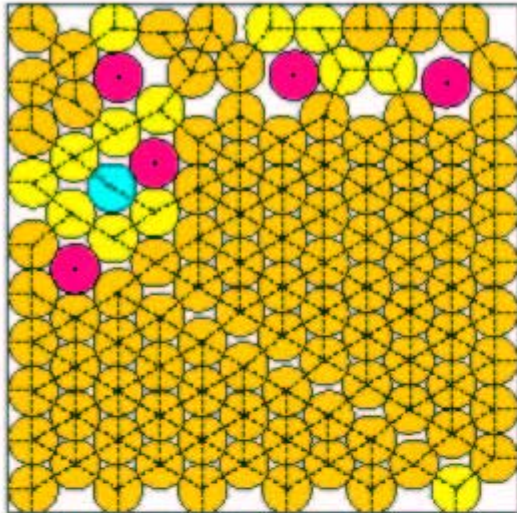
114 circles in the unit square



radius = 0.042307755523 density = 0.826055980404
distance = 0.306005965506 contacts = 260

N = 115

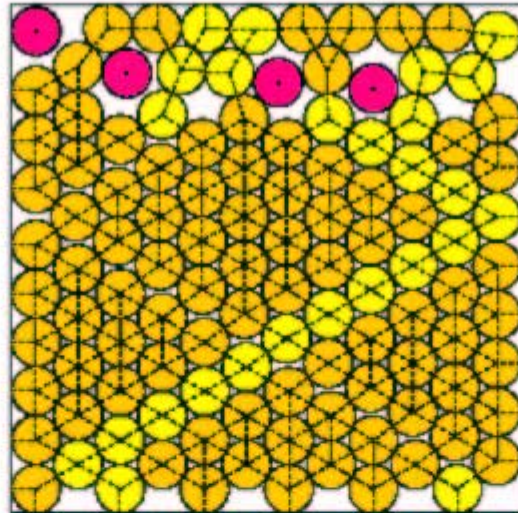
115 circles in the unit square



radius = 0.04793244420 density = 0.82442660314 # of gaps = 17
 distance = 0.10594400221 contacts = 298

N = 116

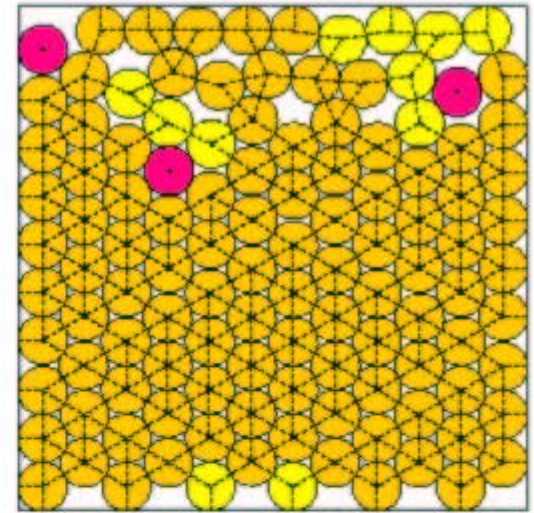
116 circles in the unit square



radius = 0.04777294245 density = 0.82470984452 # of gaps = 17
 distance = 0.10583982759 contacts = 297

N = 117

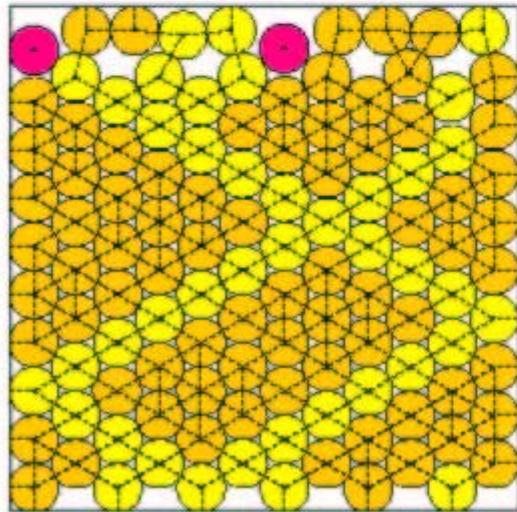
117 circles in the unit square



radius = 0.04745724255 density = 0.82424402714 # of gaps = 17
 distance = 0.10592004755 contacts = 279

N = 118

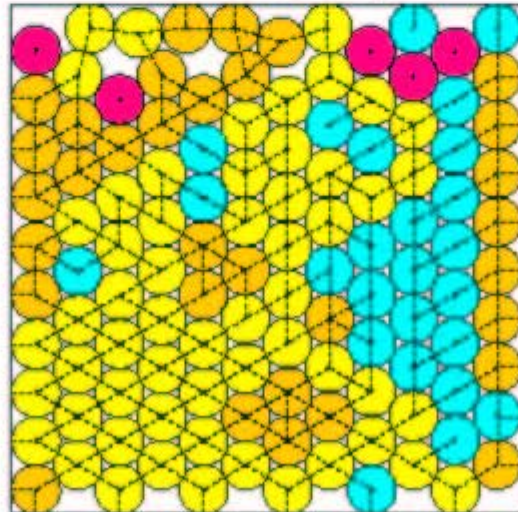
118 circles in the unit square



radius = 0.04757071220 density = 0.82440247742 # of gaps = 17
 distance = 0.10574517211 contacts = 290

N = 119

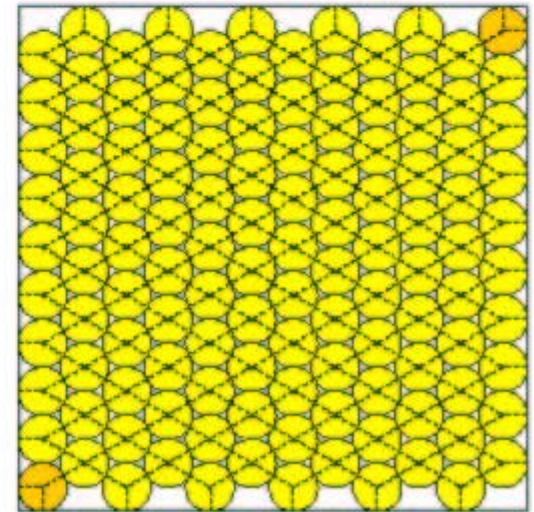
119 circles in the unit square



radius = 0.04754455022 density = 0.84508124081 # of gaps = 17
 distance = 0.10581197091 contacts = 295

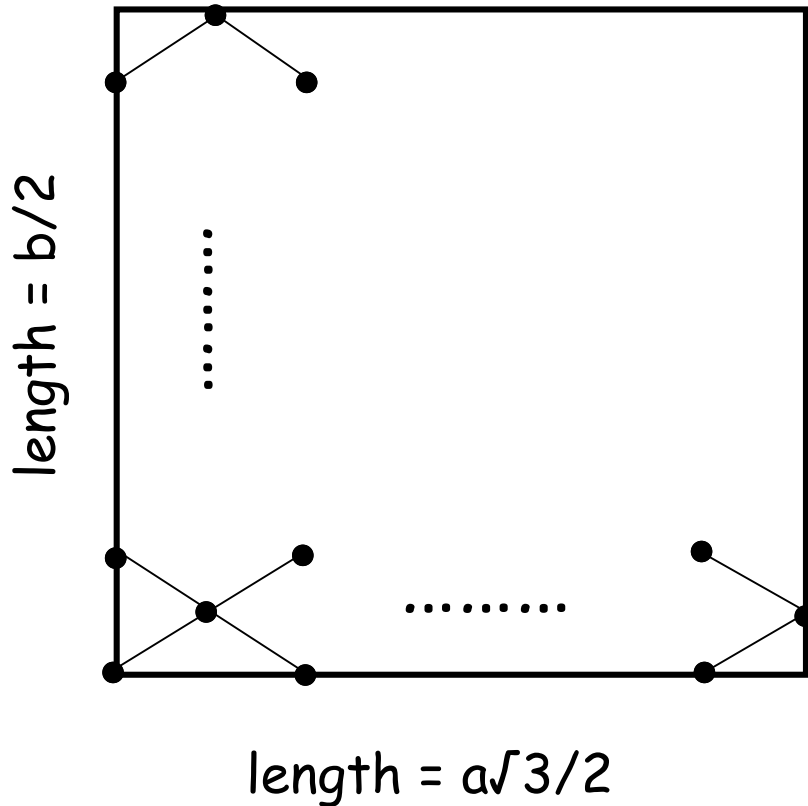
N = 120

120 circles in the unit square



radius = 0.04732461391 density = 0.83175349031 # of gaps = 17
 distance = 0.10594549130 contacts = 281

Why is the packing for 120 discs so good?



$a+1$ columns (an even #)

$b+1$ rows

number of points (= disc centers) is $(a+1)(b+1)/2$

If $b = a\sqrt{3}/2 - \varepsilon$ then can pack a slightly distorted copy of this disc arrangement into a square.

So we need to have a/b slightly less than $\sqrt{3}$.

Use the (under-)convergents to $\sqrt{3}$.

These are: $\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}, \frac{265}{103}, \dots, \frac{b_n}{a_n}$

The corresponding values of the number of discs $N = \frac{1}{2}(a_n + 1)(b_n + 1)$

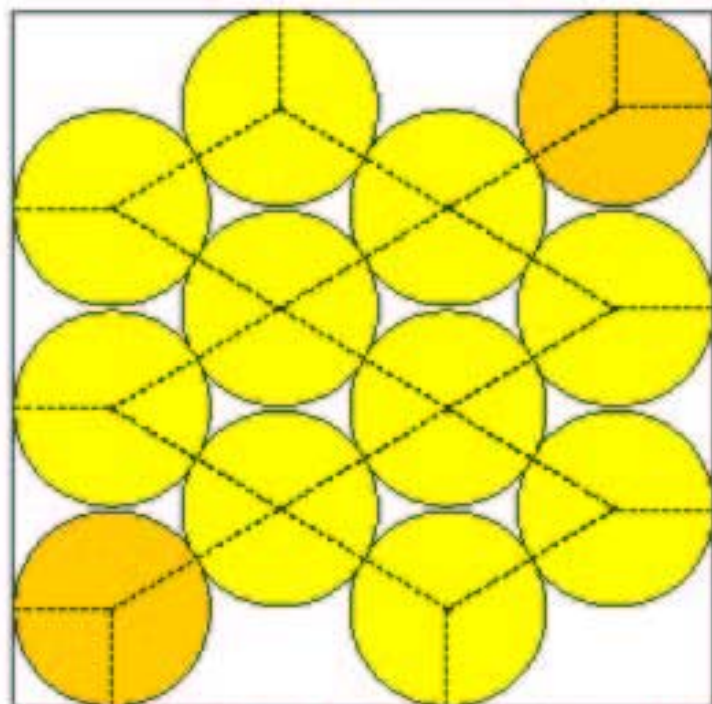
are 2, 12, 120, 1512, 13832,.....

Conjecture: (Nurmela, Östergård -- 1999)

For these values of N , the "near-hexagonal" packing of N discs is optimal.

N = 12

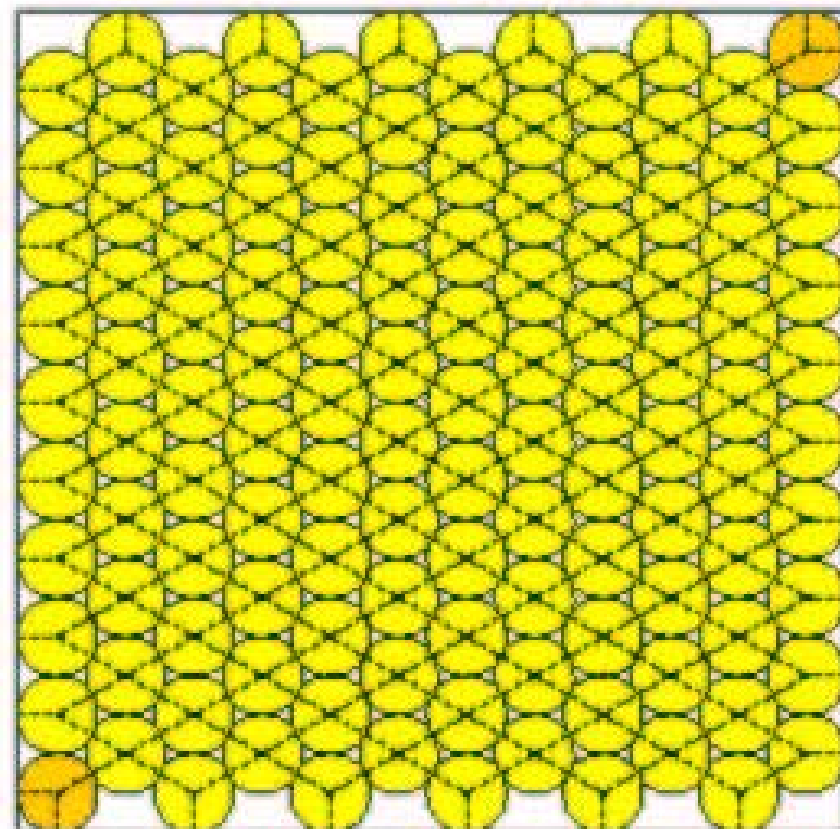
12 circles in the unit square



radius = 0.10955544035 density = 0.775465223554 # of lines = 25
distance = 0.38830126723 contacts = 25

N = 120

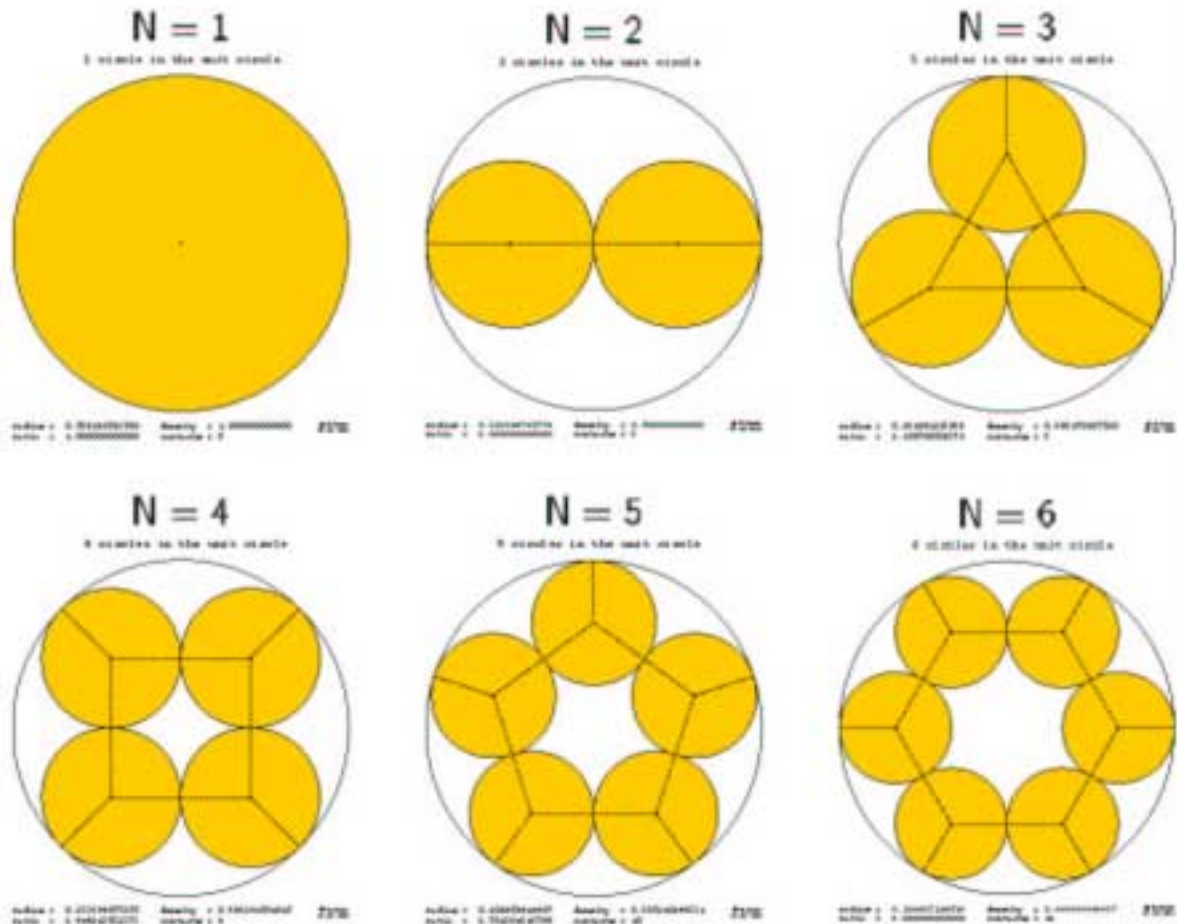
120 circles in the unit square



radius = 0.047393901593 density = 0.653655315036 # of lines = 341
distance = 0.109045440090 contacts = 341

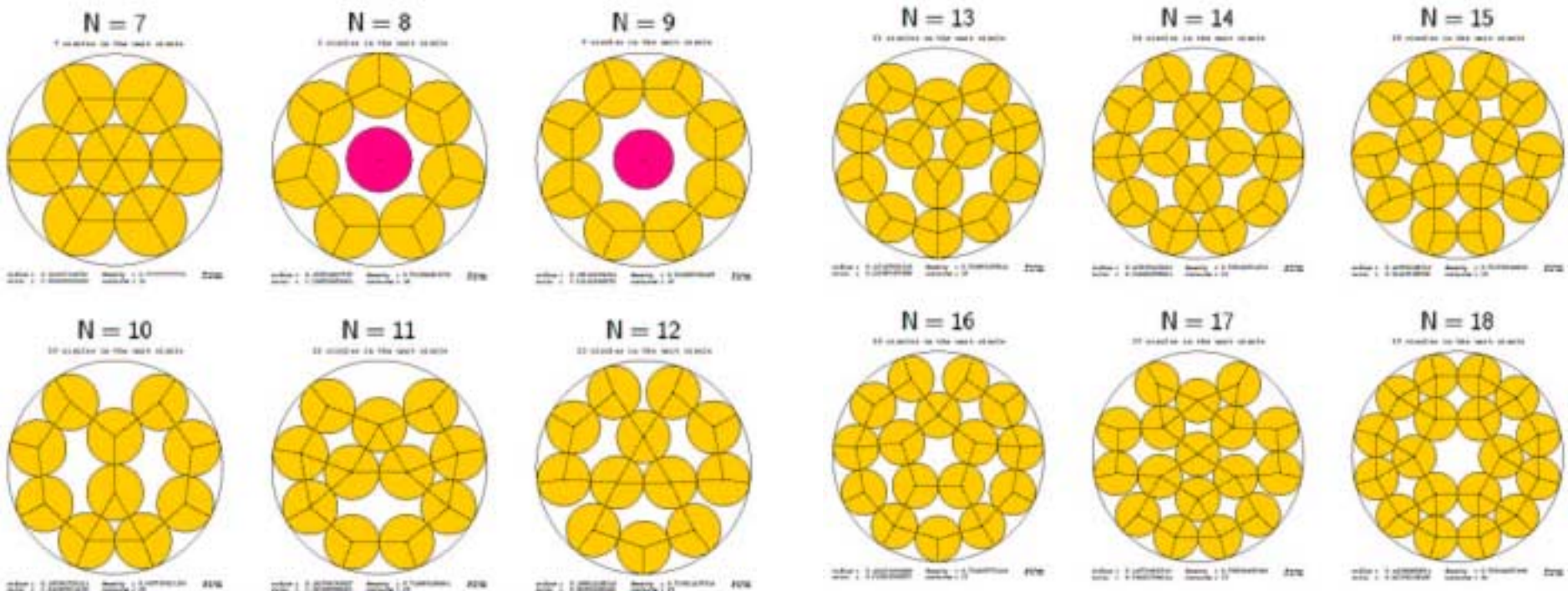
How about packing discs in circles?

Of course, people have been doing that, too, for a long time (with even less success!)

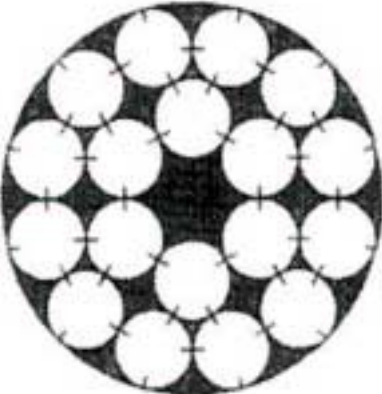


2. Earlier results

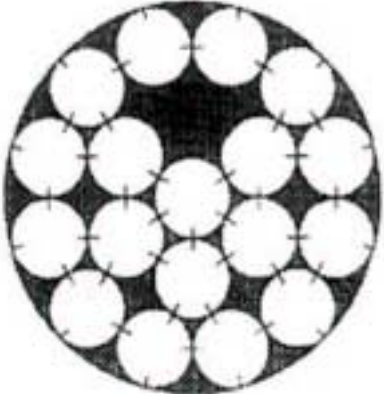
Kravitz [10] was, to our knowledge, the first to consider the problem of packing n congruent circles in a circle. In [10] packings of up to 19 circles are given without any optimality proofs.³ Graham [6] and Pirl [17] independently proved optimality of packings of up to 7 and 10 circles, respectively. Pirl also presented good packings of up to 19 circles; some of these packings (for $n = 14, 16, 17$) were later improved by Goldberg [5], who also gave a packing of 20 circles. Goldberg's packing of 17 circles was further improved by Reis [18], who extended the range of n to 25. The packing of $n = 25$ is improved in this paper. Recently, Melissen [13] proved the optimality for the case $n = 11$.



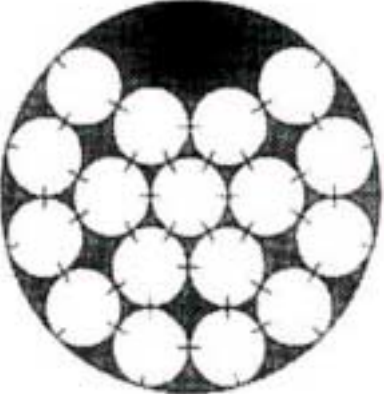
A peculiarity of the 18-circle case is that the best known packings of 18 circles have the same r as the best known packing of 19 circles. Three different, equally dense packings of 18 circles can be obtained by removing a circle in the packing of 19 circles in Fig. 2; see packings 18(a)–18(c) in Fig. 3. (A packing obtained by a congruence transformation, that is, by rotation or reflection, from another is considered the same.) In addition to these three packings, which apparently were the only ones known before, there are at least 7 more equally good packings. We suspect that there is no 11th equally good packing. At least, if one circle is removed from any of those 10 presumed best and then put back in the packing without overlaps with other circles, then one of these 10 packings is obtained. Furthermore, starting from any of these packings, all the others can be obtained with a series of such transformations.



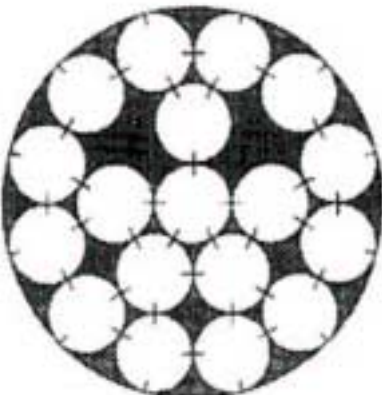
$n = 18$ (a)



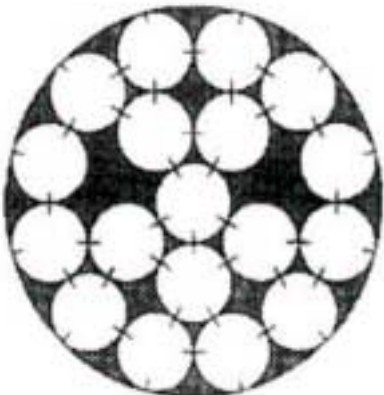
$n = 18$ (b)



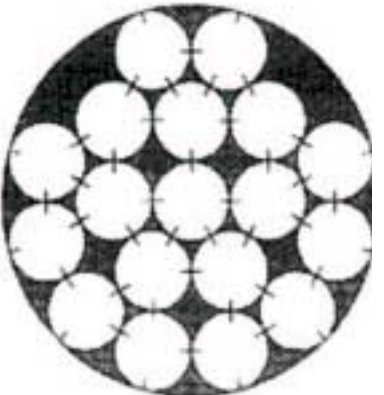
$n = 18$ (c)



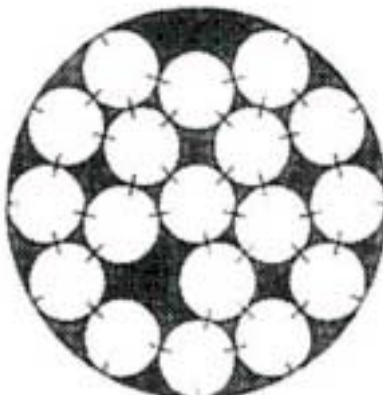
$n = 18$ (d)



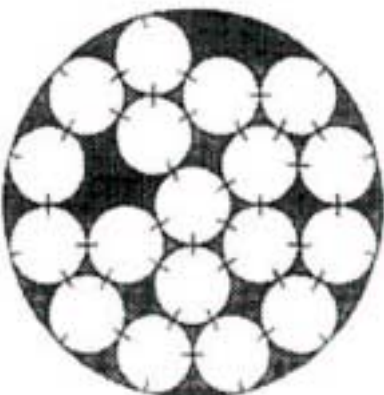
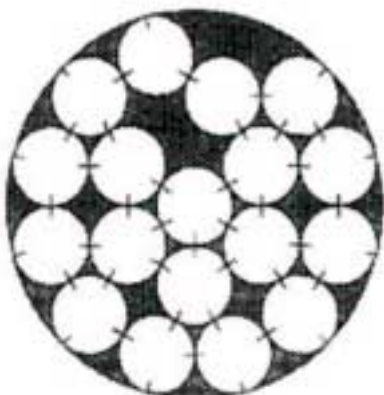
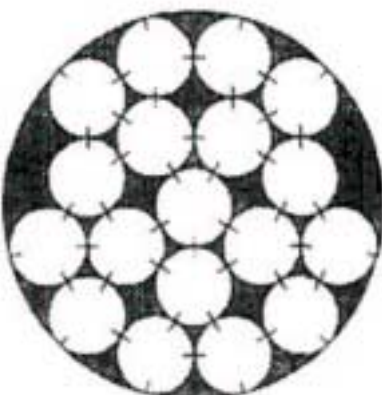
$n = 18$ (e)



$n = 18$ (f)



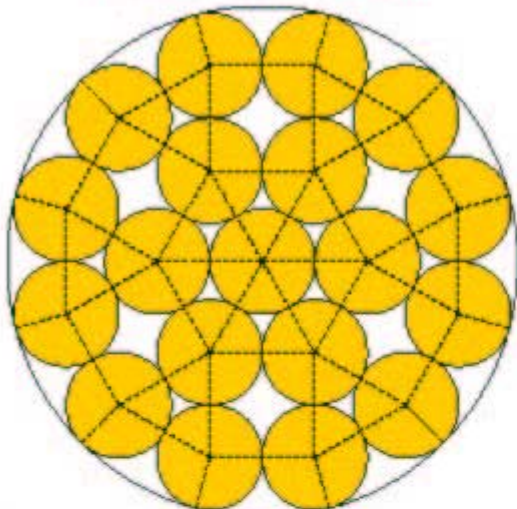
$n = 18$ (j)



N = 19



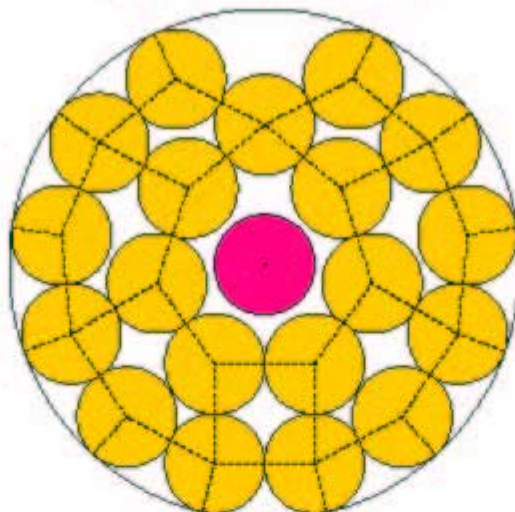
19 circles in the unit circle



radius = 0.31600000001 density = 0.80320244410
ratio = 4.36370530519E contacts = 45

N = 20

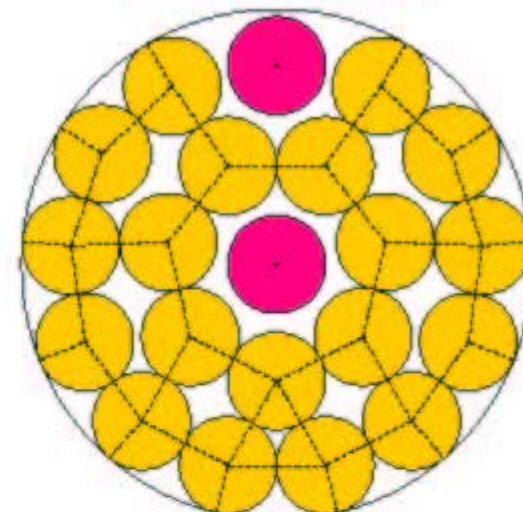
20 circles in the unit circle



radius = 0.310453333475 density = 0.79324620055
ratio = 5.12222073692 contacts = 34

N = 21

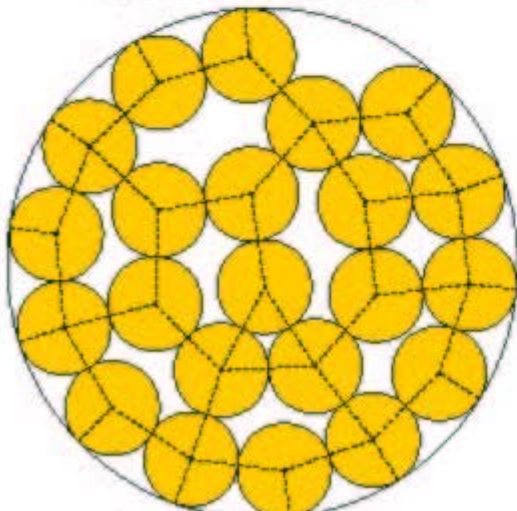
21 circles in the unit circle



radius = 0.307410265041 density = 0.78222561218
ratio = 5.25231478010 contacts = 34

N = 22

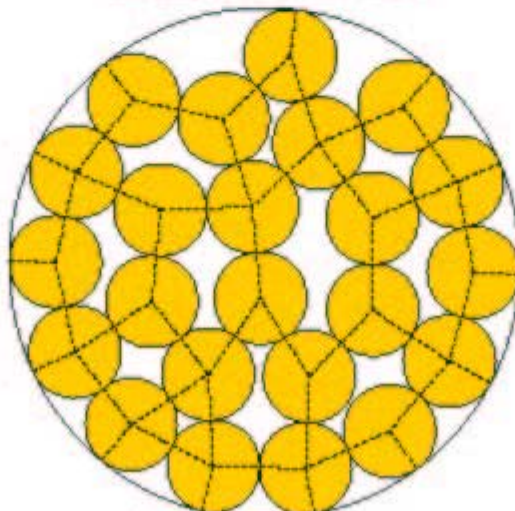
22 circles in the unit circle



radius = 0.307136678701 density = 0.740480706965
ratio = 5.42193895002 contacts = 44

N = 23

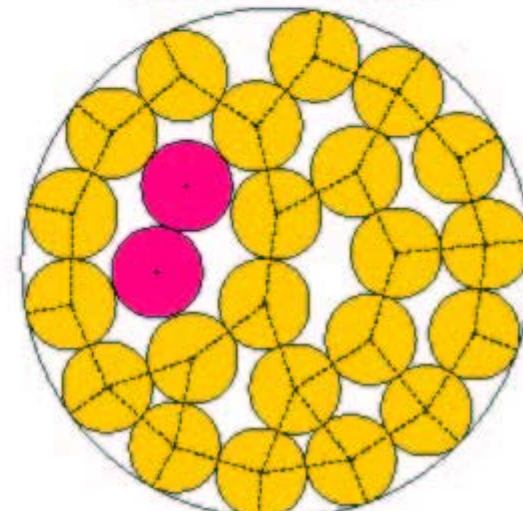
23 circles in the unit circle



radius = 0.301742610780 density = 0.747166713378
ratio = 5.56520422275 contacts = 46

N = 24

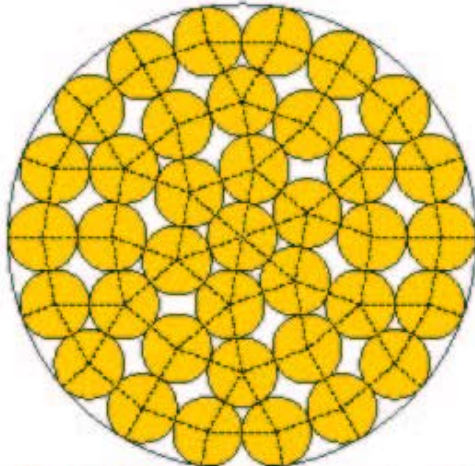
24 circles in the unit circle



radius = 0.299827218404 density = 0.751278262865
ratio = 5.55651068715 contacts = 44

N = 37

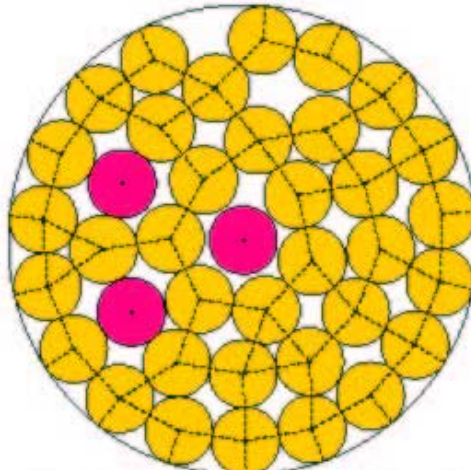
37 circles in the unit circle



radius = 0.00475300321 density = 0.0026321730 
 ratio = 4.750770402344 contacts = 90

N = 38

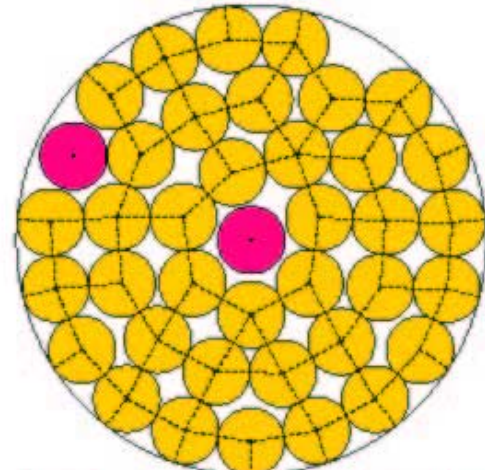
39 circles in the unit circle



radius = 0.00302750624 density = 0.754004566611 
 ratio = 4.96336465226 contacts = 70

N = 39

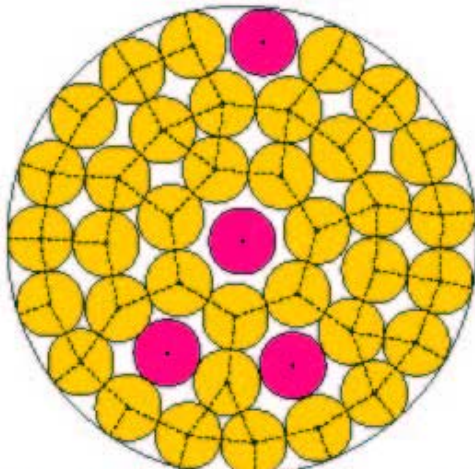
39 circles in the unit circle



radius = 0.07327435506 density = 0.750236495826 
 ratio = 7.05766436624 contacts = 74

N = 40

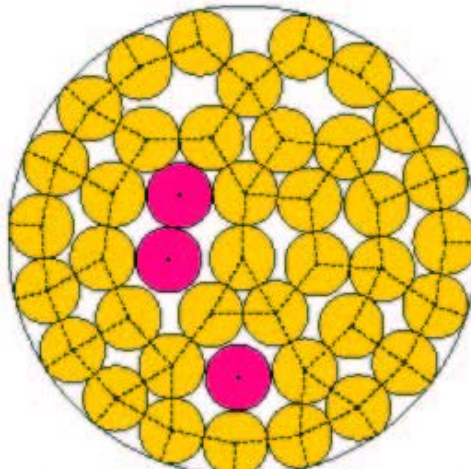
40 circles in the unit circle



radius = 0.07310732026 density = 0.756262990274 
 ratio = 7.32046455945 contacts = 70

N = 41

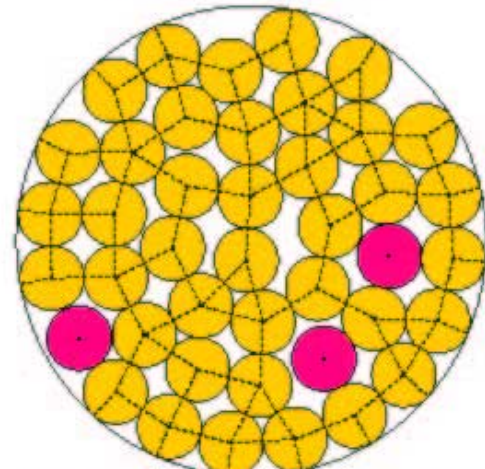
41 circles in the unit circle



radius = 0.07771102380 density = 0.77767170259 
 ratio = 7.24003226670 contacts = 76

N = 42

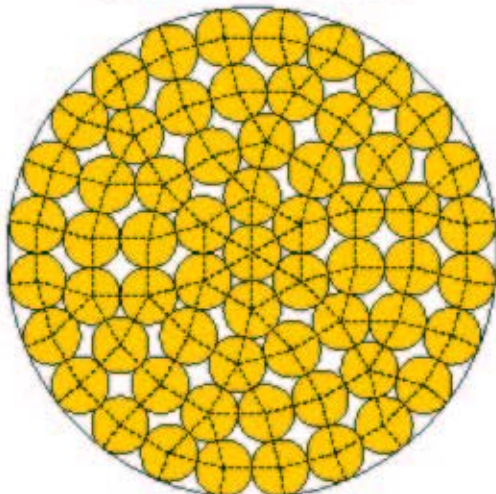
42 circles in the unit circle



radius = 0.07672292929 density = 0.776322000756 
 ratio = 7.34673640243 contacts = 76

N = 61

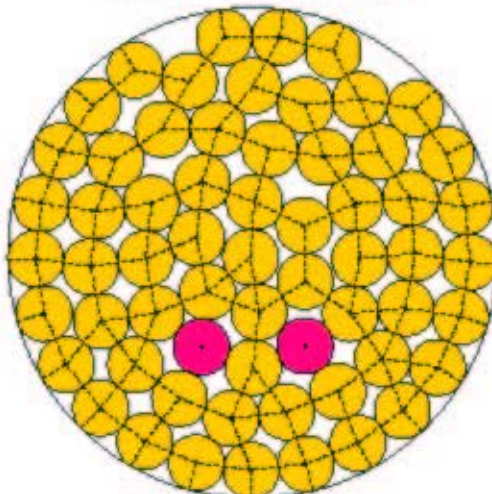
61 circles in the unit circle



radius = 0.0652032492 density = 0.8227250726 
 ratio = 3.66207575590 contacts = 116

N = 62

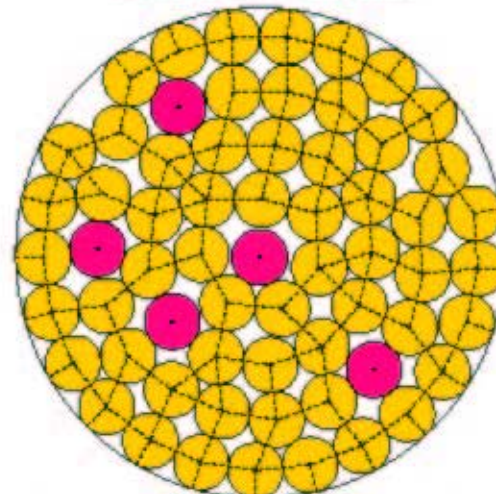
62 circles in the unit circle



radius = 0.06286727630 density = 0.79522220980 
 ratio = 3.329765408772 contacts = 120

N = 63

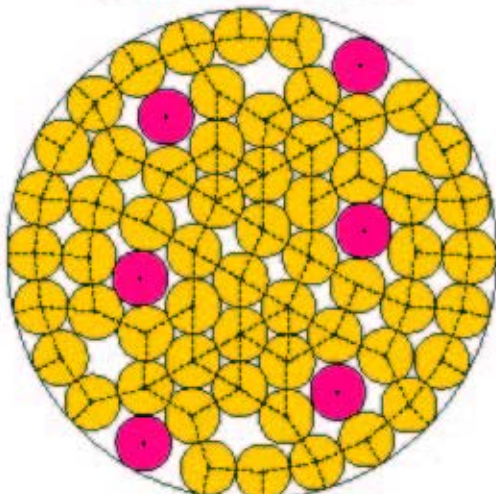
63 circles in the unit circle



radius = 0.06244660650 density = 0.78722205522 
 ratio = 3.39225227552 contacts = 121

N = 64

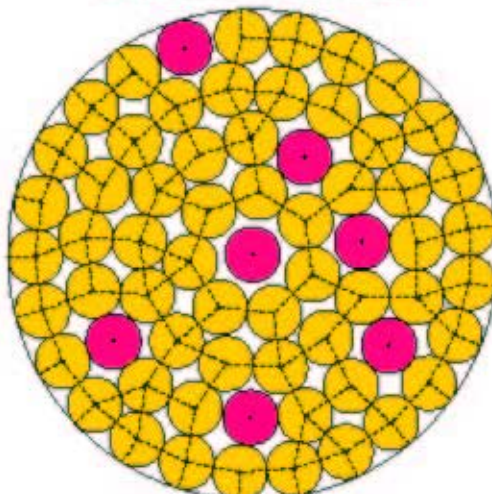
64 circles in the unit circle




radius = 0.06292712770 density = 0.79620244204 
 ratio = 3.96272208504 contacts = 125

N = 65

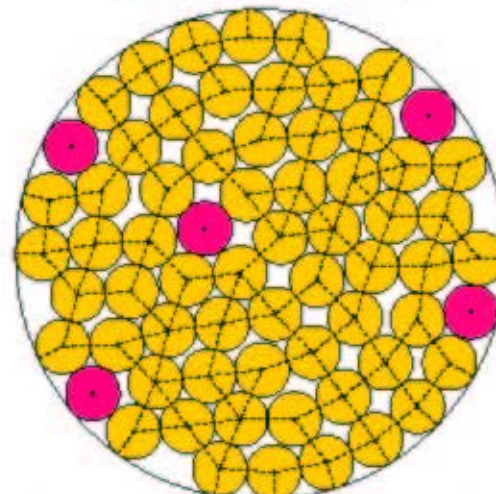
65 circles in the unit circle



radius = 0.06296720765 density = 0.79627570490 
 ratio = 3.02759722200 contacts = 126

N = 66

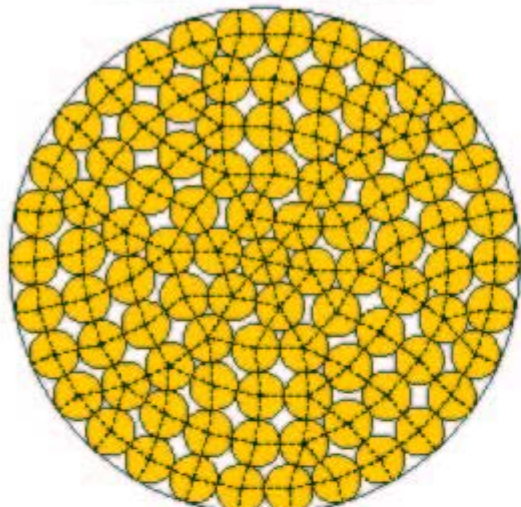
66 circles in the unit circle



radius = 0.06202279528 density = 0.797520945702 
 ratio = 3.08665207028 contacts = 122

N = 91

91 circles in the unit circle

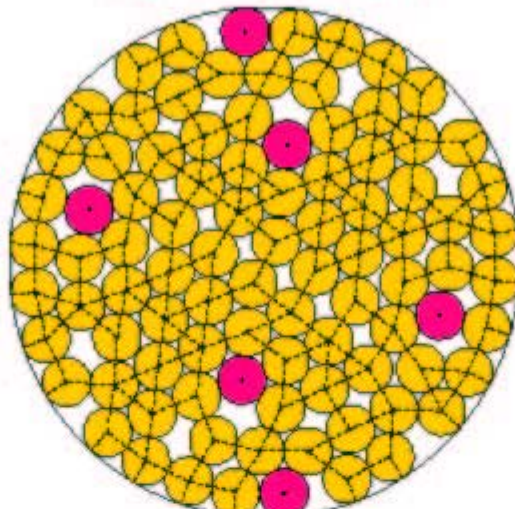


radius = 0.05234600559 density = 0.8368804602
 ratio = 30.364771237506 contacts = 230



N = 92

92 circles in the unit circle

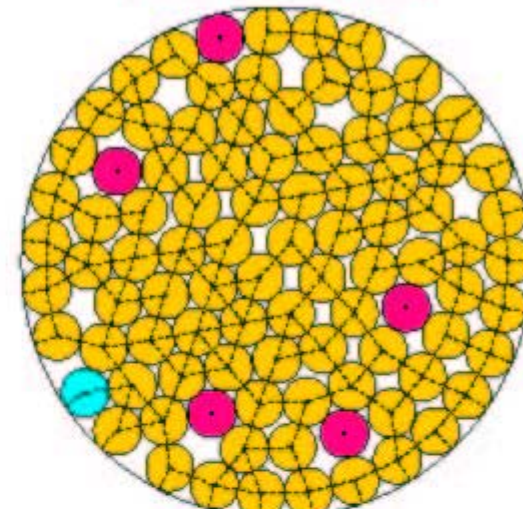


radius = 0.052600552107 density = 0.80563154066
 ratio = 30.634630750022 contacts = 272



N = 93

93 circles in the unit circle

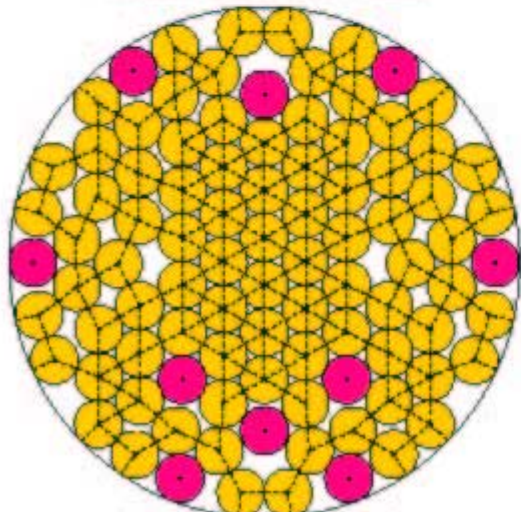


radius = 0.052544047718 density = 0.806650598005
 ratio = 30.707000402391 contacts = 276



N = 94

94 circles in the unit circle

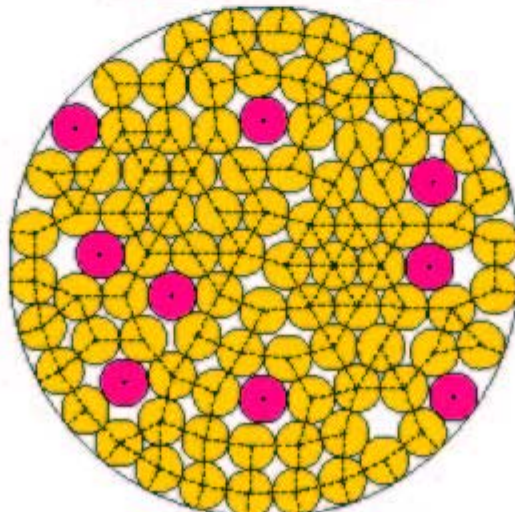


radius = 0.052346251611 density = 0.809187011151
 ratio = 30.779602314495 contacts = 278



N = 95

95 circles in the unit circle

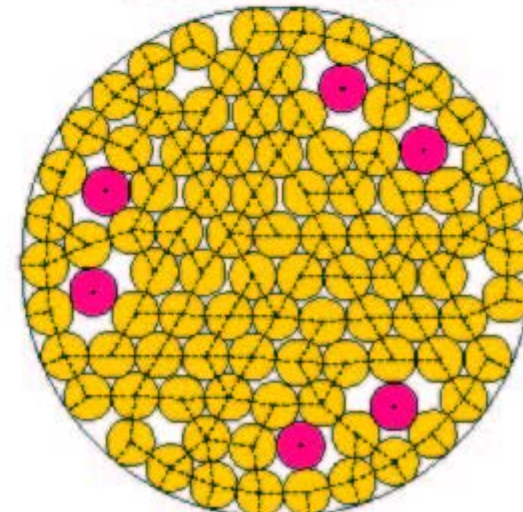


radius = 0.052046025280 density = 0.808461023637
 ratio = 30.846005002226 contacts = 272



N = 96

96 circles in the unit circle

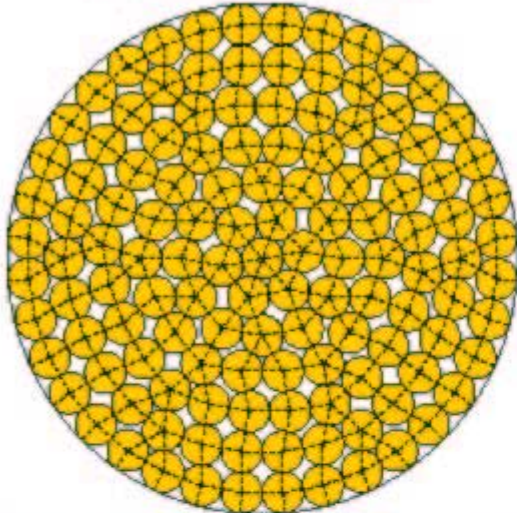


radius = 0.051626104901 density = 0.810490274112
 ratio = 30.880440897112 contacts = 280



N = 127

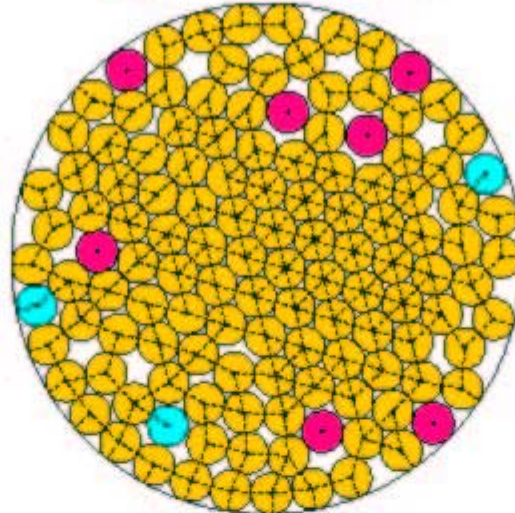
127 circles in the unit circle



radius = 0.04520000201 density = 0.82620000001
 ratio = 12.47370049070 contacts = 266

N = 128

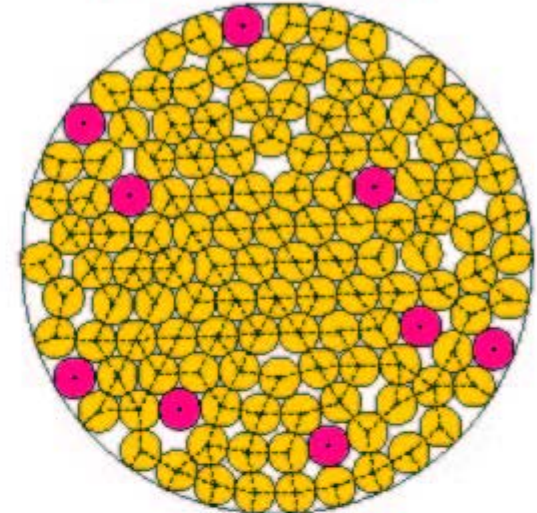
128 circles in the unit circle



radius = 0.04507000000 density = 0.82040000001
 ratio = 12.53640000000 contacts = 254

N = 129

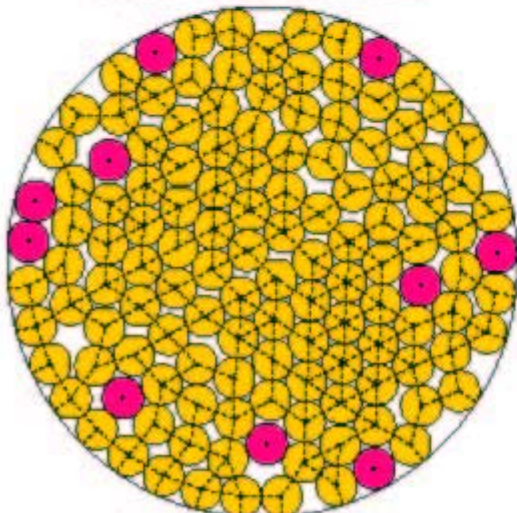
129 circles in the unit circle



radius = 0.04480000000 density = 0.81700000000
 ratio = 12.58500000000 contacts = 251

N = 130

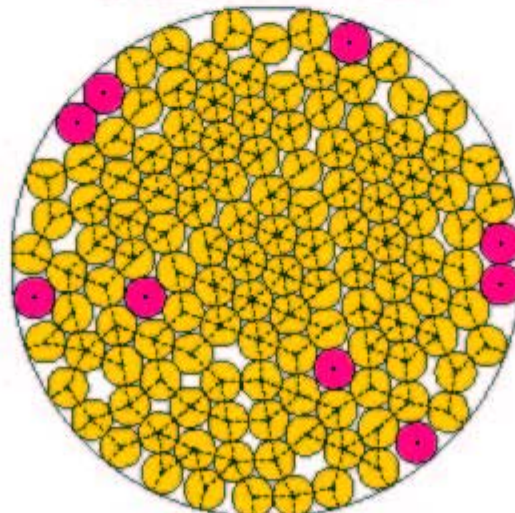
130 circles in the unit circle



radius = 0.04470000000 density = 0.81500000000
 ratio = 12.60000000000 contacts = 246

N = 131

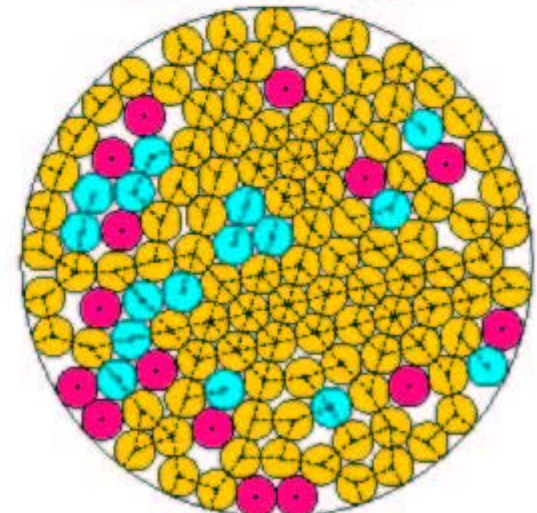
131 circles in the unit circle



radius = 0.04450000000 density = 0.81000000000
 ratio = 12.65000000000 contacts = 241

N = 132

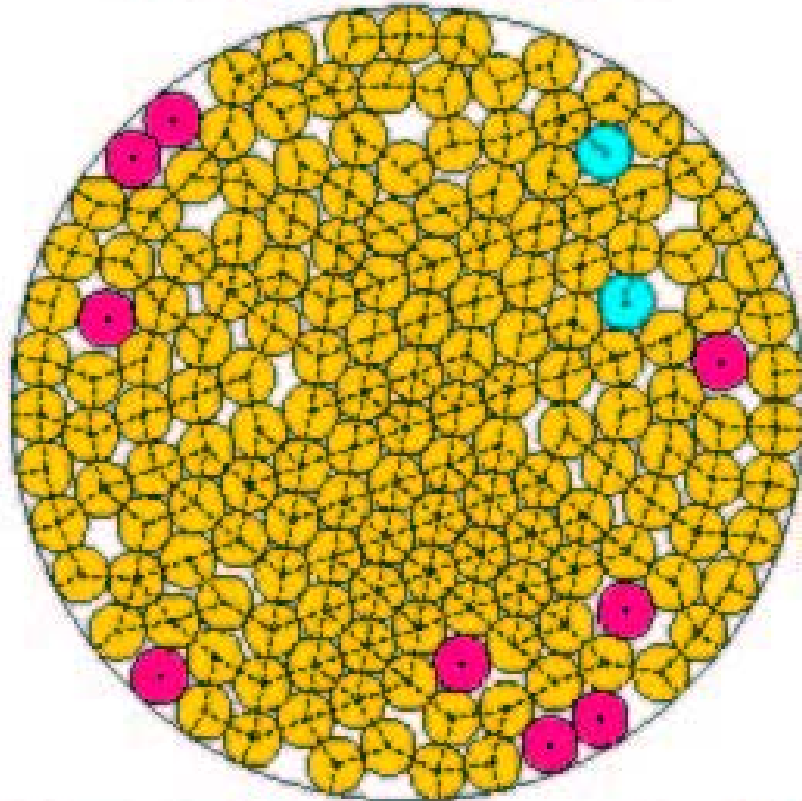
132 circles in the unit circle



radius = 0.04400000000 density = 0.80500000000
 ratio = 12.70000000000 contacts = 230

$$N = 169 \diamond$$

169 circles in the unit circle



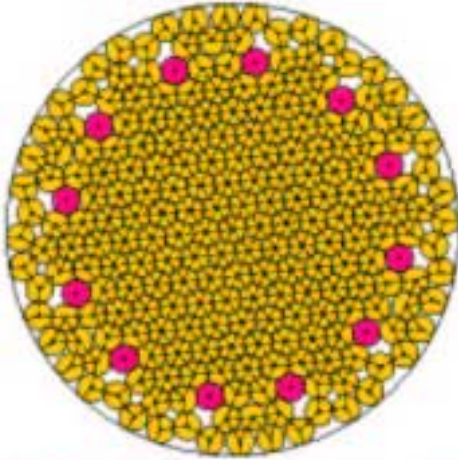
radius = 0.000127732706 density = 0.02404405140 # circles = 169
width = 0.000063866353 circumference = 0.000402110

$$169 = 3 \cdot 7 \cdot 8 + 1 = \diamond$$

Too bad!

$N = 235$

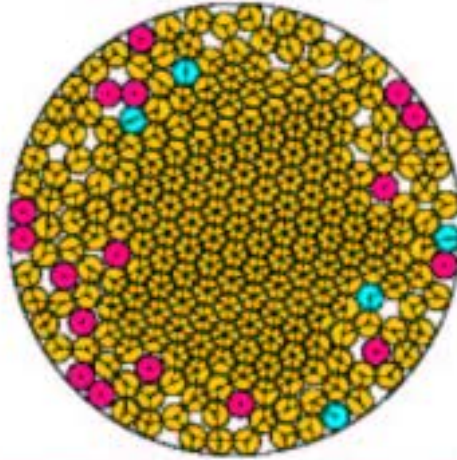
235 circles in the unit circle



radius = 0.00719313297 density = 0.9421943397 $\frac{235}{\pi}$
 order = 87.74277513184 structure = 15a

$N = 236$

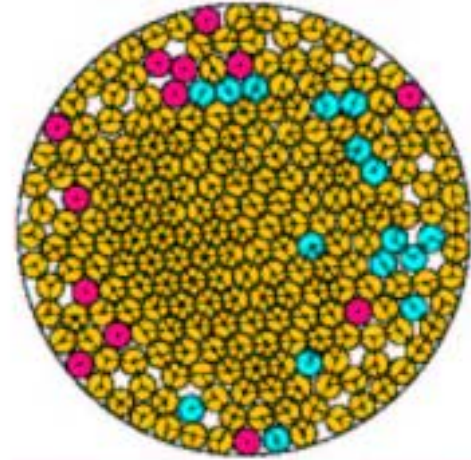
236 circles in the unit circle



radius = 0.00719313297 density = 0.9189642199 $\frac{236}{\pi}$
 order = 87.42051862195 structure = 15b

$N = 237$

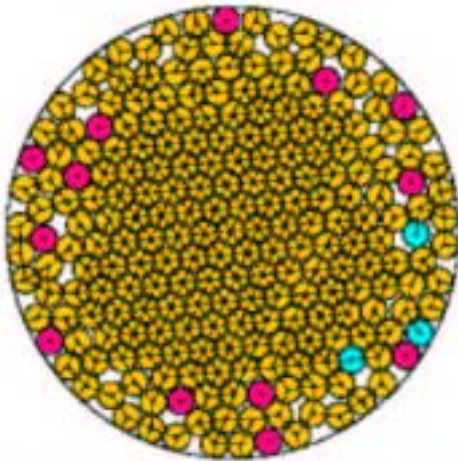
237 circles in the unit circle



radius = 0.00719313297 density = 0.9148314148 $\frac{237}{\pi}$
 order = 87.31287187729 structure = 15c

$N = 238$

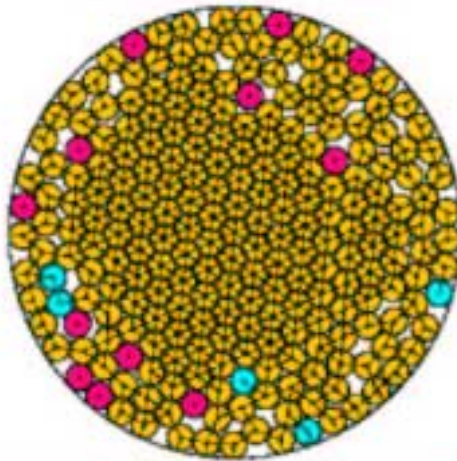
238 circles in the unit circle



radius = 0.00719313297 density = 0.9148314148 $\frac{238}{\pi}$
 order = 87.46142547132 structure = 15d

$N = 239$

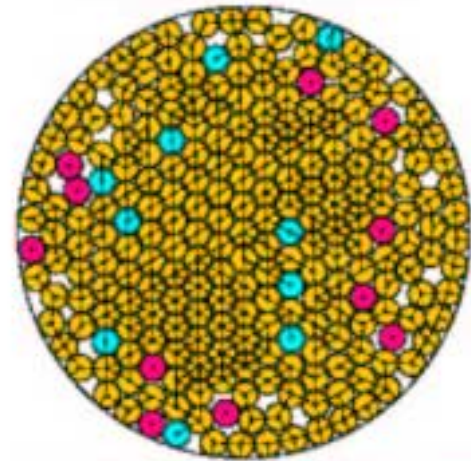
239 circles in the unit circle



radius = 0.00719313297 density = 0.9144142481 $\frac{239}{\pi}$
 order = 87.47626915845 structure = 15e

$N = 240$

240 circles in the unit circle



radius = 0.00719313297 density = 0.9140712091 $\frac{240}{\pi}$
 order = 87.392823238 structure = 15f

Any conjectures??

Dense packings of 2 sizes of discs in the plane

(A. Heppes - 2003)

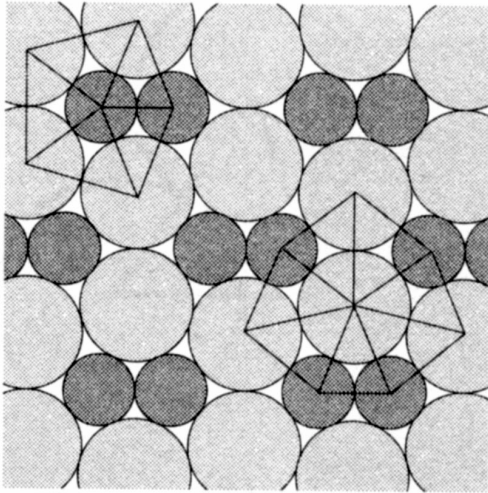


Fig. 1. \mathcal{P}_1 .

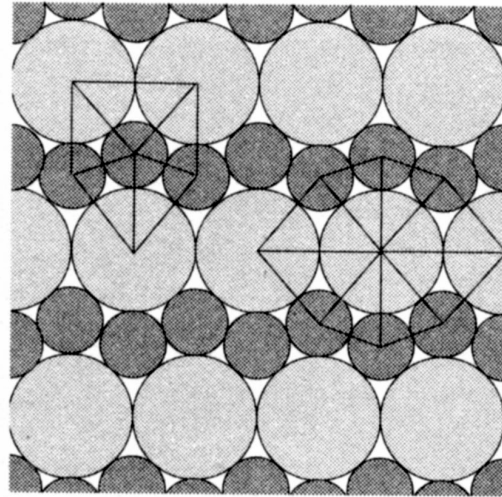


Fig. 2. \mathcal{P}_2 .

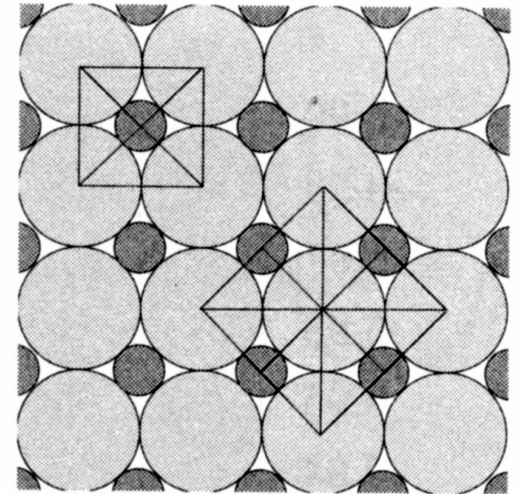


Fig. 3. \mathcal{P}_3 .

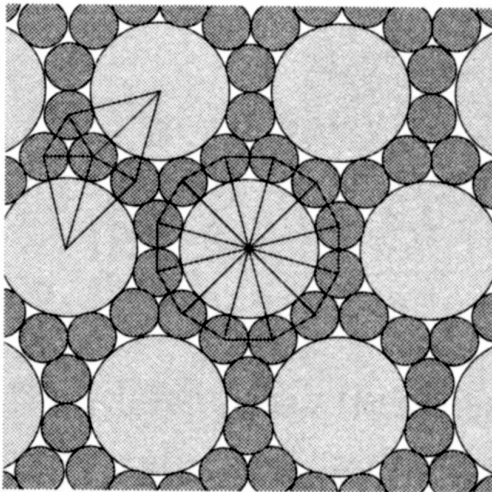


Fig. 4. \mathcal{P}_4 .

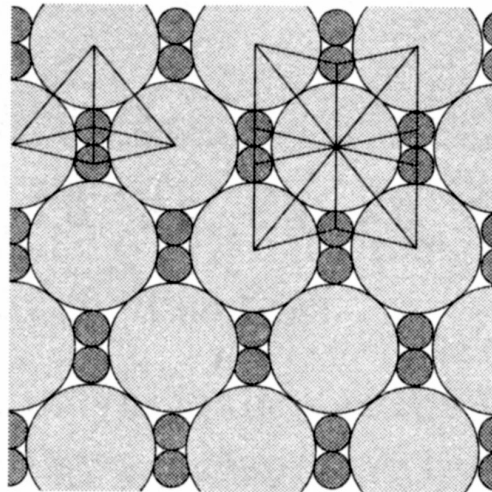


Fig. 5. \mathcal{P}_5 .

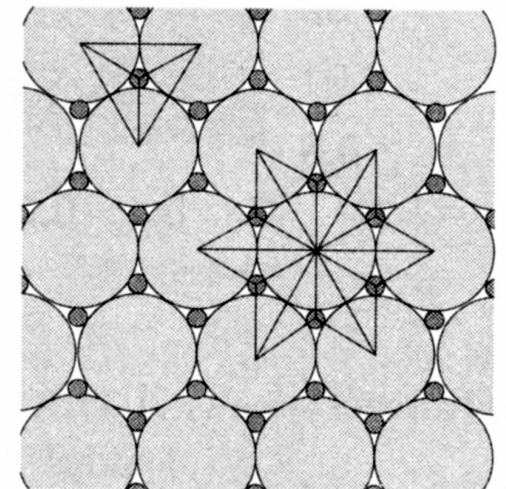
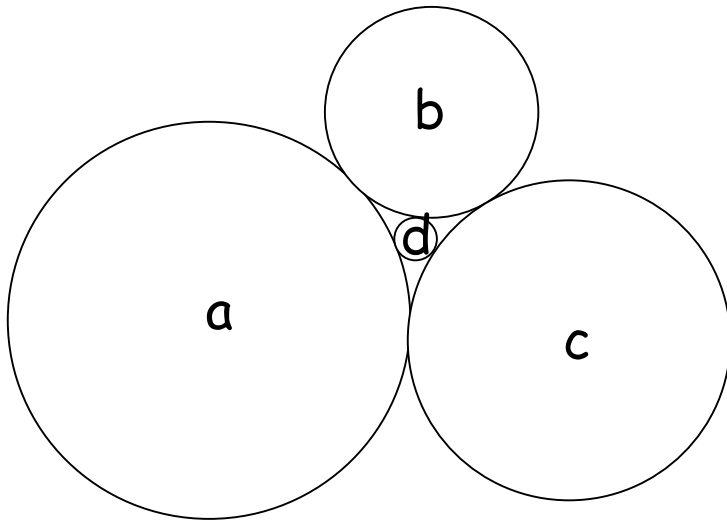


Fig. 6. \mathcal{P}_6 .

Dense packings of discs of many sizes in the plane:

Apollonian circle packings

Apollonian circle packings arise by repeatedly filling the interstices between mutually tangent circles with further tangent circles. It is possible for every circle in such a packing to have integer curvature. Such packings are called **integral Apollonian circle packings**.



a, b, c and d are **reciprocals of the radii** of the circles (also called the "bends" of the circles).

Descartes Theorem

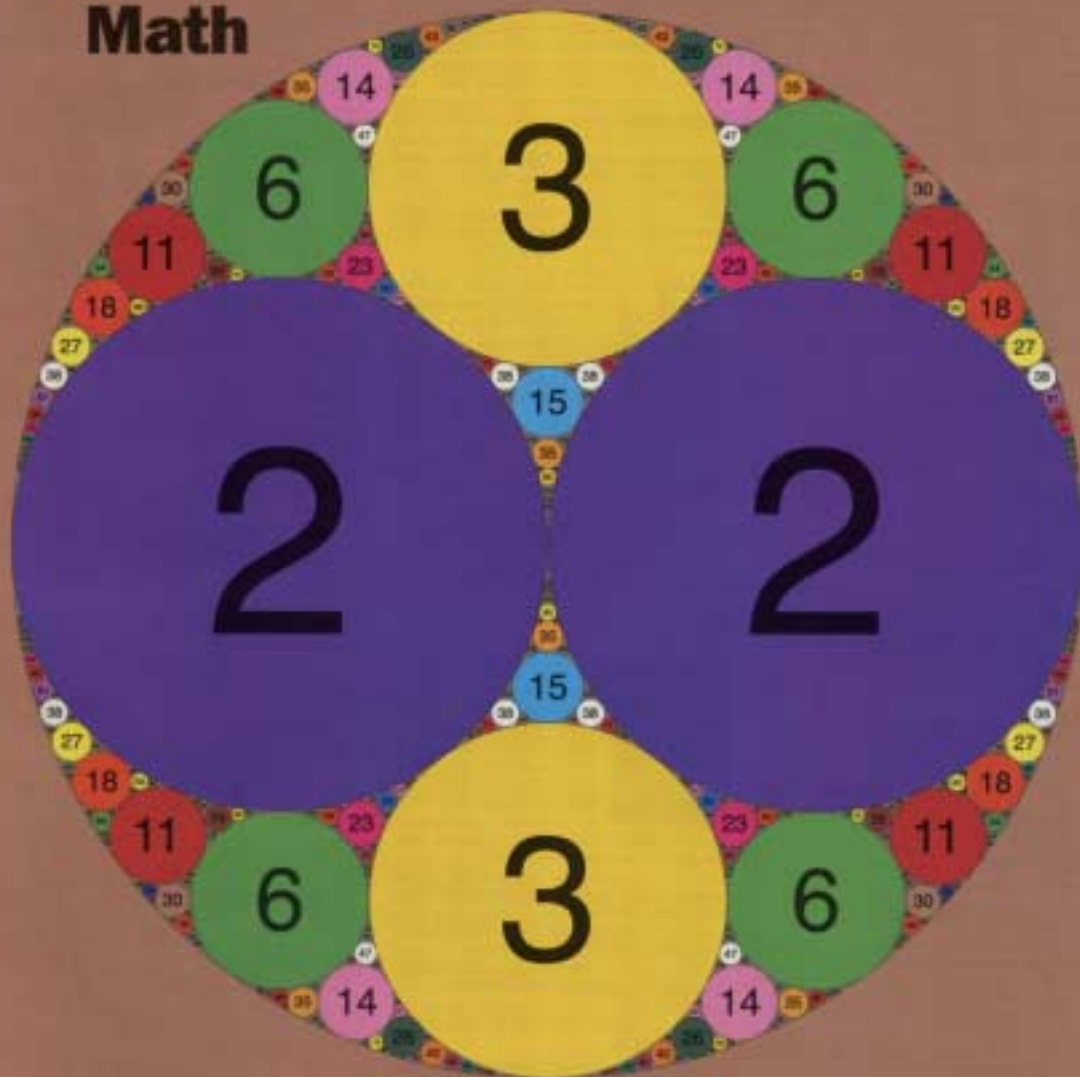
$$(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2)$$

The Weekly Newsmagazine of Science

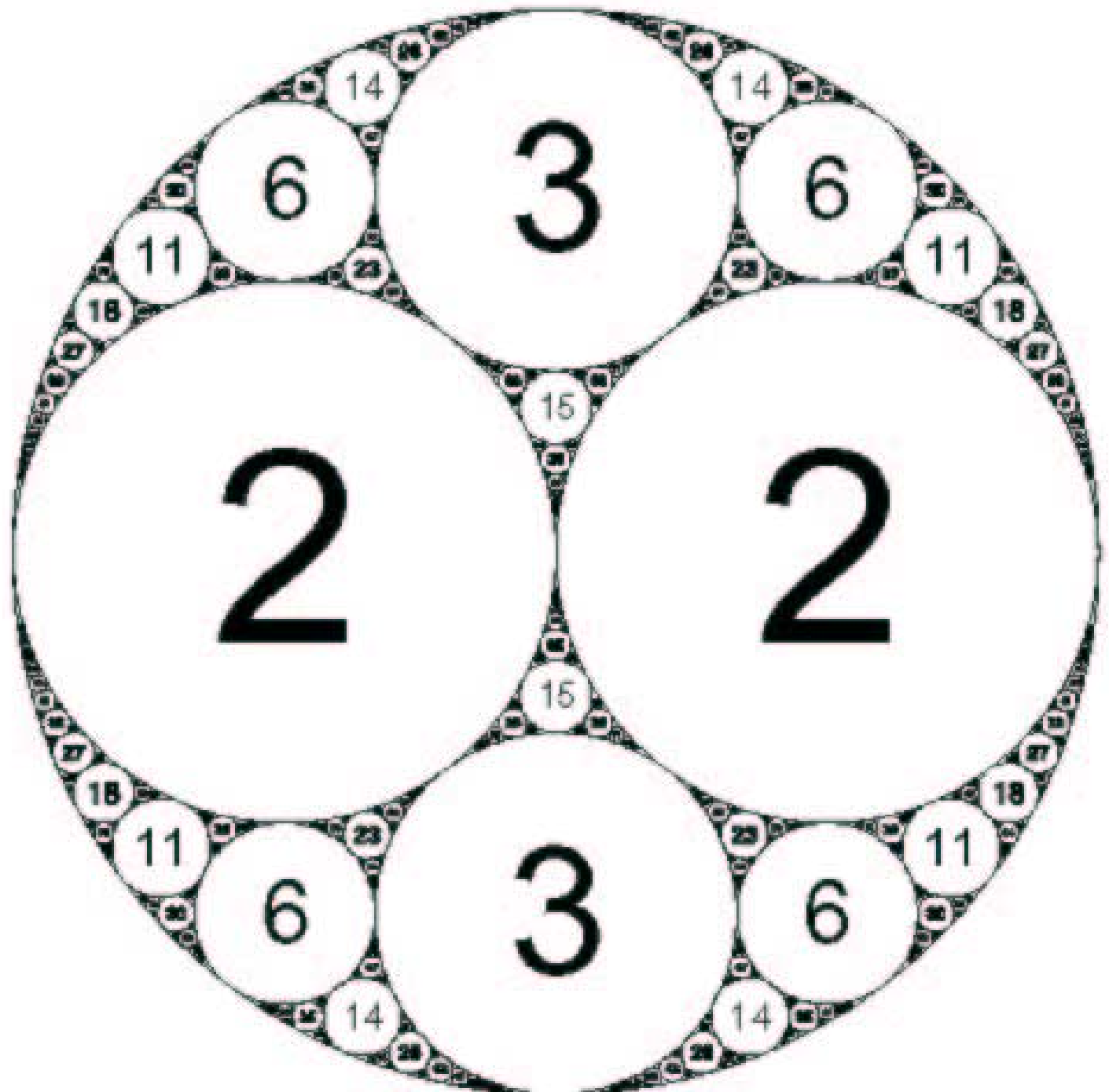
SCIENCE NEWS

April 21, 2001
Vol. 150, No. 16
Pages 241-258

Circle Math

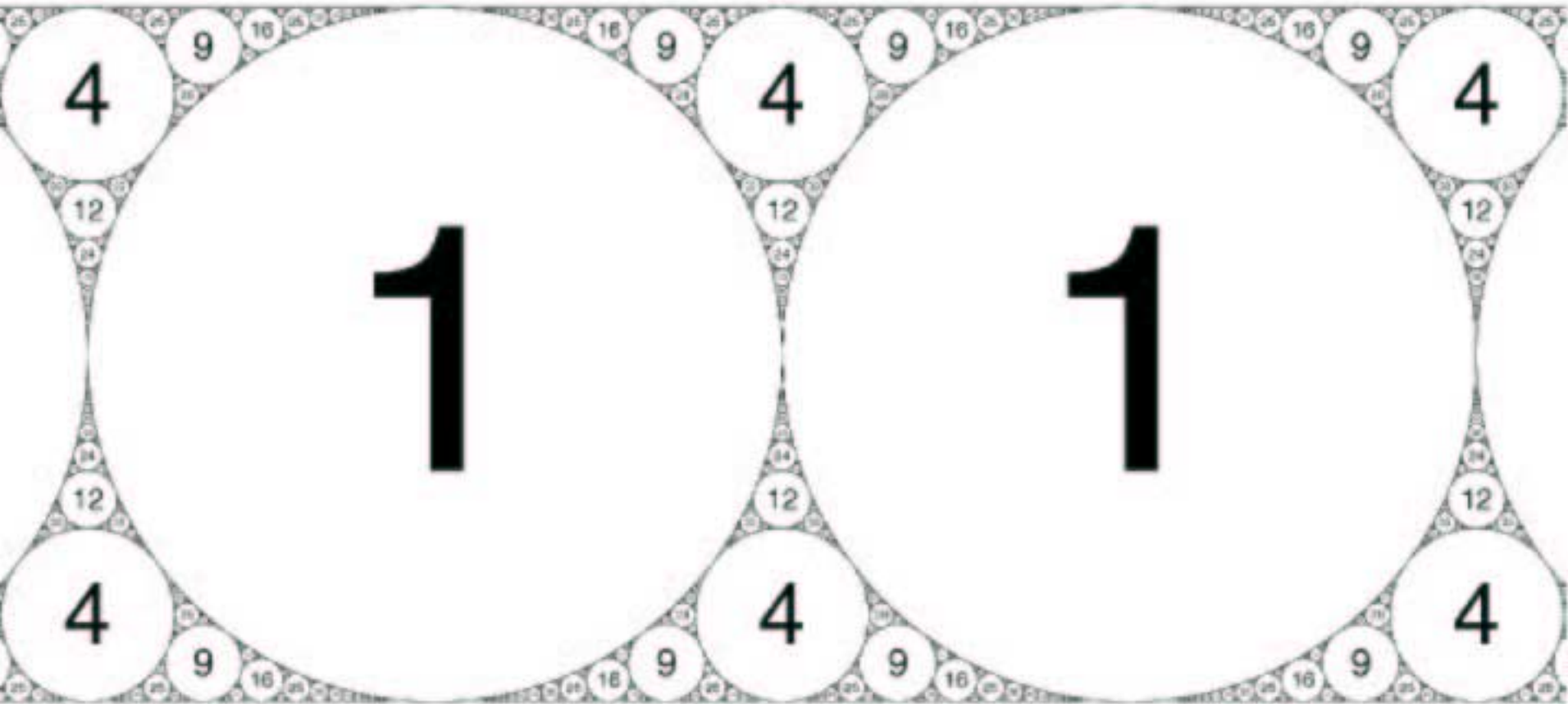


The integral Apollonian circle packing $(-1, 2, 2, 3)$



Fact: All bends in the packing $(-1,2,2,3)$ must be congruent to 2,3 6 or 11 (mod 12).

Conjecture (\$500) All sufficiently large numbers satisfying these congruence conditions occur as bends in the packing $(-1,2,2,3)$



The Apollonian circle packing $(0,0,1,1)$

Fact: All bends occurring in $(0,0,1,1)$ are congruent to $0, 1, 4, 9, 12$ or $16 \pmod{24}$

Conjecture (\$500) All sufficiently large numbers satisfying these congruence conditions occur as bends in the packing $(0,0,1,1)$

Fact: For any m with $\text{g.c.d}(m,30) = 1$, every congruence class modulo m occurs **infinitely often** as a bend in **every** integral Apollonian circle packing..

Conjecture: The above statement is true for all m with $\text{g.c.d.}(m,6) = 1$.

Conjecture: All congruential restrictions on bends in integral Apollonian circle packings can be expressed modulo 24.

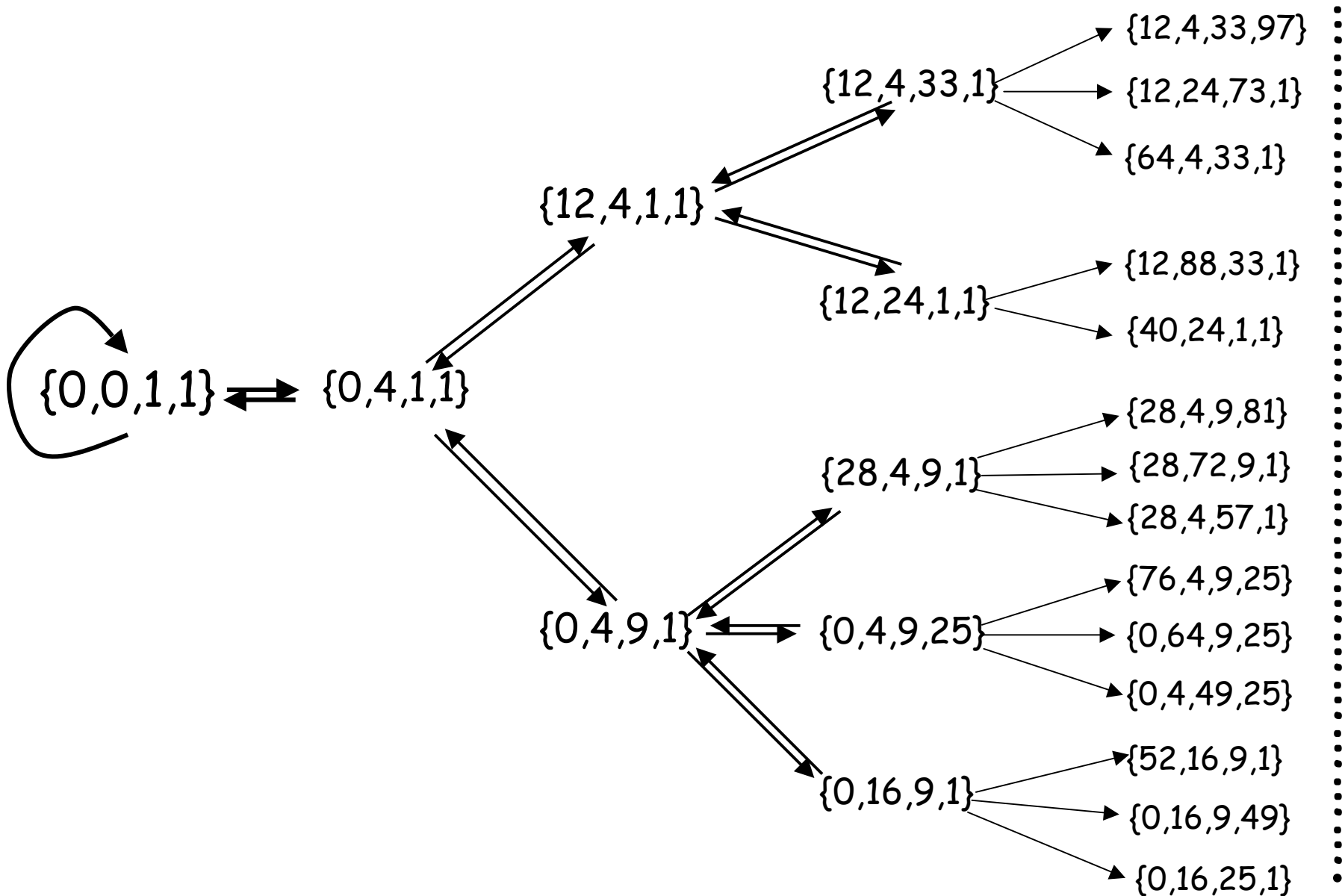
An equivalent number theory problem

Starting with some multiset $S = \{a,b,c,d\}$ of integers, repeatedly perform the following transformation:

Replace any element, say d , in S , by $d' = 2(a+b+c) - d$, forming $S' = \{a,b,c,d'\}$

Question: Which integers can ever be generated by this process?

For example, if we start with $\{0,0,1,1\}$, is it true that all sufficiently large integers congruent to $0,1,4,9,12$ and $16 \pmod{24}$ occur?





ACADEMIC
PRESS

Available at
WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®

Journal of Number Theory 100 (2003) 1–45

**JOURNAL OF
Number
Theory**

<http://www.elsevier.com/locate/jnt>

Apollonian circle packings: number theory

Ronald L. Graham,^{a,*}¹ Jeffrey C. Lagarias,^a² Colin
L. Mallows,^a³ Allan R. Wilks,^a and Catherine H. Yan^b

Now for something **completely different**.....

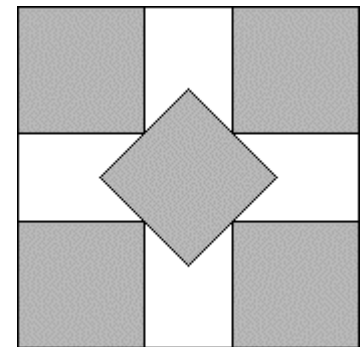
Packing **squares** in squares.

Let $s(n)$ denote the side length of the **smallest** square into which n non-overlapping unit squares can be packed.

Of course, $s(m^2) = m$, for any integer m .

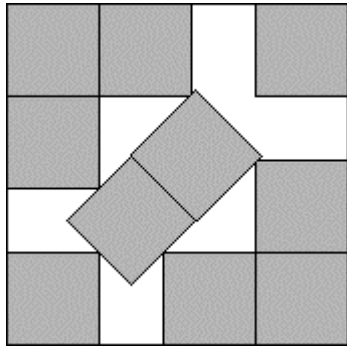
n	2	3	5	6	7	8	14	15	24	35
$s(n)$	2	2	$2 + \frac{1}{\sqrt{2}}$	3	3	3	4	4	5	6

All currently known optimal values of $s(n)$ for $n \neq \square$

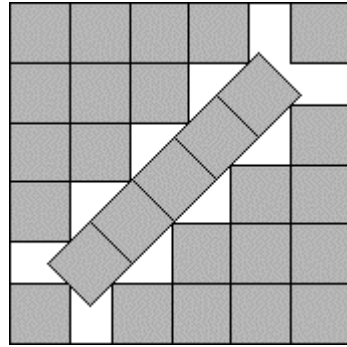


$$s(5) = 2 + \frac{1}{\sqrt{2}}$$

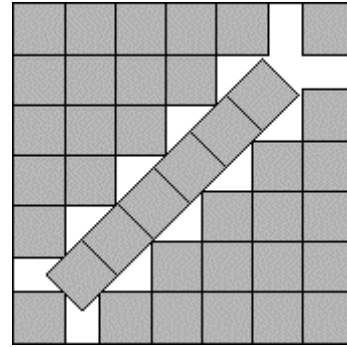
Some other currently best known packings



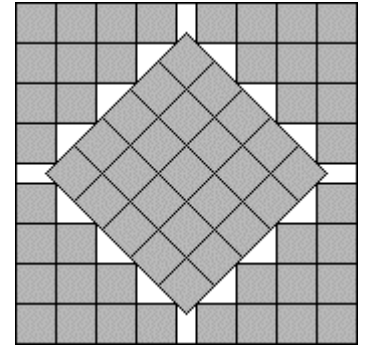
10



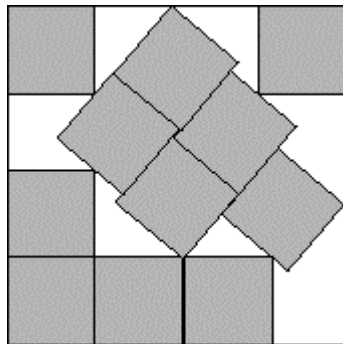
27



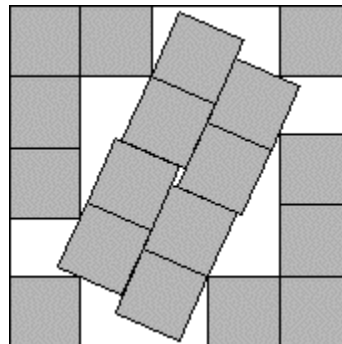
38



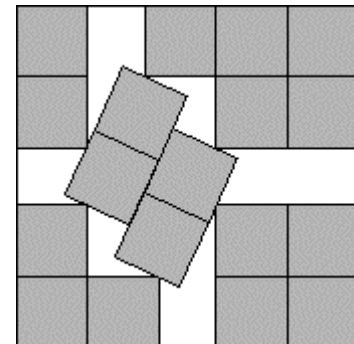
65



11

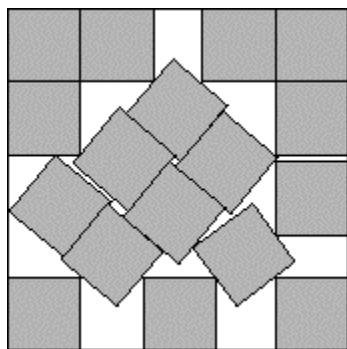


18

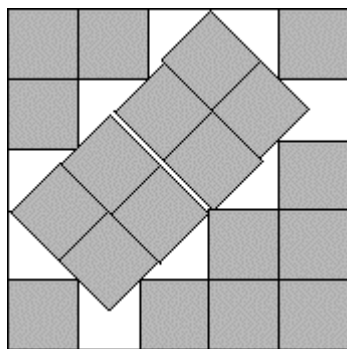


18b

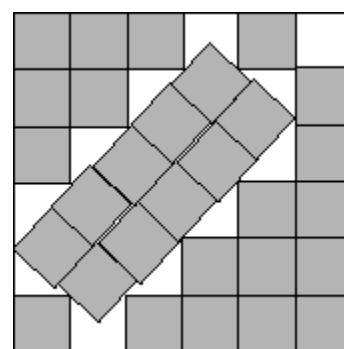
It starts getting harder...



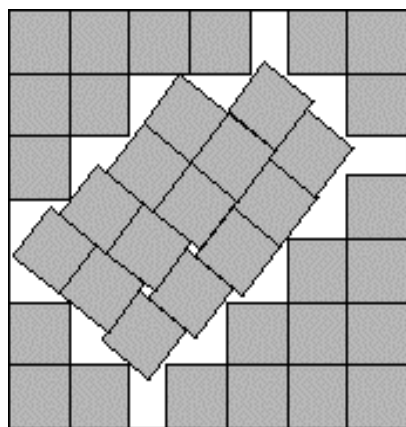
17



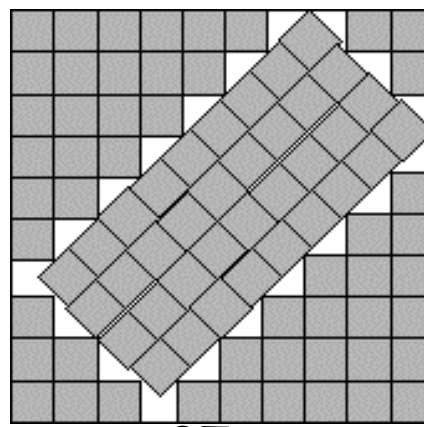
19



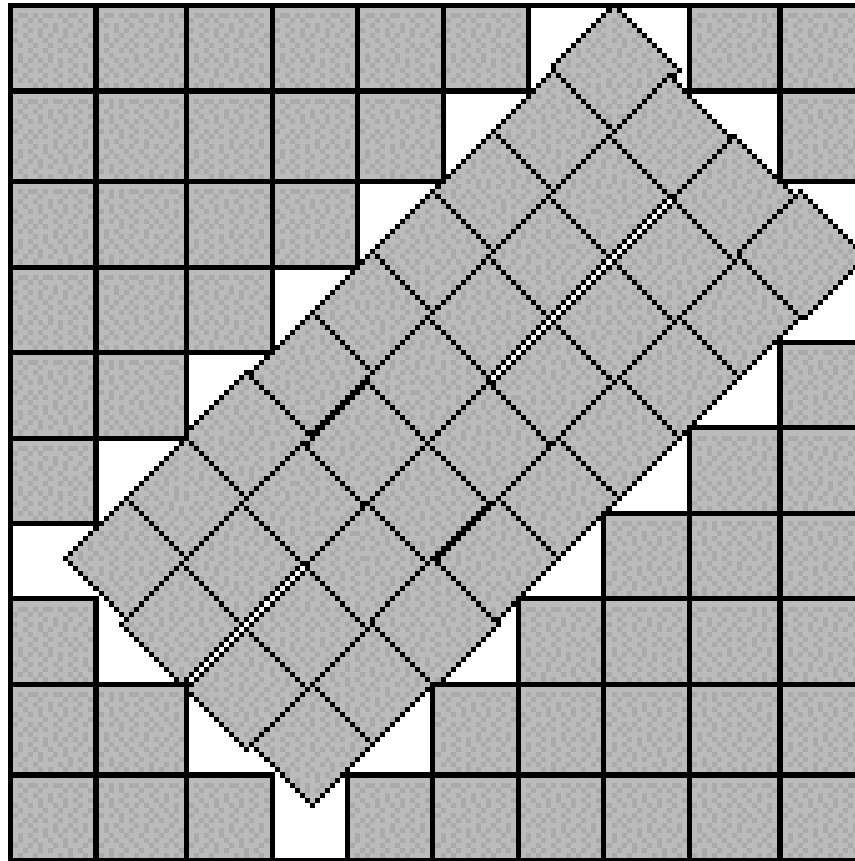
29



37



87

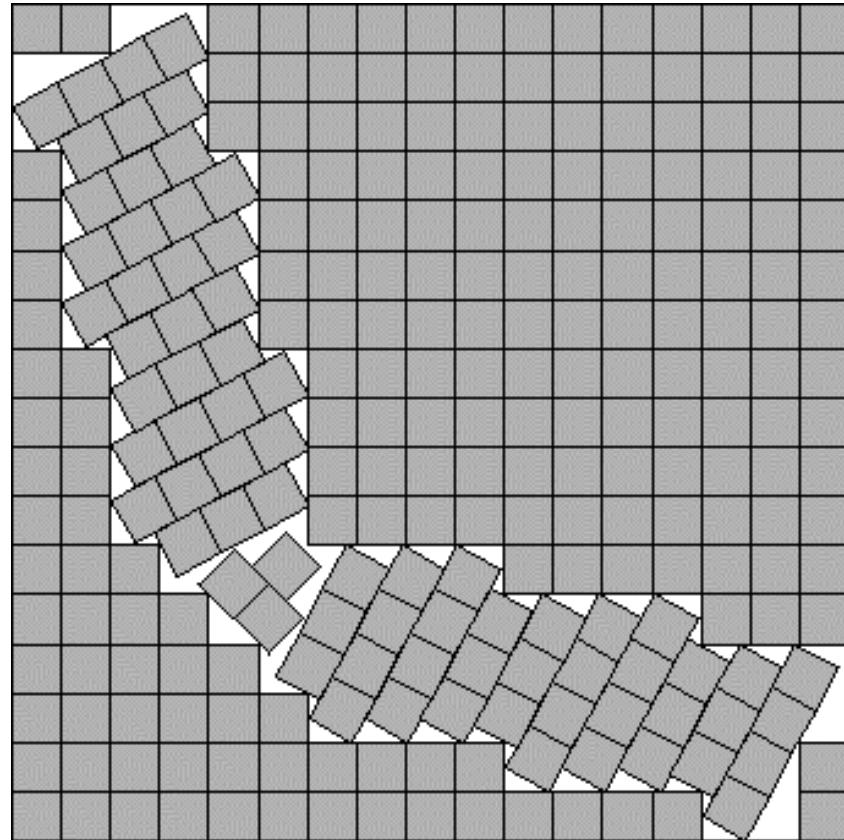


87

Could this really be "the truth" ?

Old conjecture: $s(n^2 - n) = n$

(New) counterexample:



$$s(17^2 - 17) < 17 \text{ (L. Cleemann)}$$

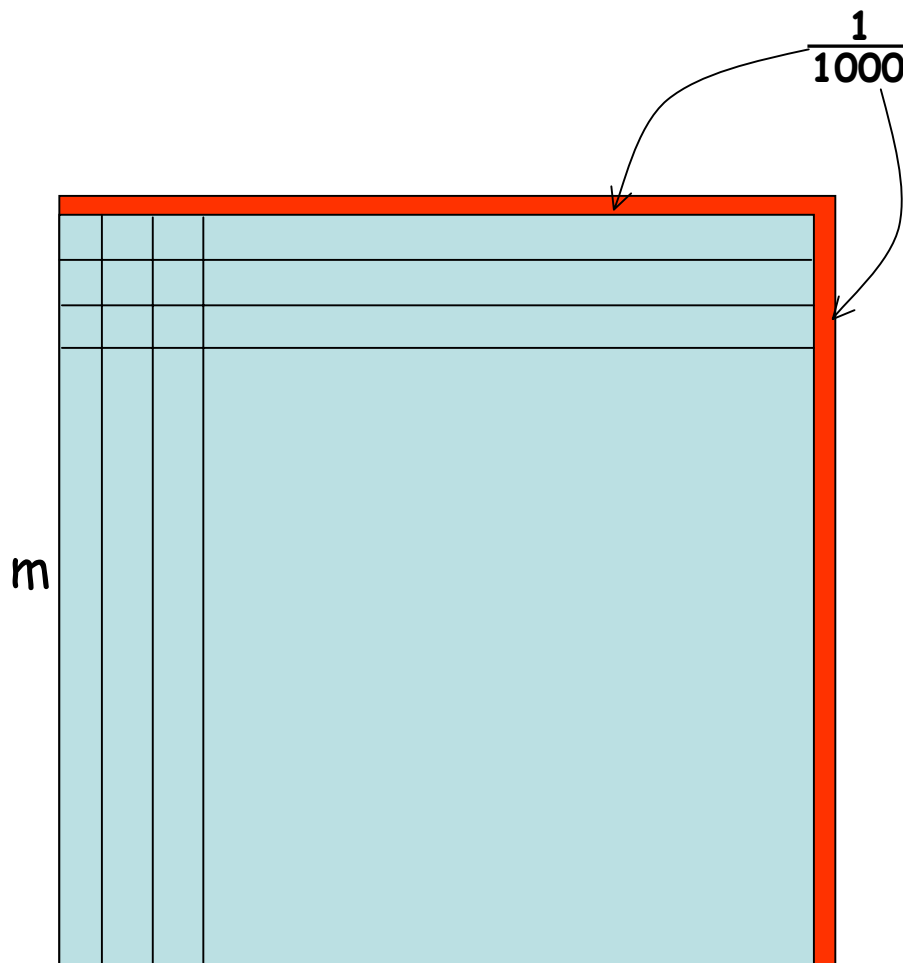
Define the wasted space $W(s)$ in a packing of an $s \times s$ square by:

$$W(s) := s^2 - \max(n : s(n) \leq s)$$

$W(m) = 0$ if m is an integer.

What is $W(m + \frac{1}{1000})$?

$W(m + \frac{1}{1000}) \approx \frac{m}{500}$?



Define the wasted space $W(s)$ in a packing of an $s \times s$ square by:

$$W(s) := s^2 - \max(n : s(n) \leq s)$$

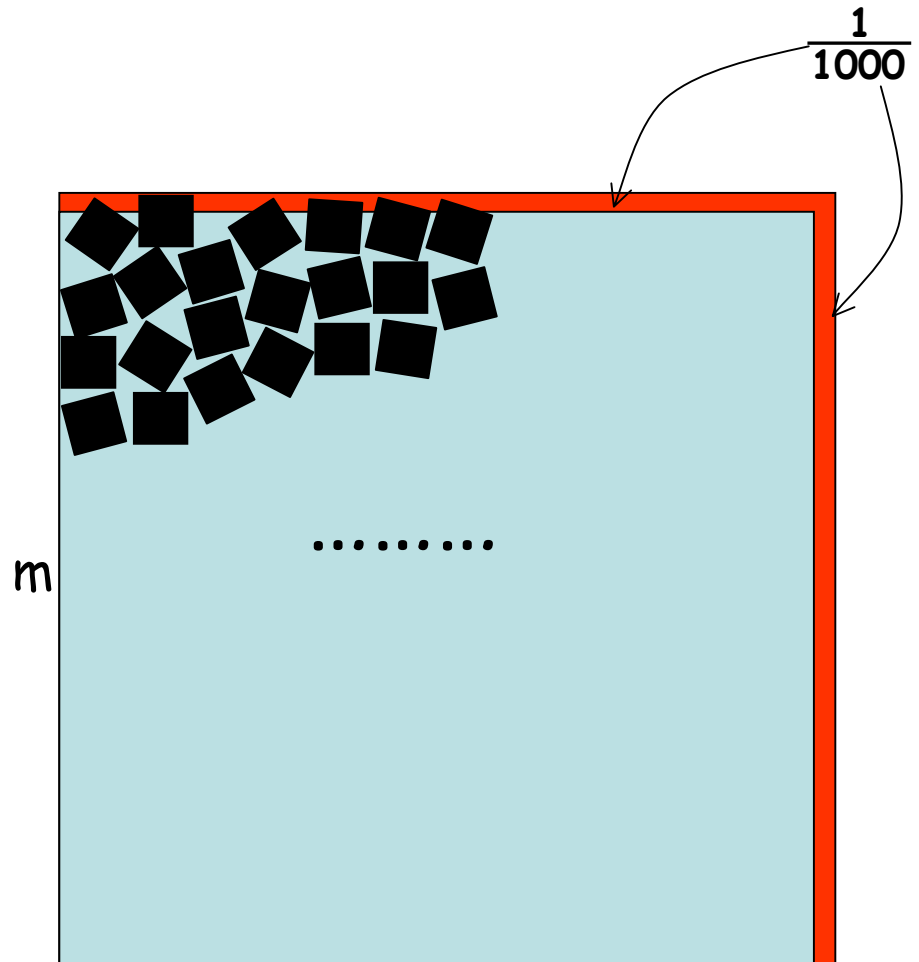
$W(m) = 0$ if m is an integer.

What is $W(m + \frac{1}{1000})$?

A non-obvious packing!

Theorem (Erdős-RLG - 1975)

$$W(s) = O(s^{\frac{7}{11}})$$



Theorem: (H. Montgomery)

For any $\varepsilon > 0$,

$$W(s) = O(s^{\frac{3-\sqrt{3}}{2}+\varepsilon})$$

Note: $\frac{3-\sqrt{3}}{2} = 0.63397\dots < 0.63636\dots = \frac{7}{11}$

What about a lower bound?

Theorem: (K. F. Roth-R. C. Vaughan - 1978)

Suppose $s(s - \lfloor s \rfloor) > \frac{1}{6}$.

Then

$$W(s) > 10^{-100} \sqrt{s |s - \lfloor s + \frac{1}{2} \rfloor|}$$

Thus, $W(s) \gg s^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$, (for s bounded away from integers)

Conjecture: (\$1000)

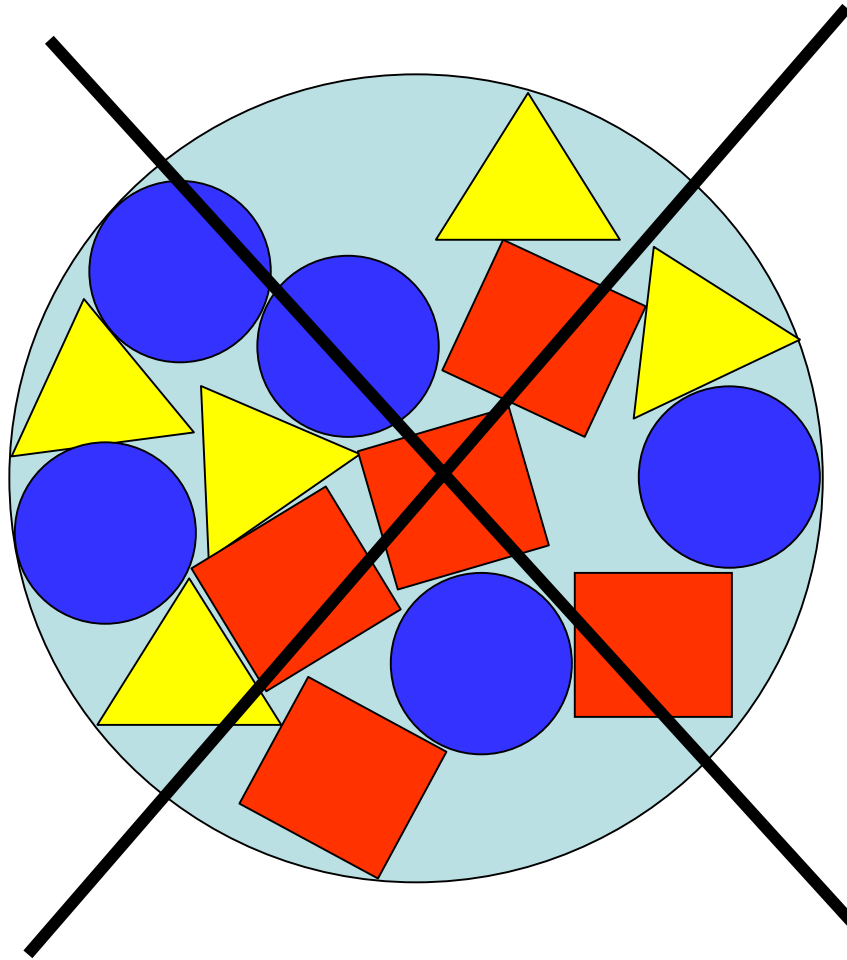
For some $\varepsilon > 0$, $W(s) \gg s^{\frac{1}{2}+\varepsilon}$ (for s bounded away from integers)

E. Friedman, Packing unit squares in squares: a survey and new results,
Electronic J. Combinatorics 7 (2000) DS #7

P. Erdős and R. L. Graham, On packing squares with equal squares,
J. Combin. Theory Ser. A **19** (1975) 119-123

K. F. Roth and R. C. Vaughan, Inefficiency in packing squares
with unit squares, *J. Combin. Theory Ser. A* **24** (1978) 170-186

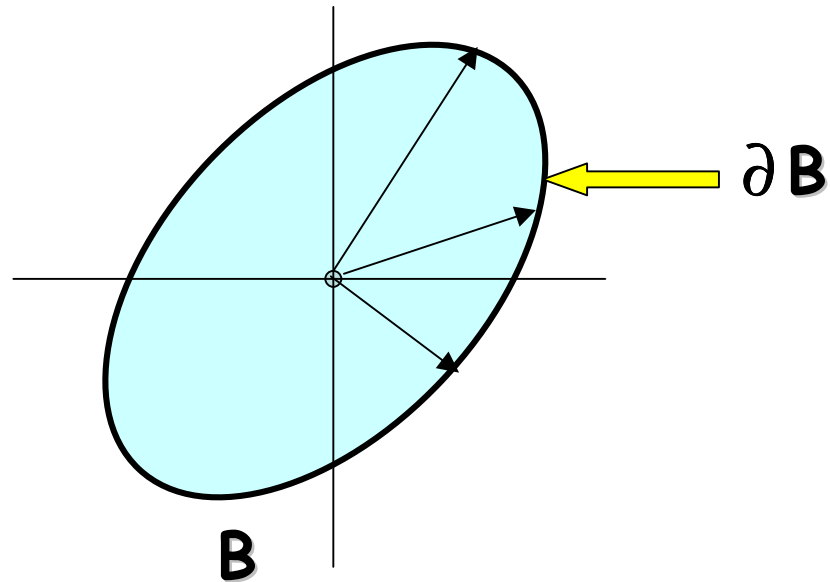
What next?



What next?

A different metric

The Minkowski plane — unit ball determined by a compact convex centrally symmetric domain B .



Theorem (Folkman, RLG—1969)

If K is a simplicial complex in the plane, and $d(x,y) \geq 1$ for $x \neq y$ in K ,

then
$$\alpha_0(K) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K)$$

Theorem (RLG, H. Witsenhausen, H. Zassenhaus - 1972)

For any Minkowski plane, and any finite simplicial complex K in the plane, we have

$$\alpha_0(K) \leq \frac{1}{2\Delta^*} A(K) + \frac{1}{2} P(K) + \chi(K)$$

where Δ^* is the minimum (by compactness) over all areas of triangles with unit side lengths.

Instead of **packings**, we consider the same questions for **coverings**.

In general, these seem to be more difficult.

For example, G. Fejes Tóth (2003) has managed to find the **thinnest covering** of a strip of width w by unit circles where

$$\sqrt{3} \leq w \leq \sqrt{3} + \varepsilon$$

for a suitable (very small) positive ε

Instead of **packings**, we consider the same questions for **coverings**.

In general, these seem to be more difficult.

For example, G. Fejes Tóth (2003) has managed to find the **thinnest covering** of a strip of width w by unit circles where

$$\sqrt{3} \leq w \leq \sqrt{3} + \varepsilon$$

for a suitable (very small) positive ε .

Instead of **packings**, we consider the same questions for **coverings**.

In general, these seem to be more difficult.

For example, G. Fejes Tóth (2003) has managed to find the **thinnest covering** of a strip of width w by unit circles where

$$\sqrt{3} \leq w \leq \sqrt{3} + \varepsilon$$

for a suitable (very small) positive ε .

Finally, how about all of these questions
in **three** (or more) dimensions !

Even the first question we started with, namely
determining the densest packing of Euclidean 3-space
with unit balls still seems to be rather challenging!

(Kepler conjecture, Hales/Ferguson, Hsiang,)

To be continued.....