Folding & Unfolding: Unfolding Polyhedra

Joseph O'Rourke
Smith College

Folding and Unfolding Talks

Linkage folding	Tuesday	Erik Demaine
Paper folding	Wednesday	Erik Demaine
Folding polygons into convex polyhedra	Saturday_1	Joe O'Rourke
Unfolding polyhedra	**Saturday_2**	**Joe O'Rourke**

Outline

- Edge-Unfolding Polyhedra
- Geodesics & Closed Geodesics
- Unrestricted Unfoldings

Outline$_1$

- Edge-Unfolding Polyhedra
  - History (Dürer) ; Open Problem; Applications
  - Evidence For
  - Evidence Against
  - Ununfoldable Polyhedra

Outline$_2$

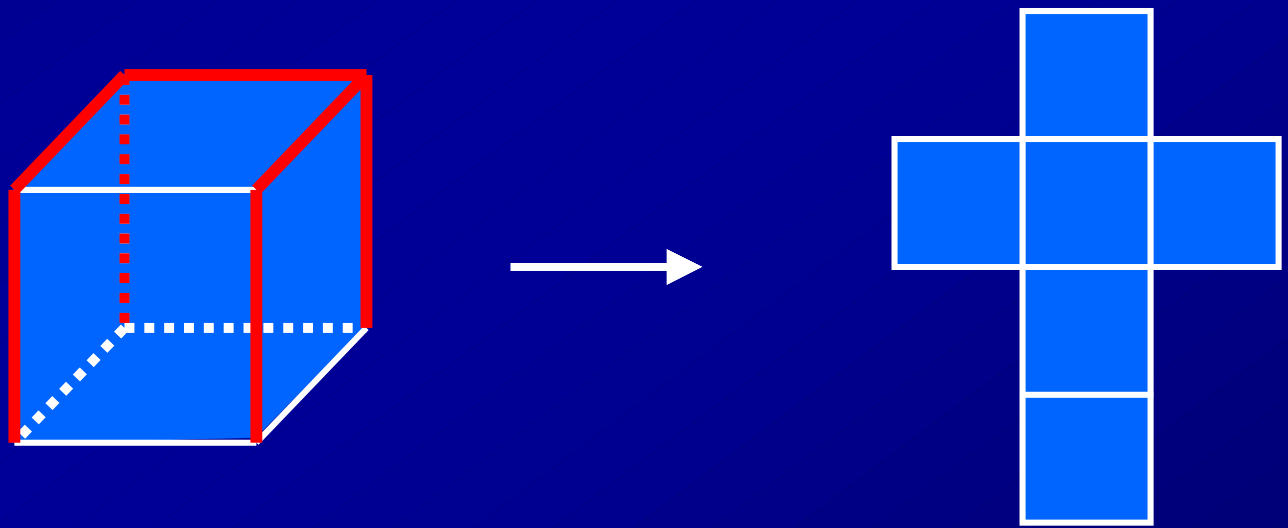
- Geodesics & Closed Geodesics
  - Lyusternick-Schnirelmann Theorem
  - Gage-Hamilton-Grayson Curve Shortening
  - Exponential Number of Closed Geodesics

Outline$_3$

- Unrestricted Unfoldings
  - Vertex Unfolding
  - Orthogonal Polyhedra
  - Open: Nonoverlapping Unfolding for Nonconvex Polyhedra

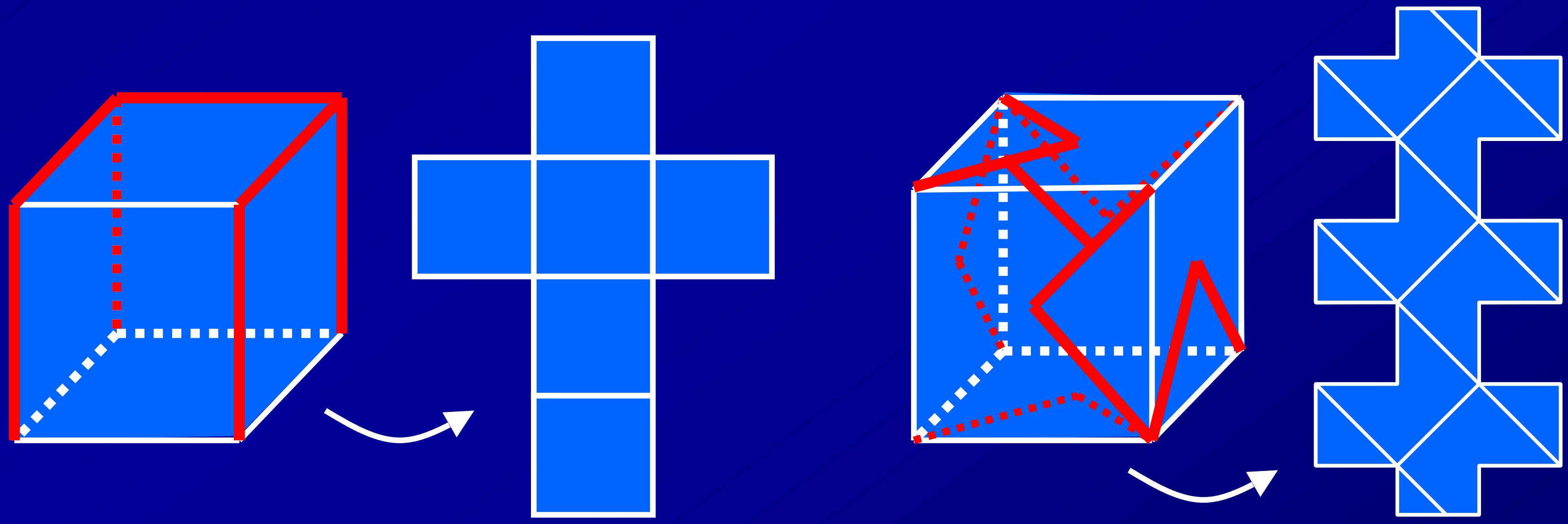
Unfolding Polyhedra

- Cut along the surface of a polyhedron
- Unfold into a simple planar polygon without overlap

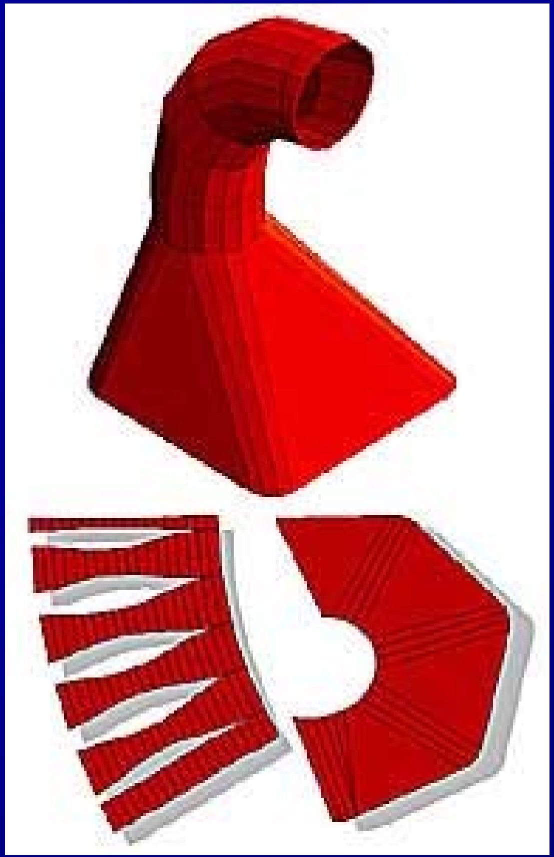


Edge Unfoldings

- Two types of unfoldings:
  - **Edge unfoldings**: Cut only along edges
  - **General unfoldings**: Cut through faces too



Commercial Software

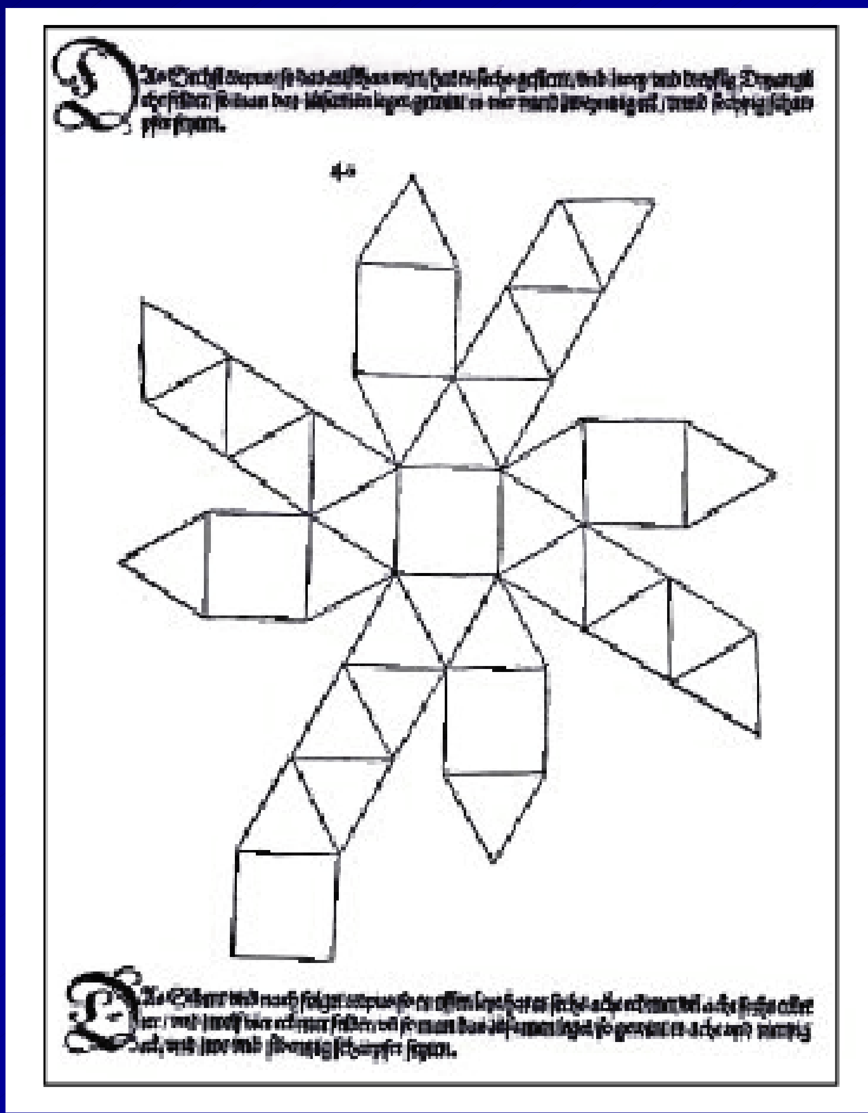


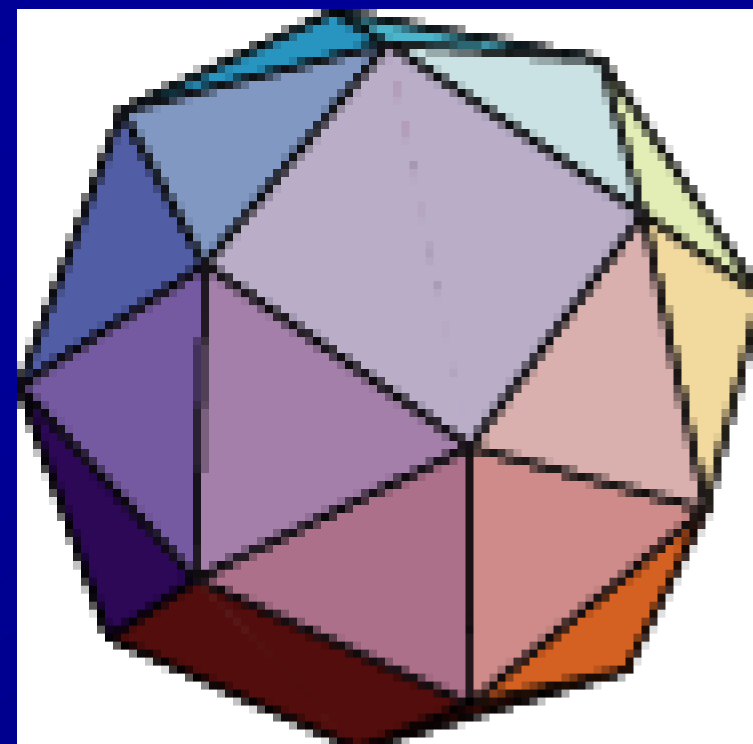
Lundström Design,
http://www.algonet.se/~ludesign/index.html

Albrecht Dürer, 1425

Melancholia I

Albrecht Dürer, 1425





Snub Cube

Open:
Edge-Unfolding Convex Polyhedra

Does every convex polyhedron have an edge-unfolding to a simple, nonoverlapping polygon?

[Shephard, 1975]

Cut Edges form Spanning Tree

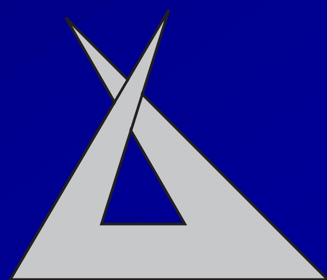
Lemma: The cut edges of an edge unfolding of a convex polyhedron to a simple polygon form a spanning tree of the 1-skeleton of the polyhedron.

- spanning: to flatten every vertex
- forest: cycle would isolate a surface piece
- tree: connected by boundary of polygon

Cut Edges (revisited)

Lemma: The cut edges of an edge unfolding of a convex polyhedron to a simple polygon form a spanning tree of the 1-skeleton of the polyhedron.

Nonsimple Polygons



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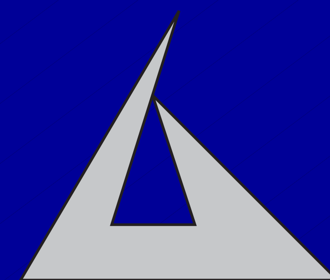

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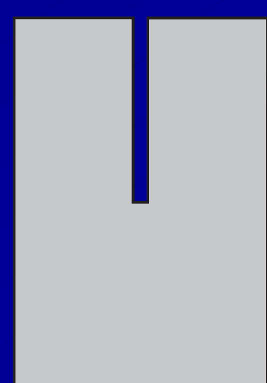
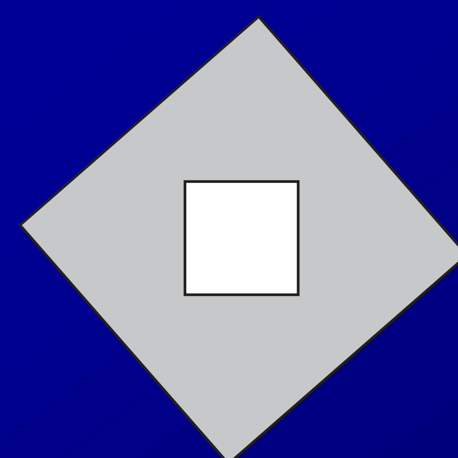
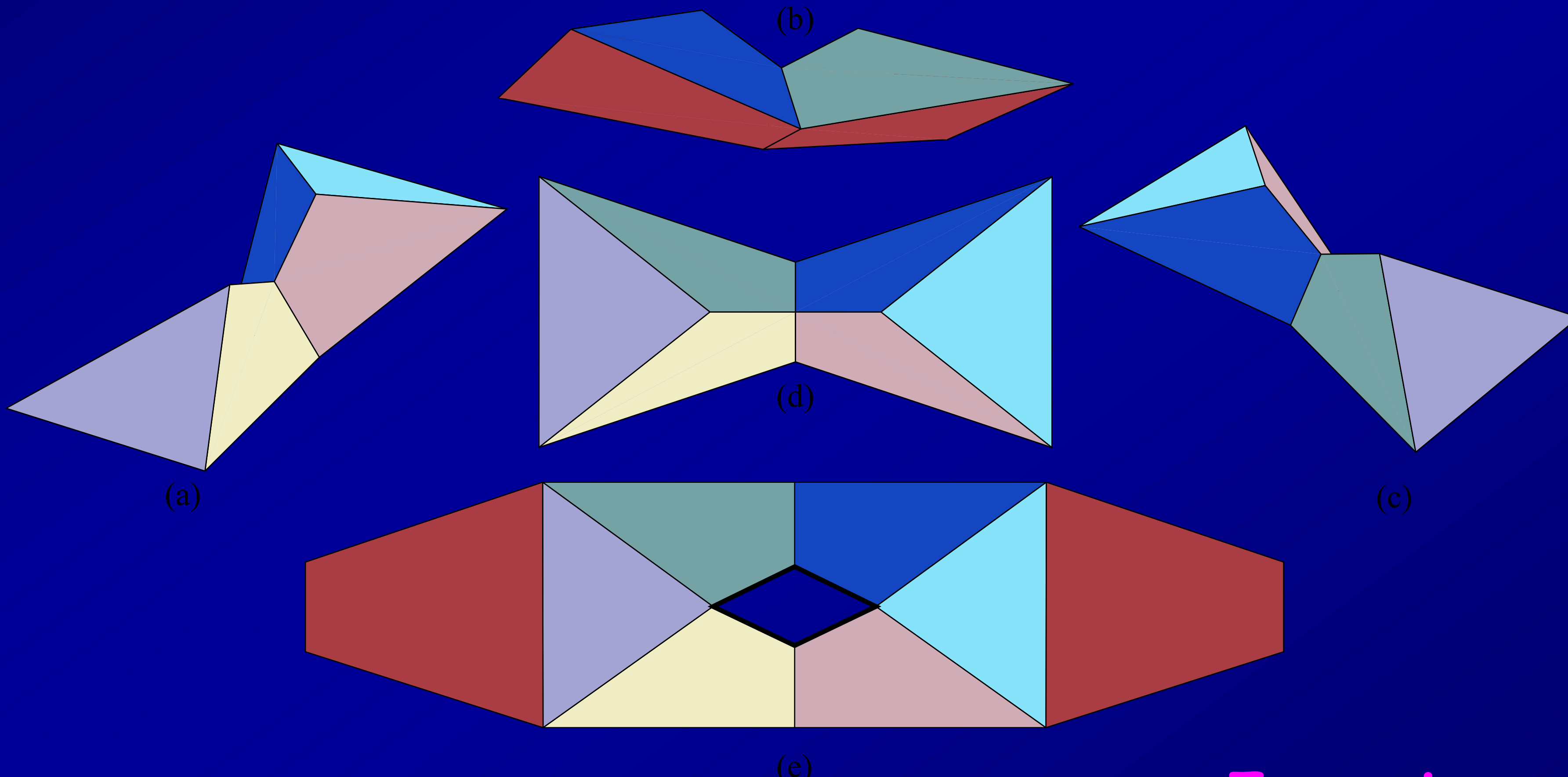


(c)



(d)

Andrea Mantler example



Javaview

Cut edges: strengthening

Lemma: The cut edges of an edge unfolding of a convex polyhedron to a single, connected piece form a spanning tree of the 1-skeleton of the polyhedron.

[Bern, Demaine, Eppstein, Kuo, Mantler, O'Rourke, Snoeyink 01]

Outline$_1$

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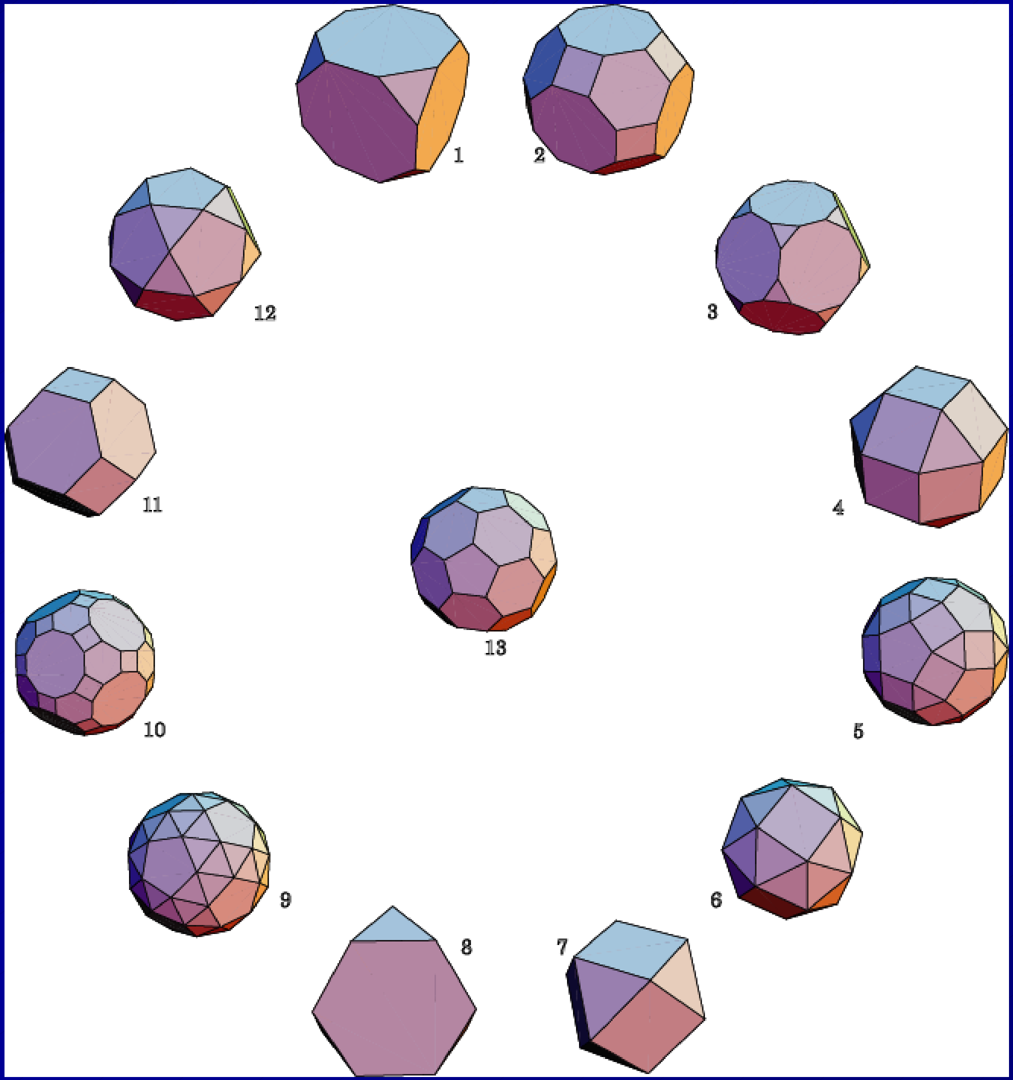
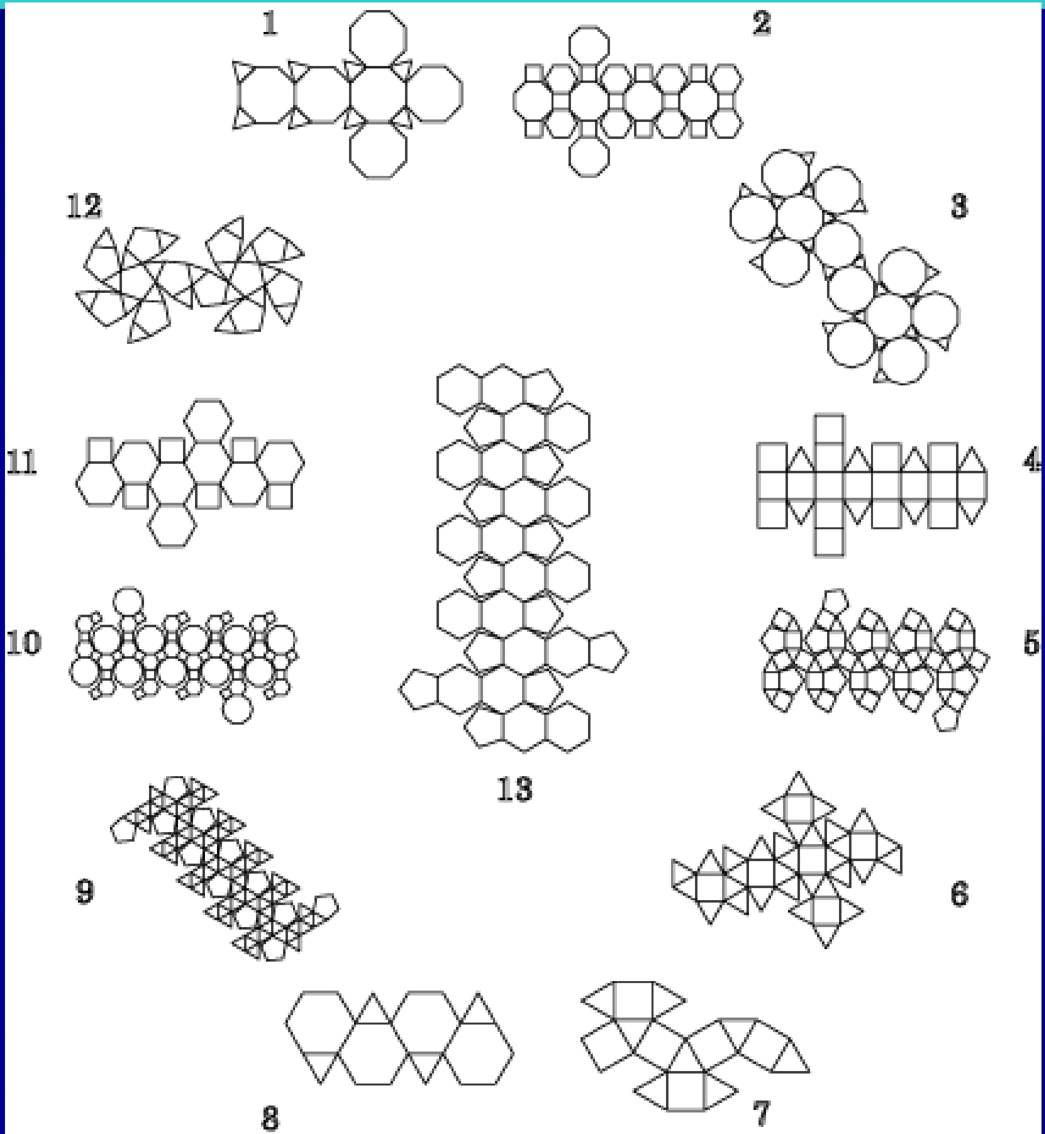
  - History (Dürer) ; Open Problem; Applications
  - Evidence For
  - Evidence Against
  - Ununfoldable Polyhedra

Archimedian Solids

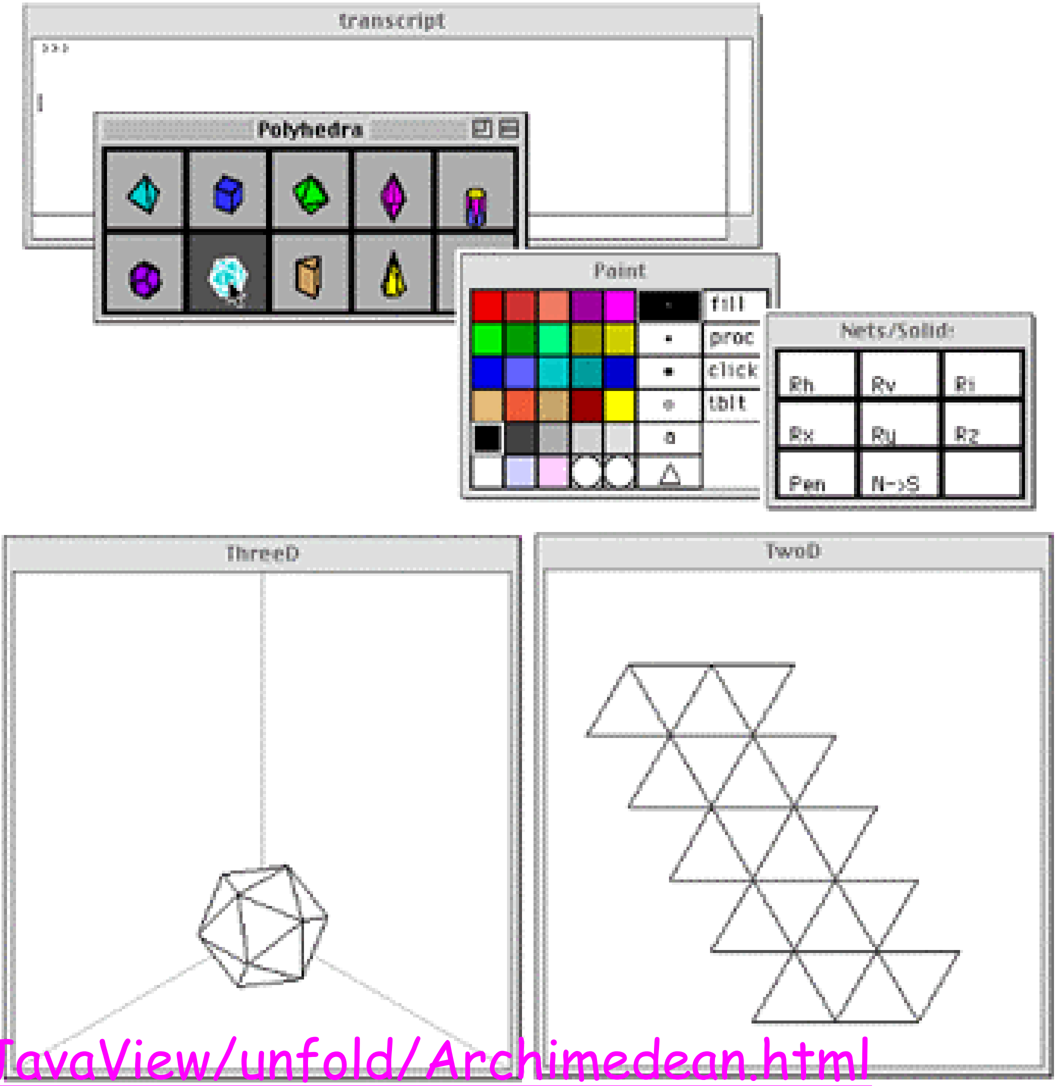


Nets for Archimedian Solids



Successful Software

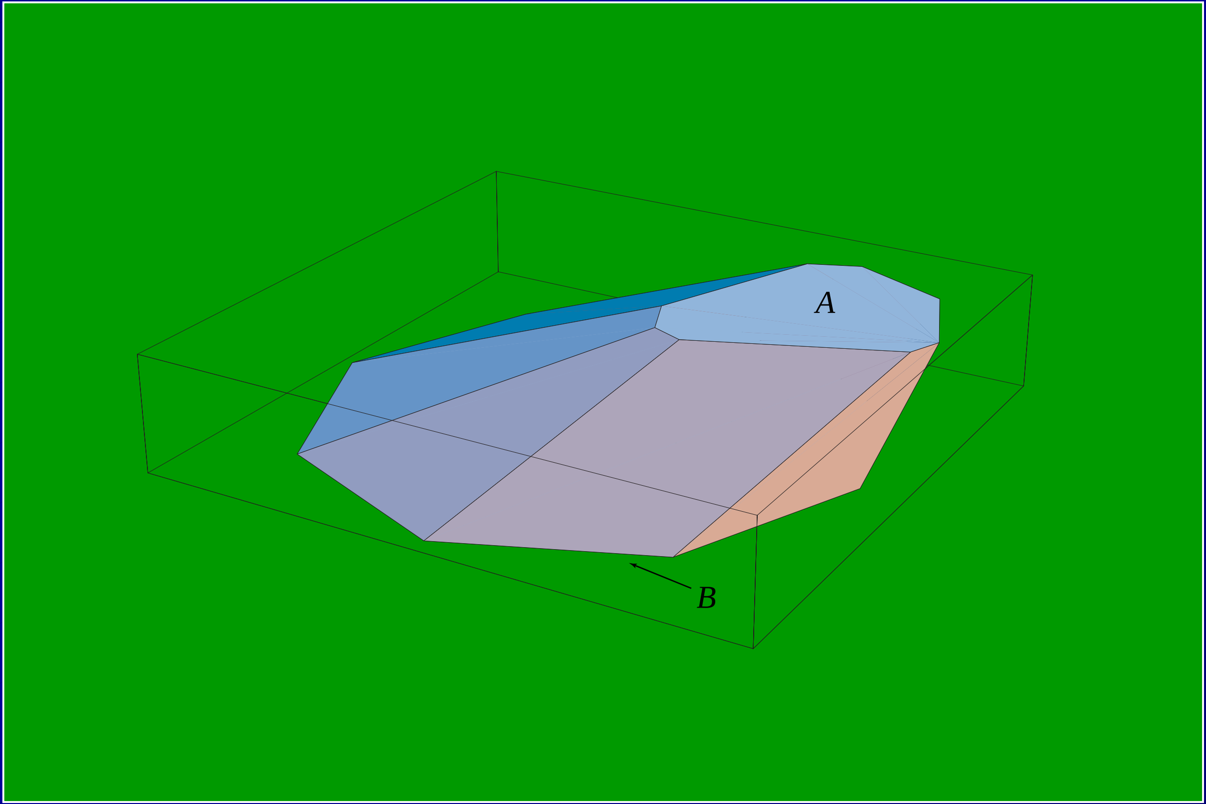
- Nishizeki
- Hypergami →
- Javaview Unfold
- ...



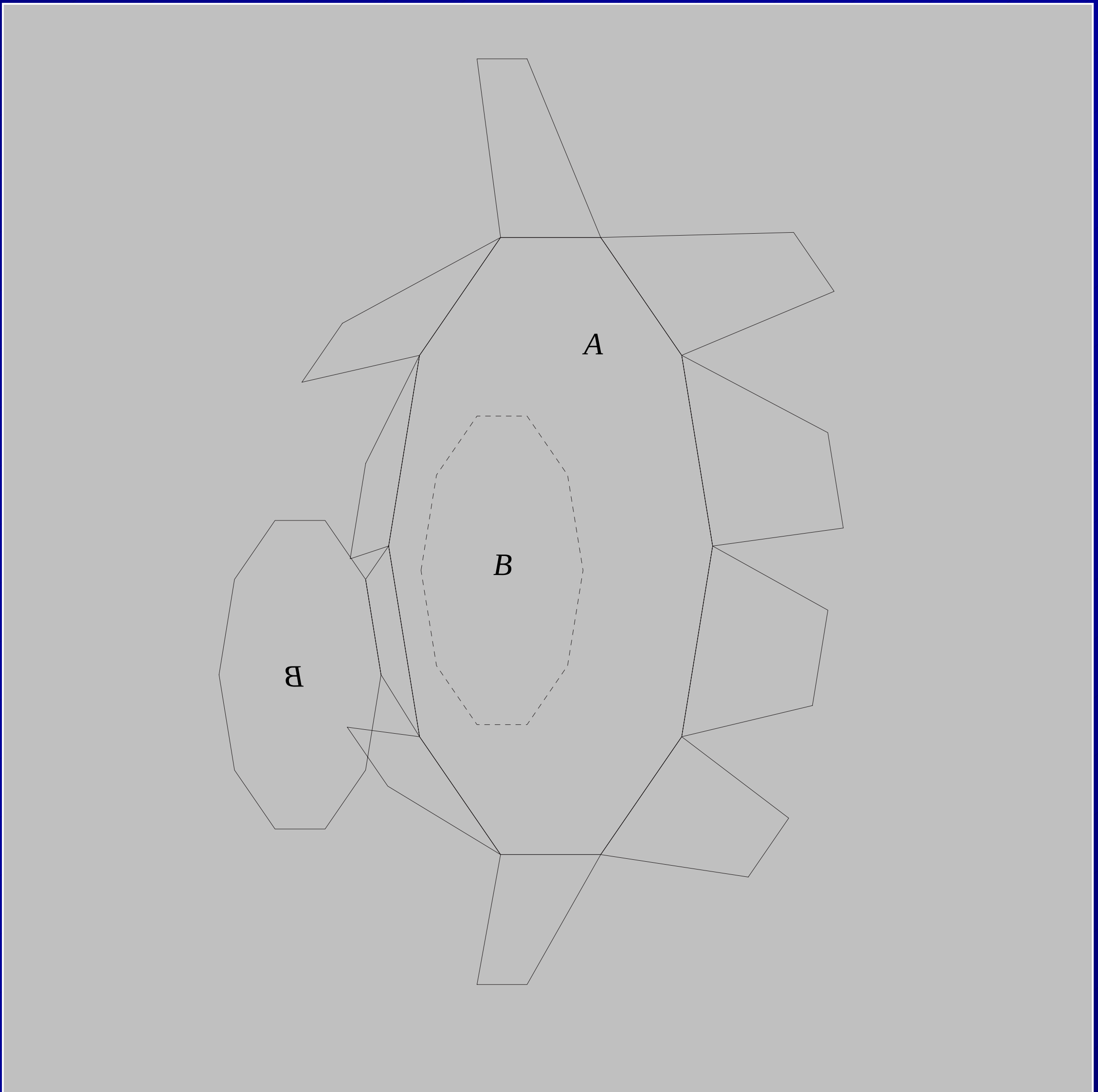
http://www.fucg.org/PartIII/JavaView/unfold/Archimedean.html

Prismoids

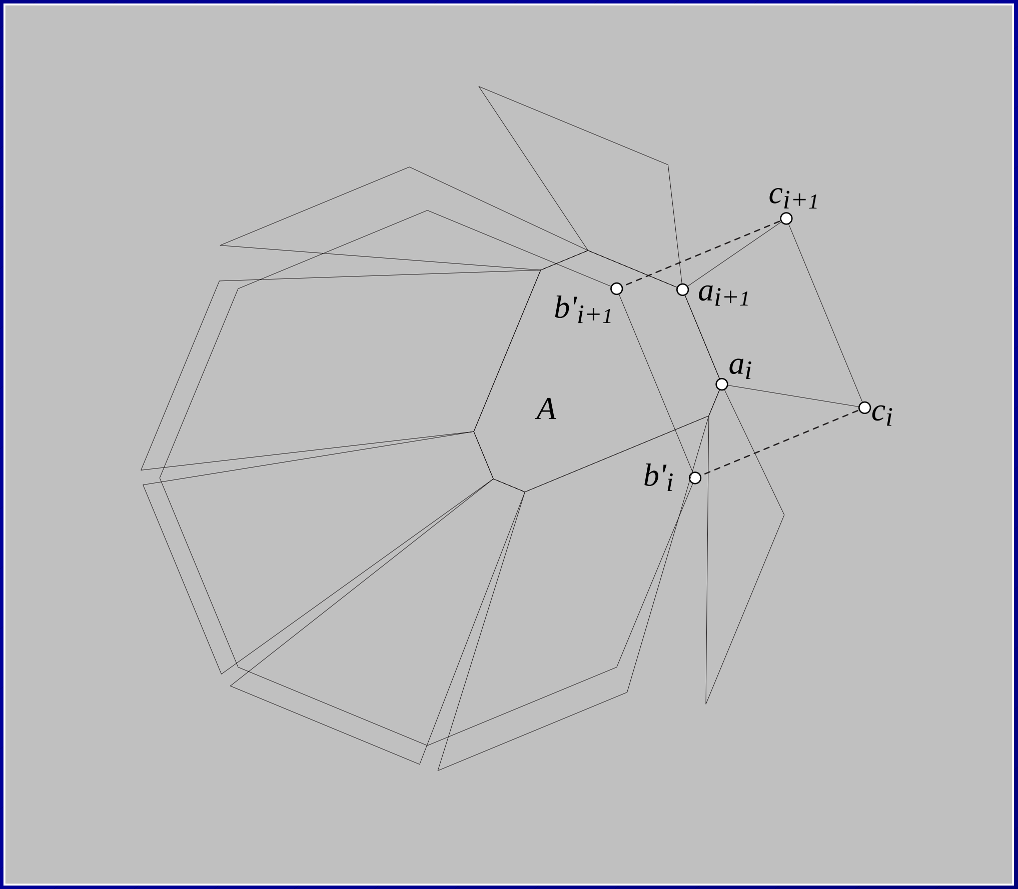
Convex top A and bottom B, equiangular.
Edges parallel; lateral faces quadrilaterals.



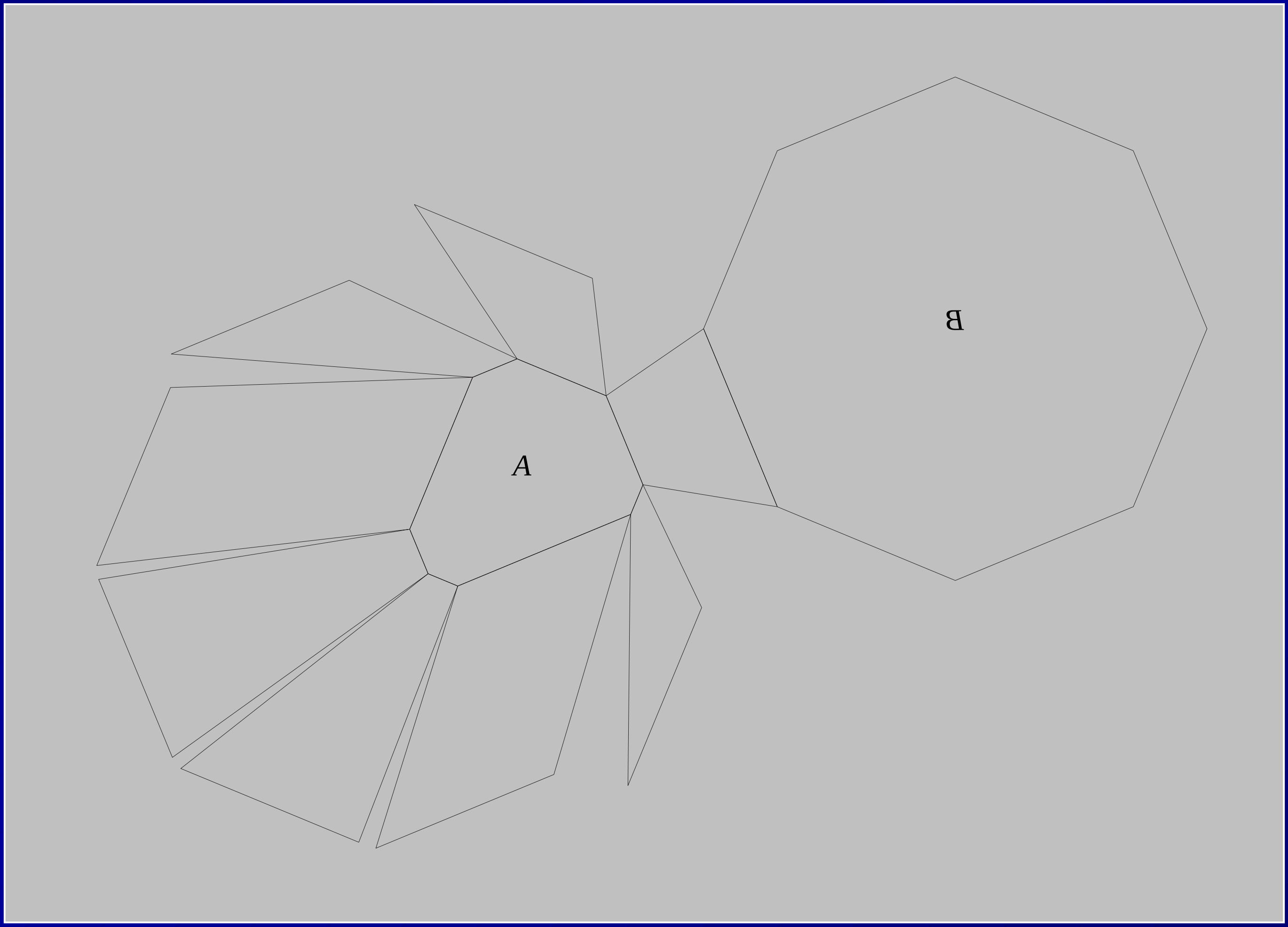
Overlapping Unfolding



Splay Unfolding (top view)



Splay Unfolding



Outline$_1$

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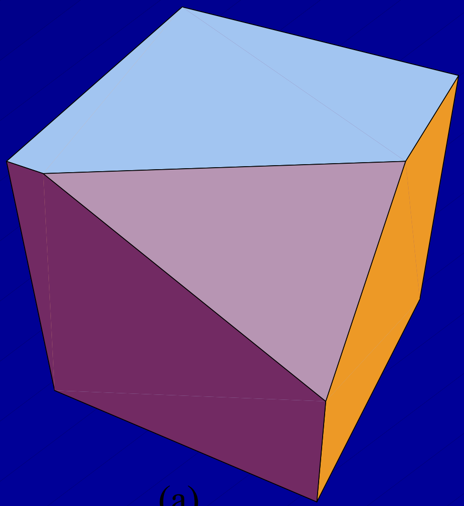
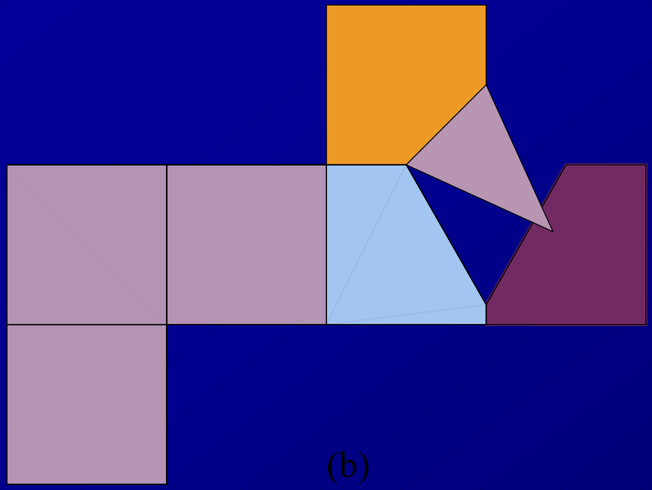
  - Unfoldable Polyhedra

Cube with one corner truncated

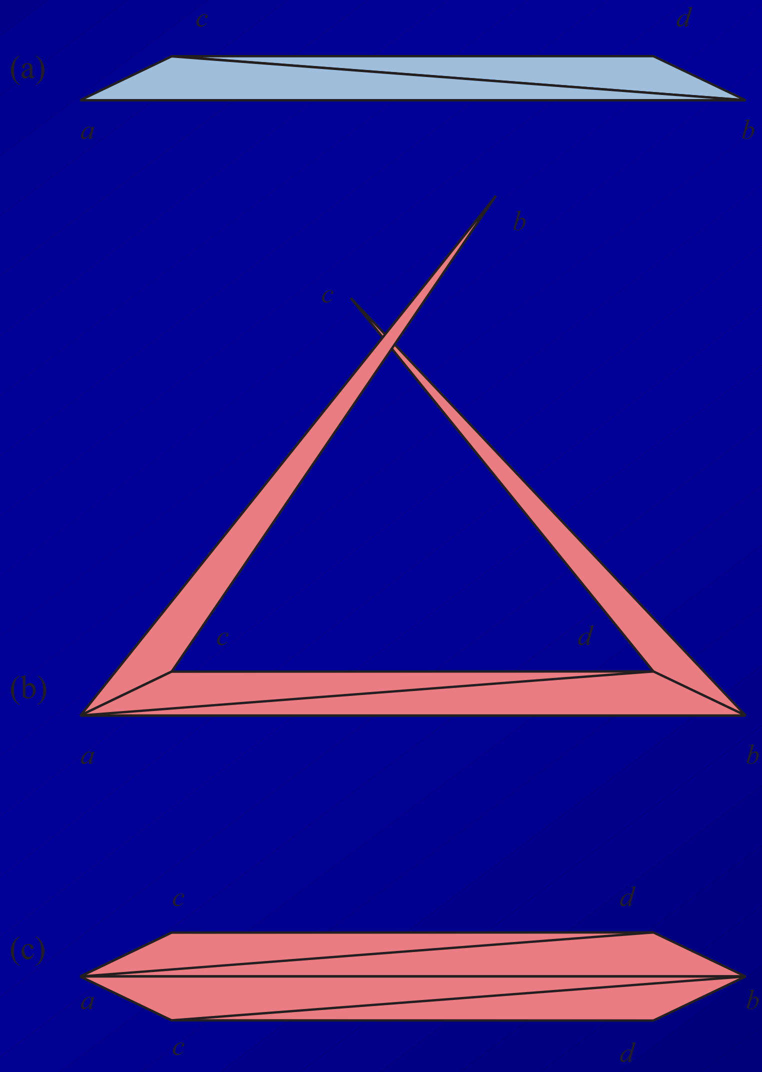


(a)



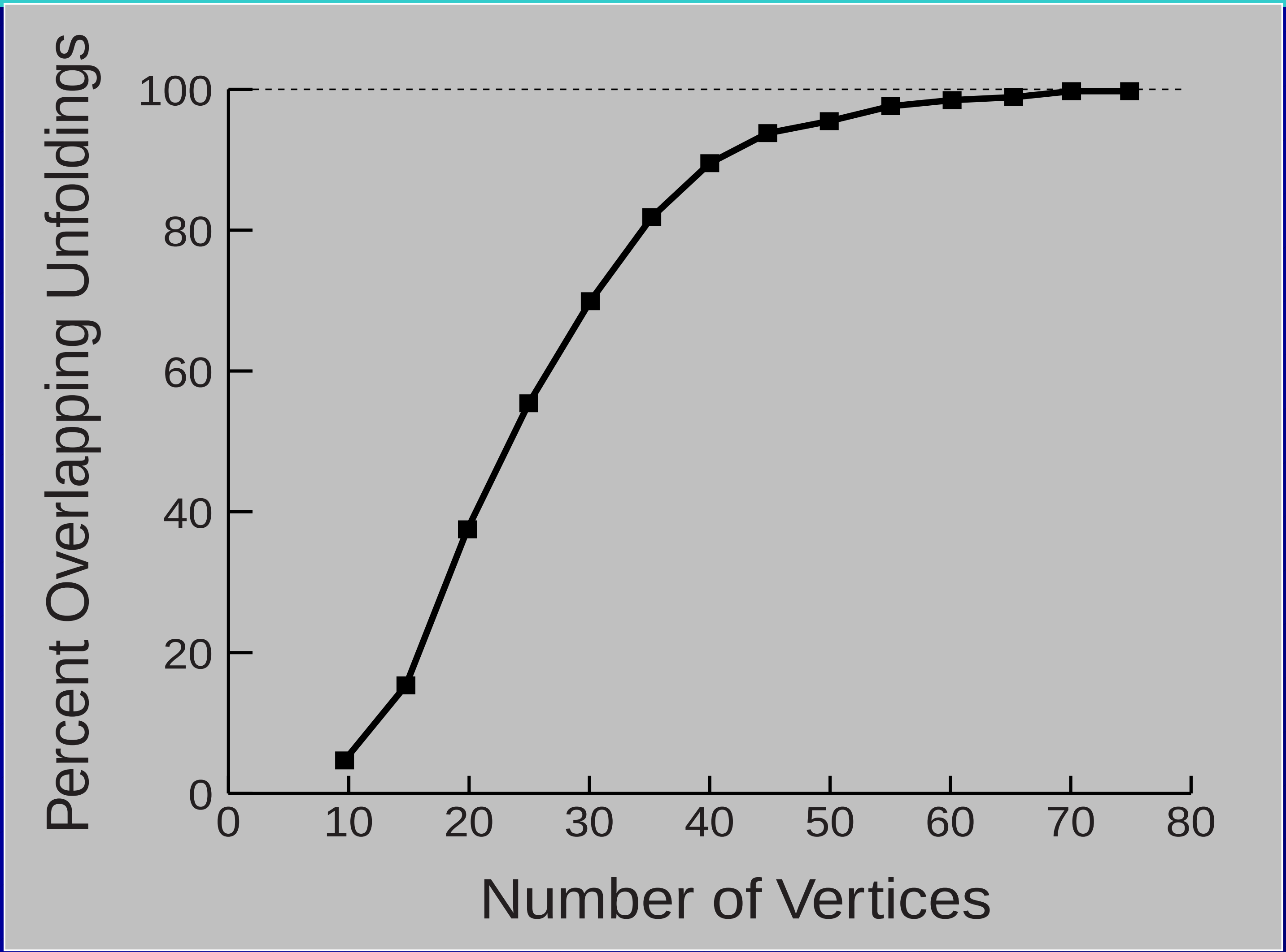
(b)

“Sliver” Tetrahedron



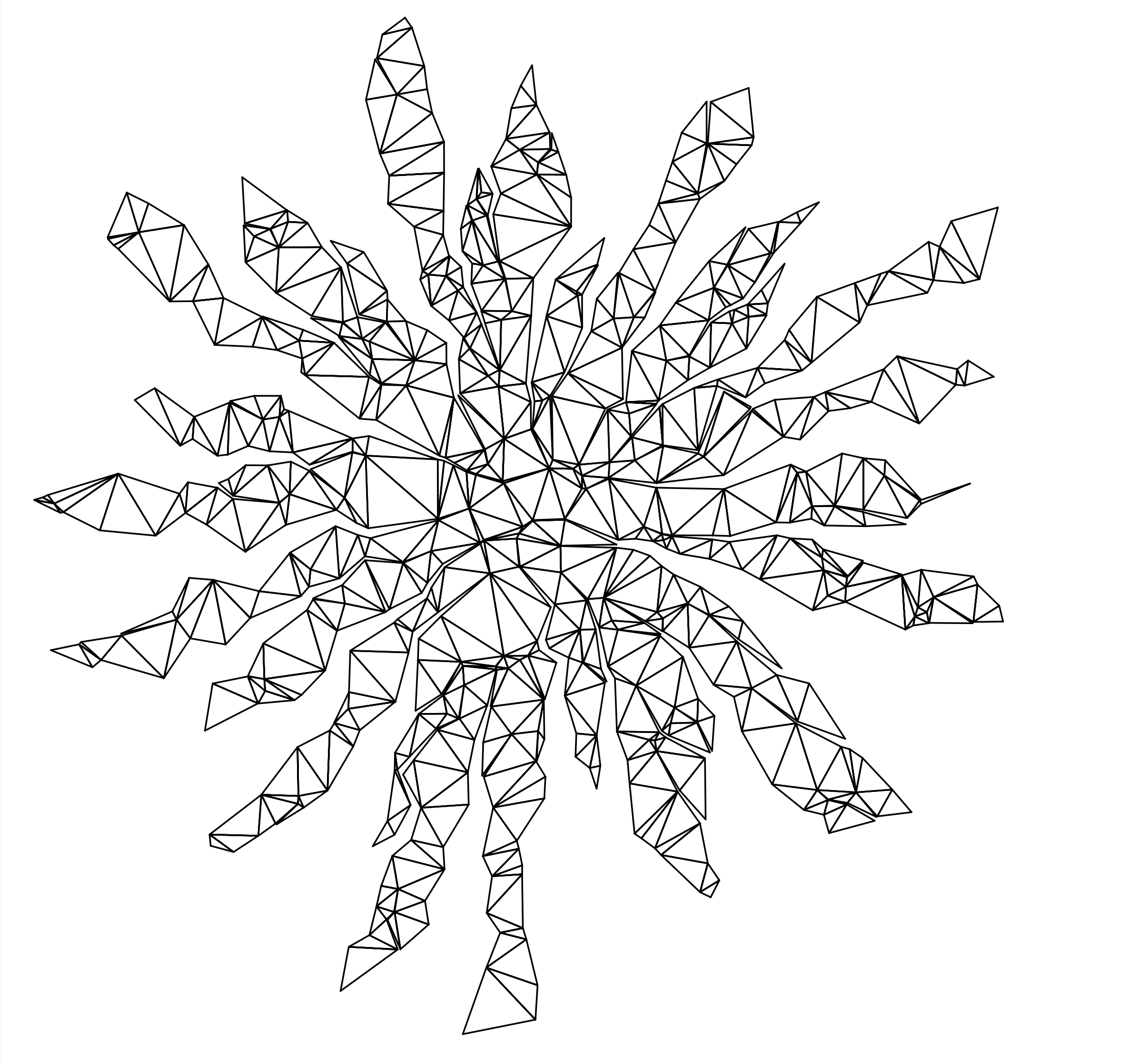
Percent Random Unfoldings that Overlap

[O'Rourke, Schevon 1987]

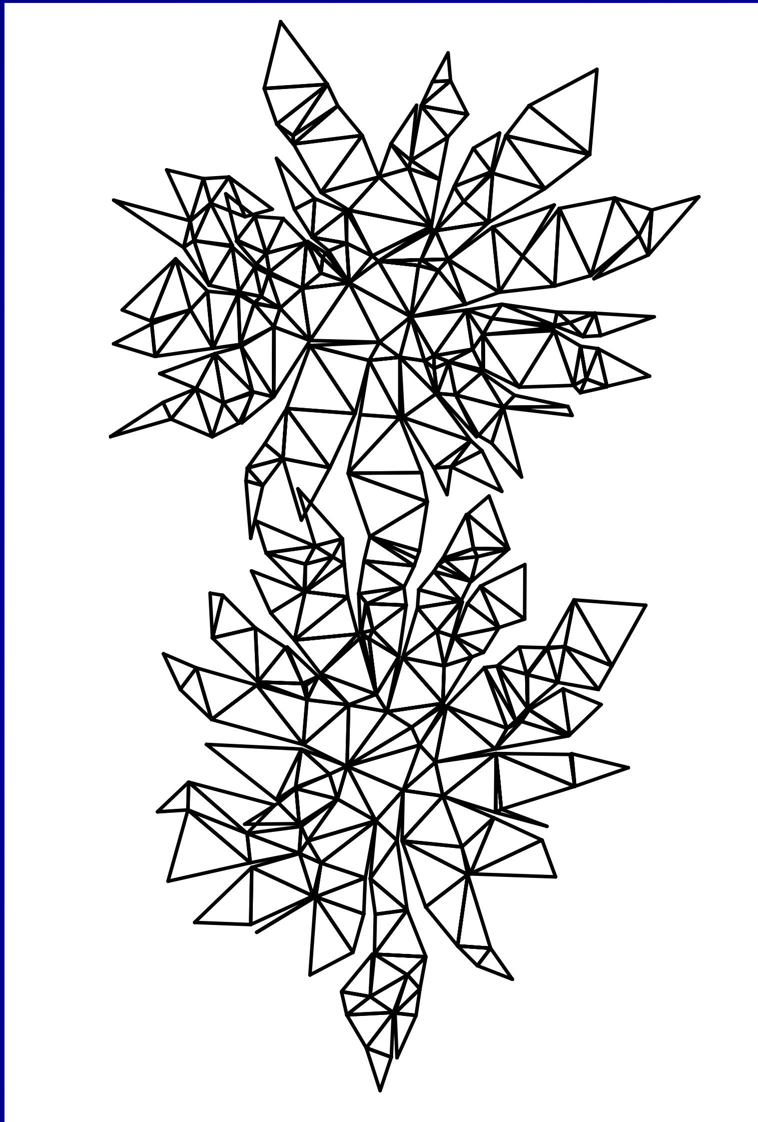


Sclickenrieder$_1$: steepest-edge-unfold

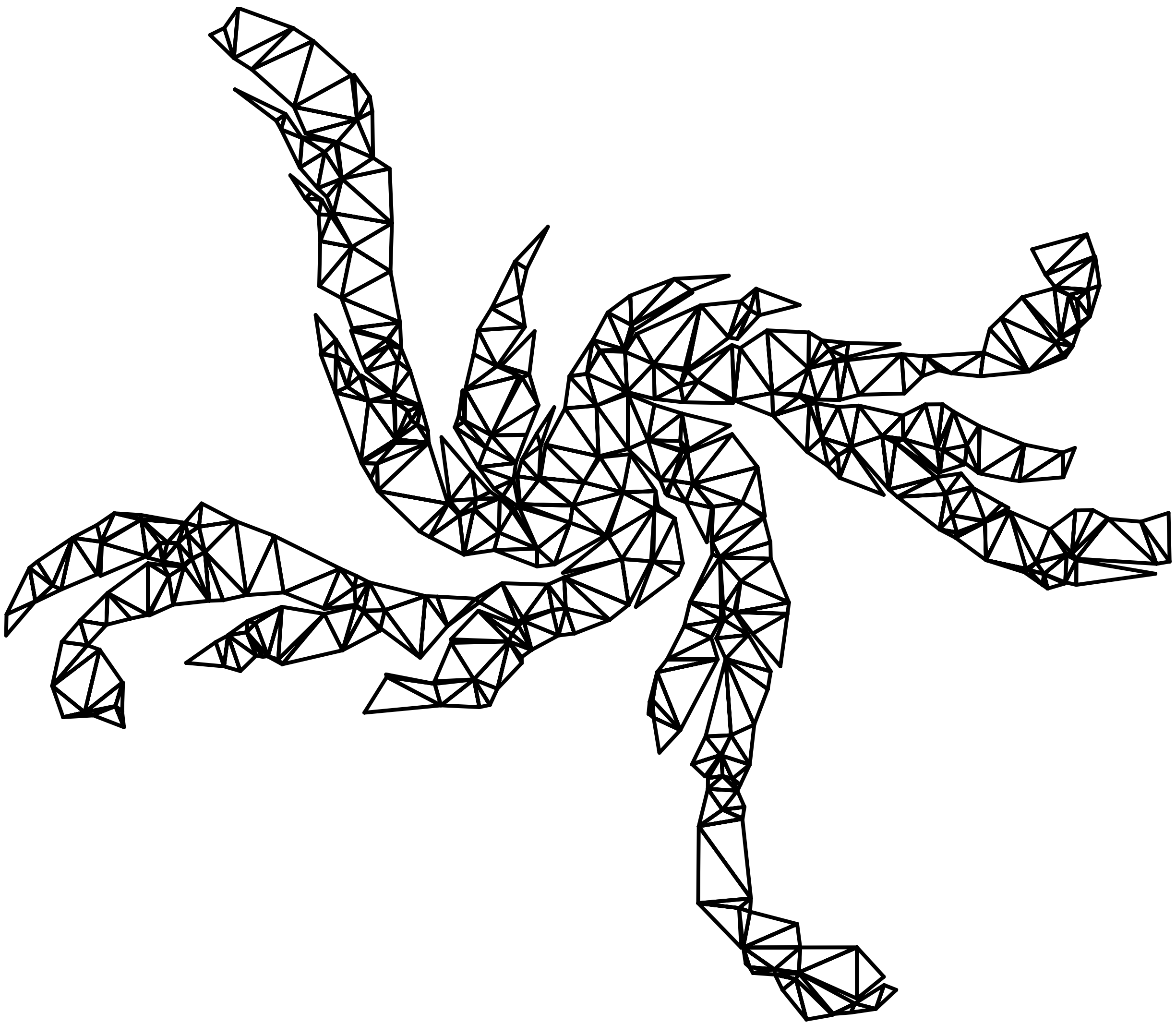
"Nets of Polyhedra"
TU Berlin, 1997



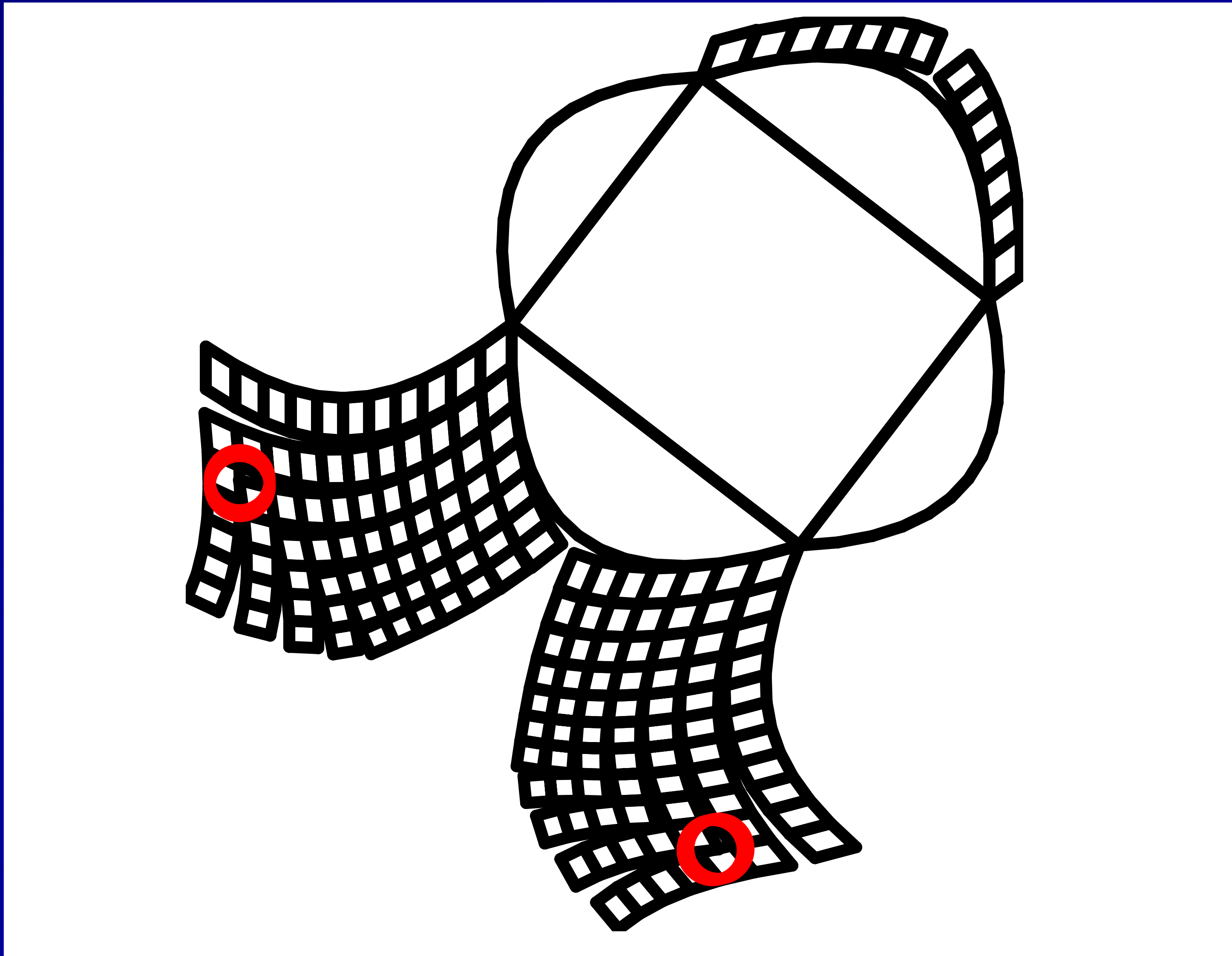
Sclickenrieder$_2$: flat-spanning-tree-unfold



Sclickenrieder$_3$: rightmost-ascending-edge-unfold



Sclickenrieder$_4$: normal-order-unfold



Open: Edge-Unfolding Convex Polyhedra (revisited)

Does every convex polyhedron have an edge-unfolding to a net (a simple, nonoverlapping polygon)?

Open: Fewest Nets

For a convex polyhedron of n vertices and F faces, what is the fewest number of nets (simple, nonoverlapping polygons) into which it may be cut along edges?

- $\leq F$
- Simplicial polyhedra: $\leq F/2$
- Simple polyhedra: $\leq (2/3)(F-2)$

Outline$_1$

- Edge-Unfolding Polyhedra
  - History (Dürer) ; Open Problem; Applications
  - Evidence For
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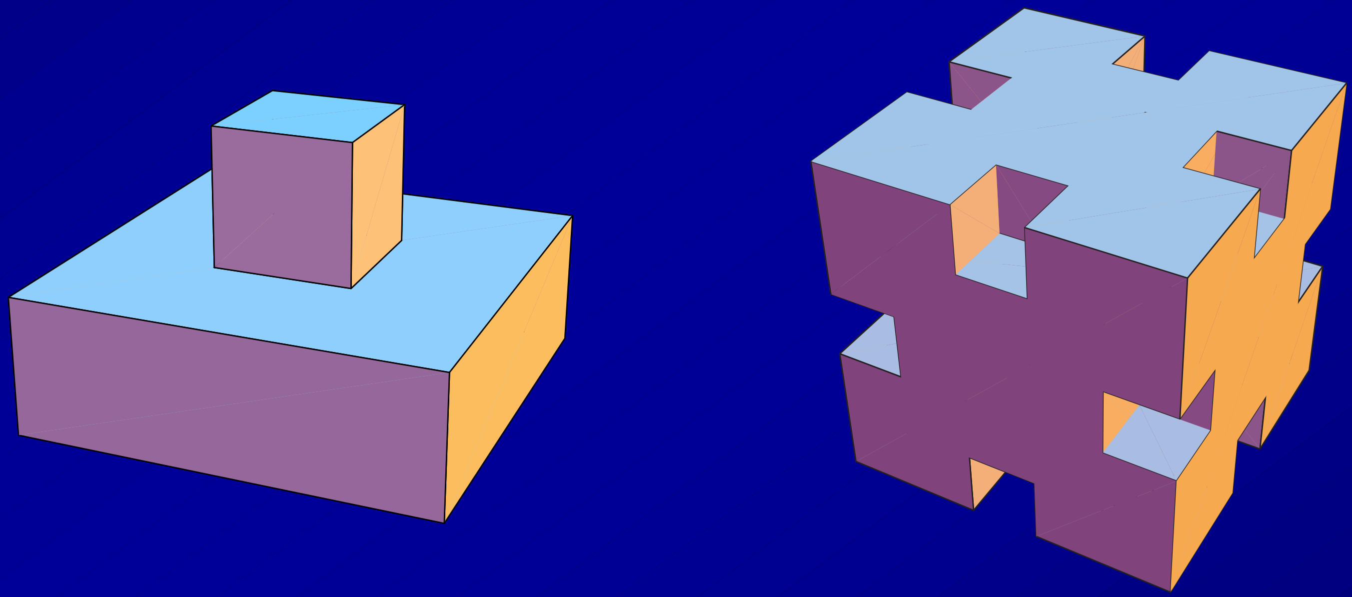
  - Ununfoldable Polyhedra

Edge-Ununfoldable Orthogonal Polyhedra



Biedl, Demaine, Demaine, Lubiw, O'Rourke, Overmars, Robbins, Whitesides [CCCG98]

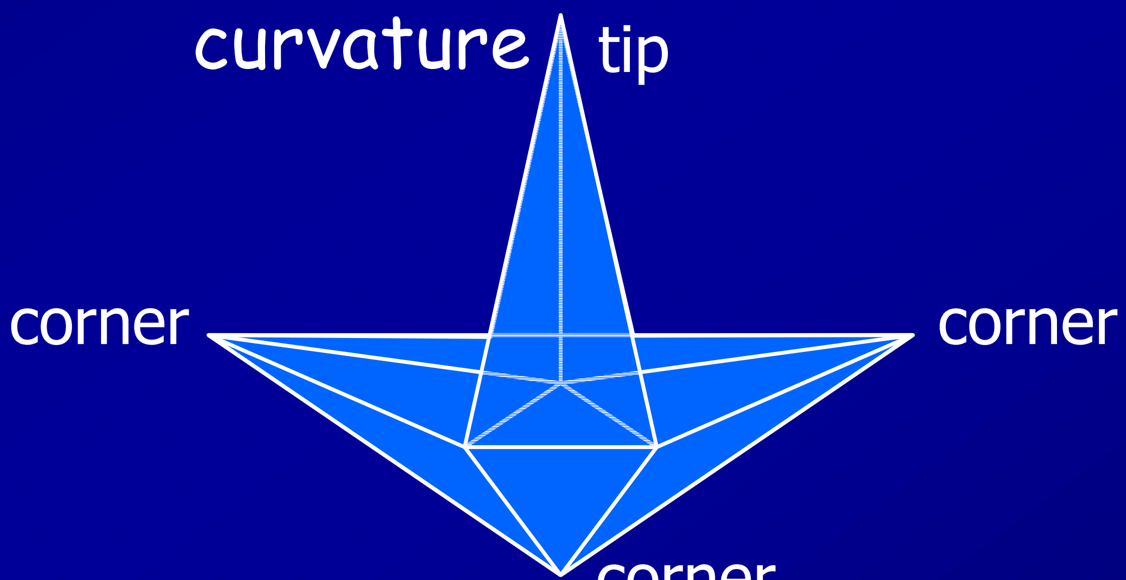
Topologically Convex Polyhedra

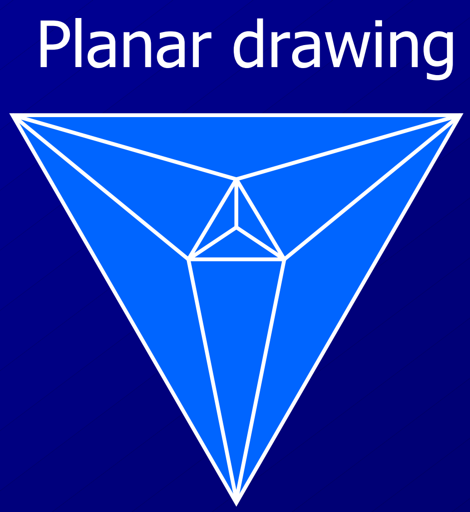
(Bern, Demaine, Eppstein, Kuo '99)

- A polyhedron is **topologically convex** if its 1-skeleton is that of a convex polyhedron
  - Steinitz's theorem: iff 3-connected and planar
- Natural question: Can all topologically convex polyhedra be edge unfolded?
- Subclass: **Convex-faced** polyhedra (every face is convex)
  - Schevon (1987): Are they all edge-unfoldable?

Triangulated Hat

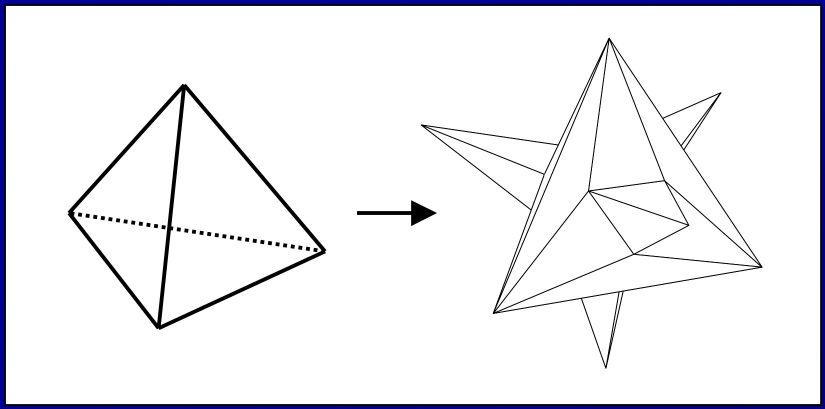
- 9 triangles
  - 6 **base** triangles, lying just above a plane
  - 3 **spike** triangles, with base angle > 60° so that middle vertices have negative curvature



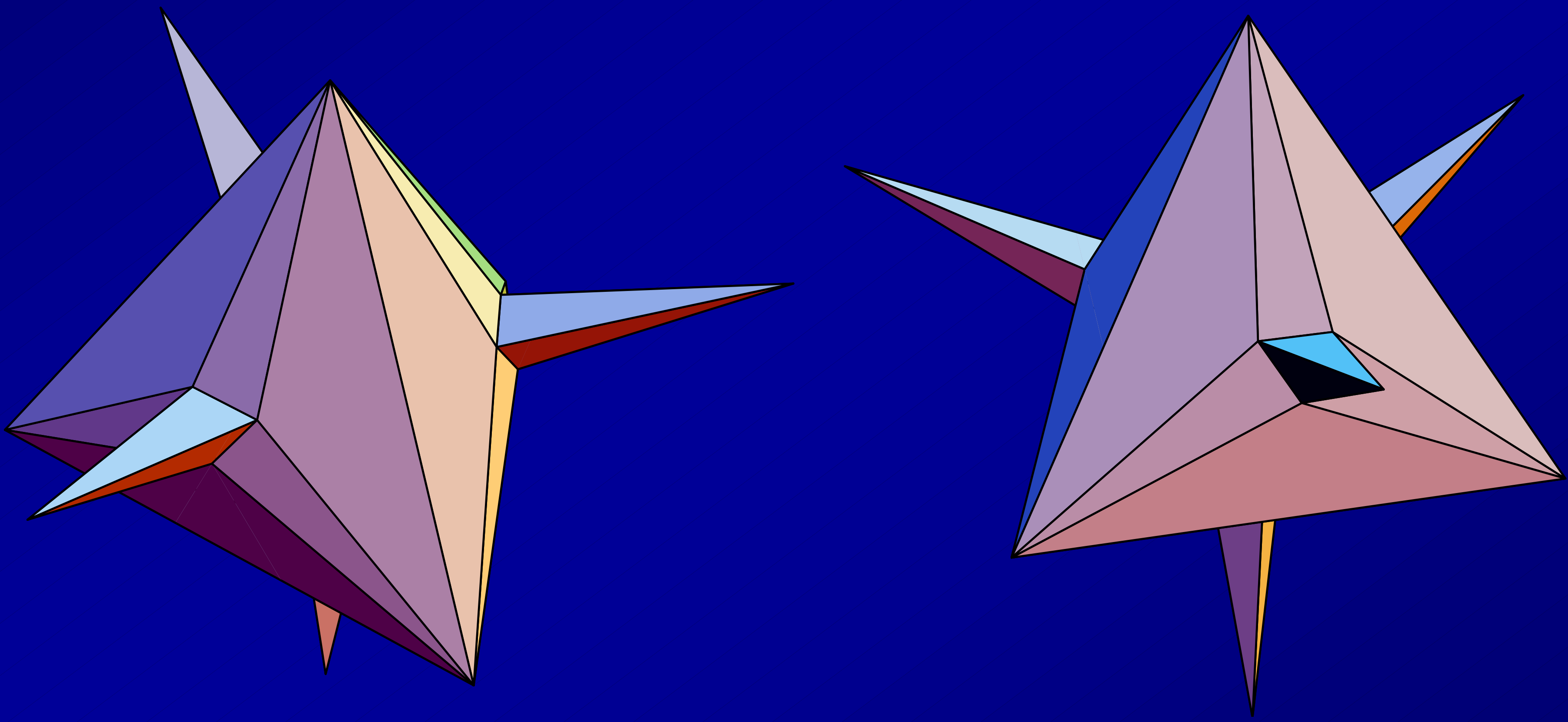


Spiked Tetrahedron

- Place a hat on each face of a regular tetrahedron



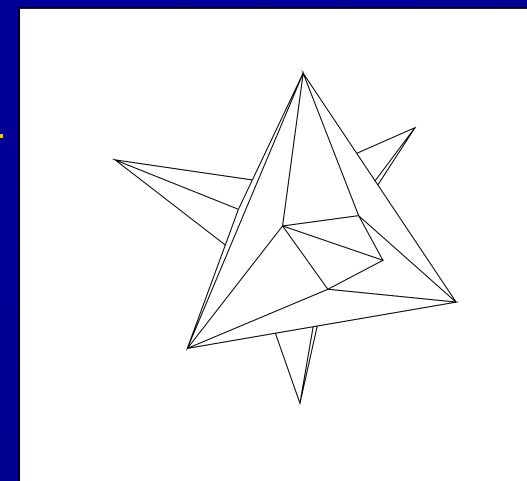
Spiked Tetrahedron

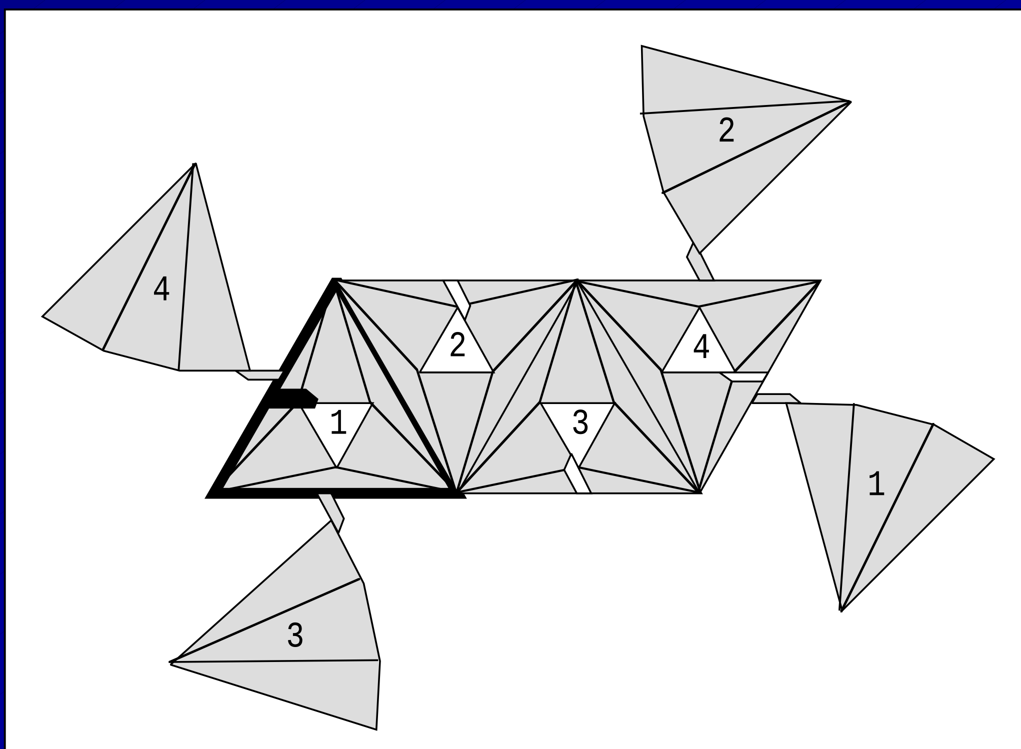


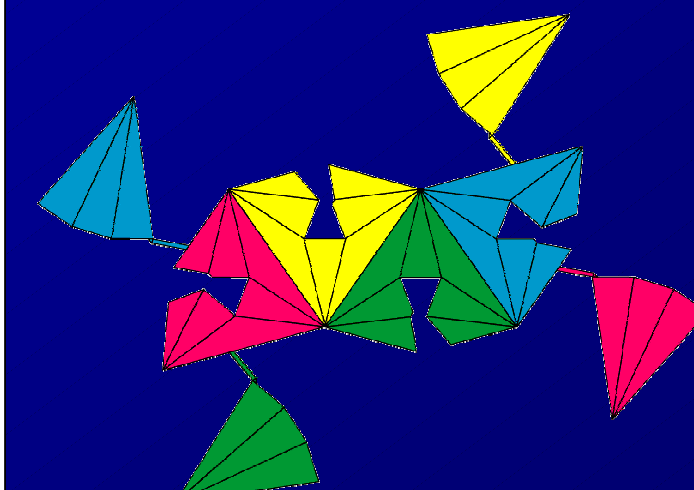
JavaView

Unfoldability of Spiked Tetrahedron (BDEKMS '99)

- **Theorem:** Spiked tetrahedron is edge-ununfoldable







Outline$_2$

- Geodesics & Closed Geodesics
  - Lyusternick-Schnirelmann Theorem
  - Gage-Hamilton-Grayson Curve Shortening
  - Exponential Number of Closed Geodesics

Geodesics & Closed Geodesics

- Geodesic: locally shortest path; straightest lines on surface
- Simple geodesic: non-self-intersecting
- Simple, closed geodesic:
  - Closed geodesic: returns to start w/o corner
  - Geodesic loop: returns to start at corner

(closed geodesic = simple, closed geodesic)

Lyusternick-Schnirelmann Theorem

Theorem: Every closed surface homeomorphic to a sphere has at least three, distinct closed geodesics.

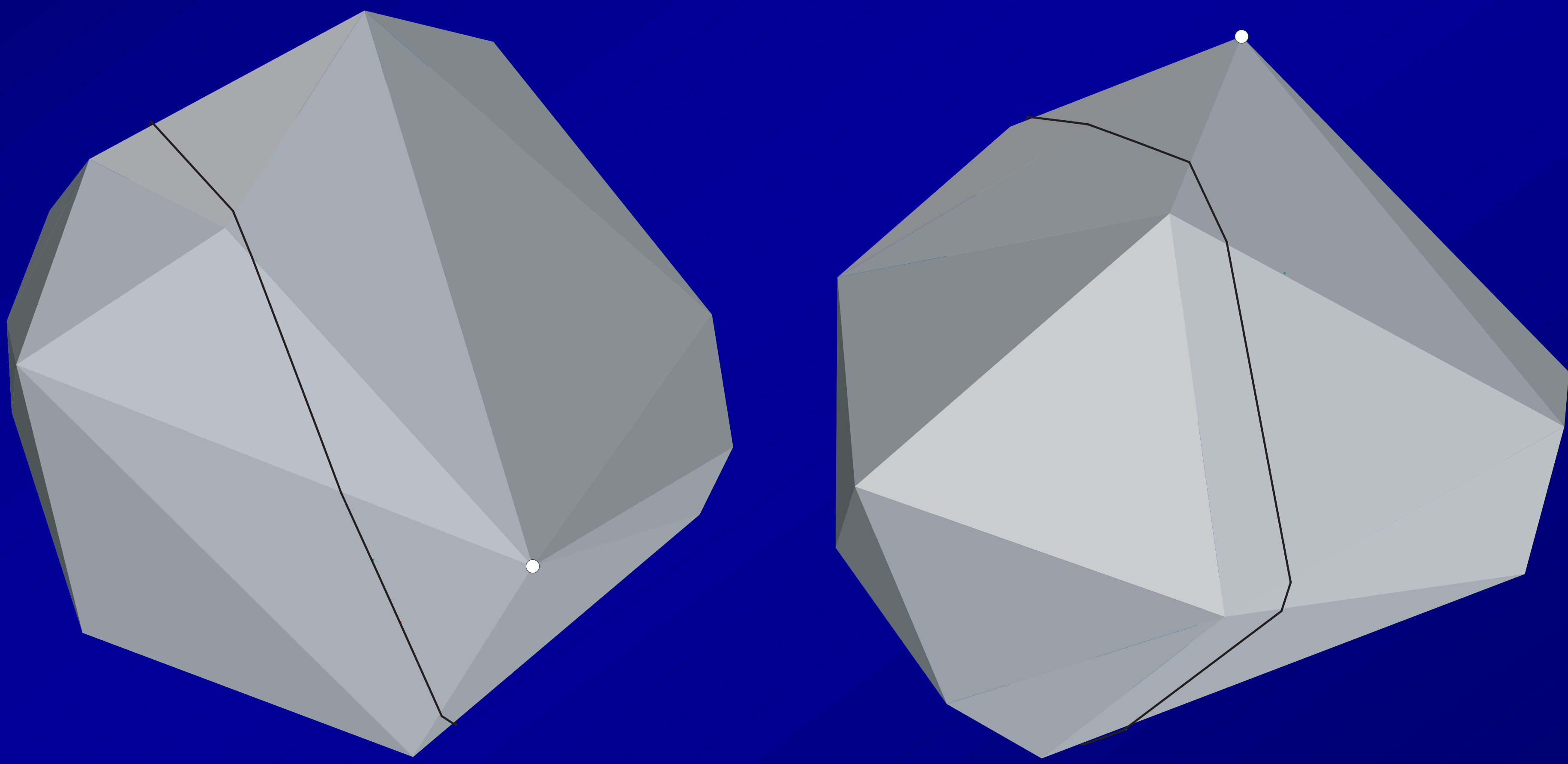
- Birkoff 1927: at least one closed geodesic
- LS 1929: at least three
- "gaps" filled in 1978 [BTZ83]
- Pogorelov 1949: extended to polyhedral surfaces

Quasigeodesic

- Aleksandrov 1948
- Define left(p) and right(p) turn angle at point p on curve
- left(p) = π – total incident face angle from left
- quasigeodesic: curve s.t.
  - left(p) ≥ 0
  - right(p) ≥ 0

  at each point p of curve.

Closed Quasigeodesic

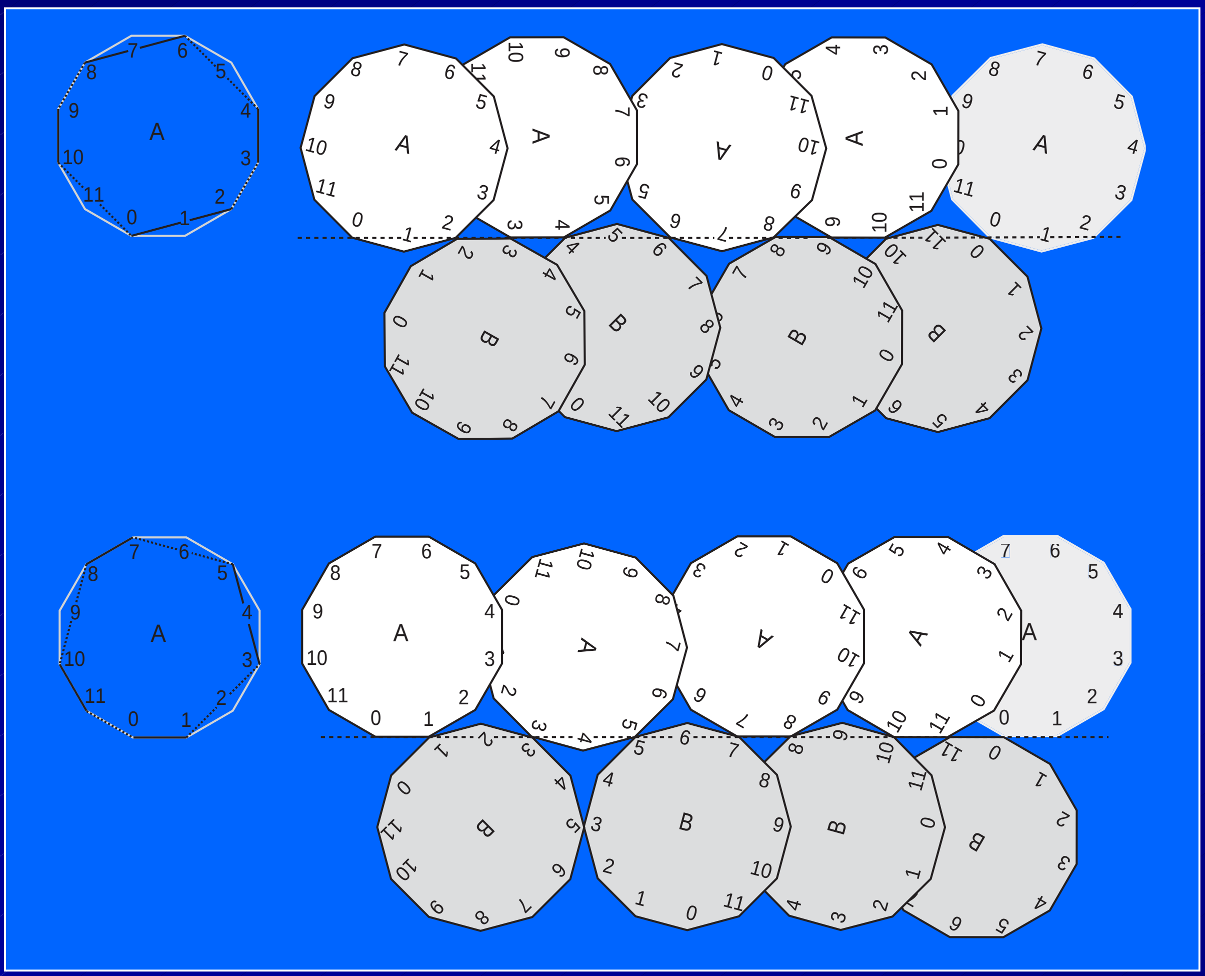


Open: Find a Closed Quasigeodesic

Is there an algorithm
- polynomial time
- or efficient numerical algorithm

for finding a closed quasigeodesic on a (convex) polyhedron?

Exponential Number of Closed Geodesics

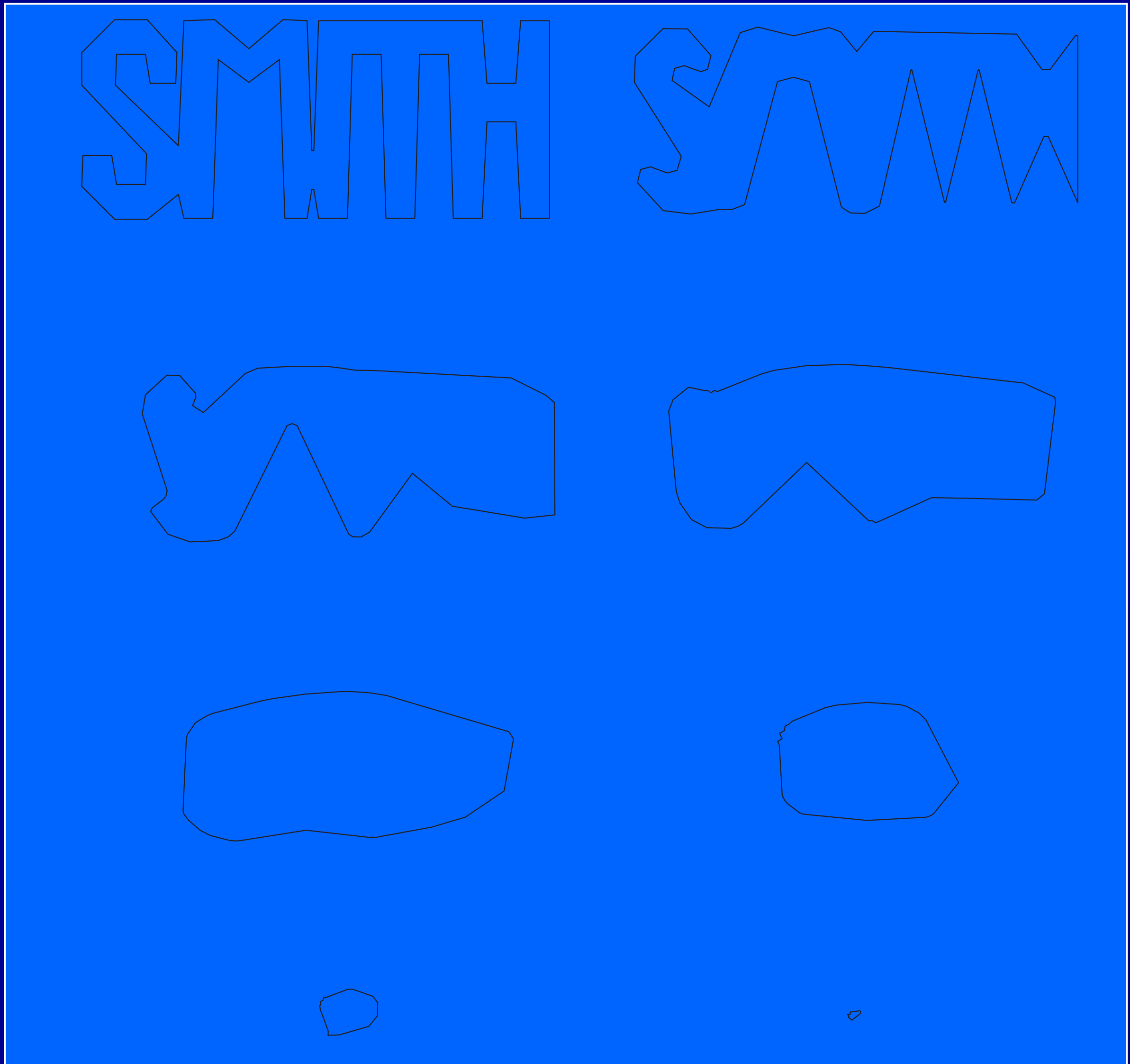


Theorem: $2^{\Omega(n)}$ distinct closed quasigeodesics.

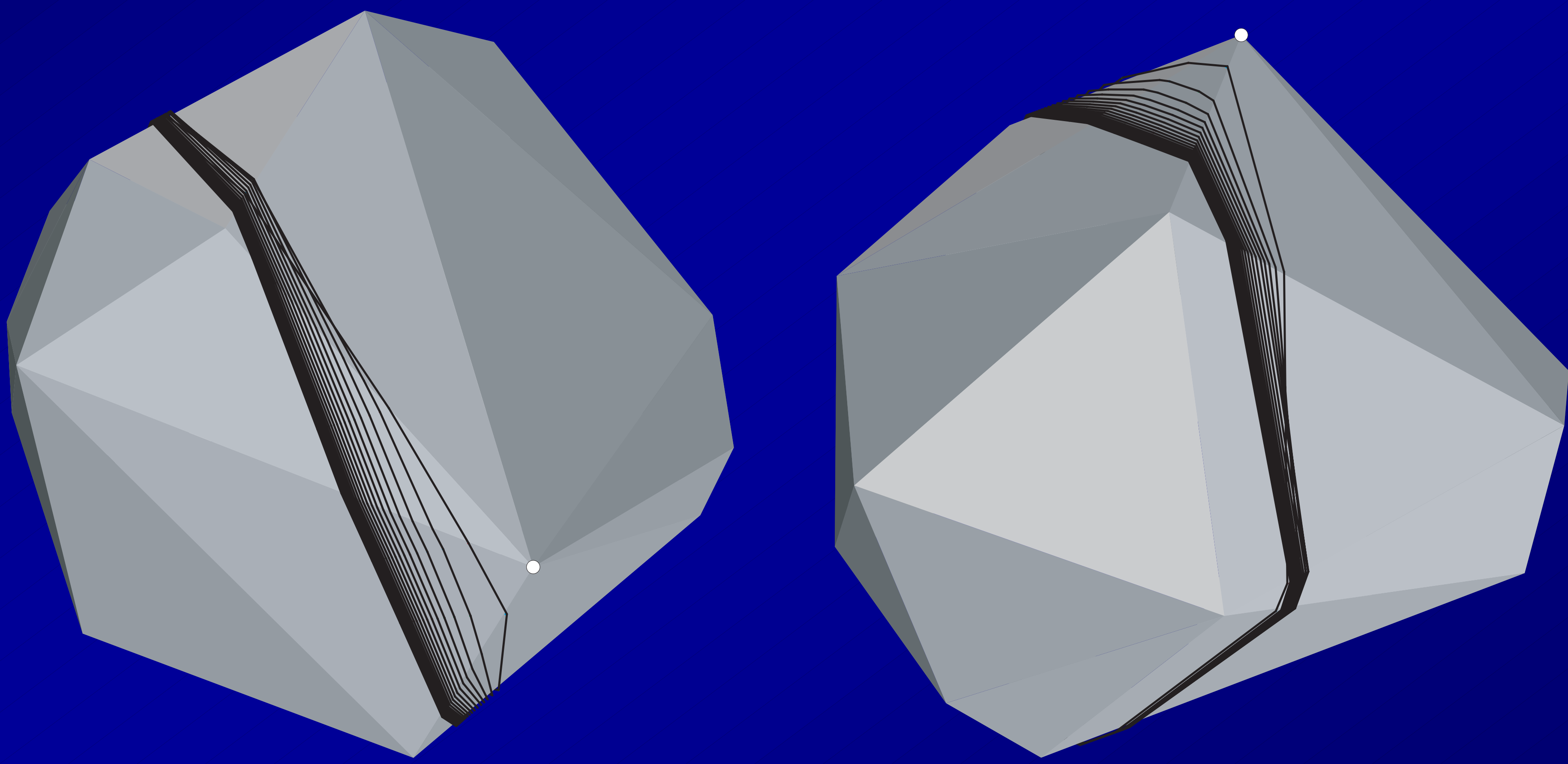
Aronov & O'Rourke 2002

Gage & Hamilton Curve Shortening

Each point p evolves along normal to curve, at speed proportional to curvature at p.

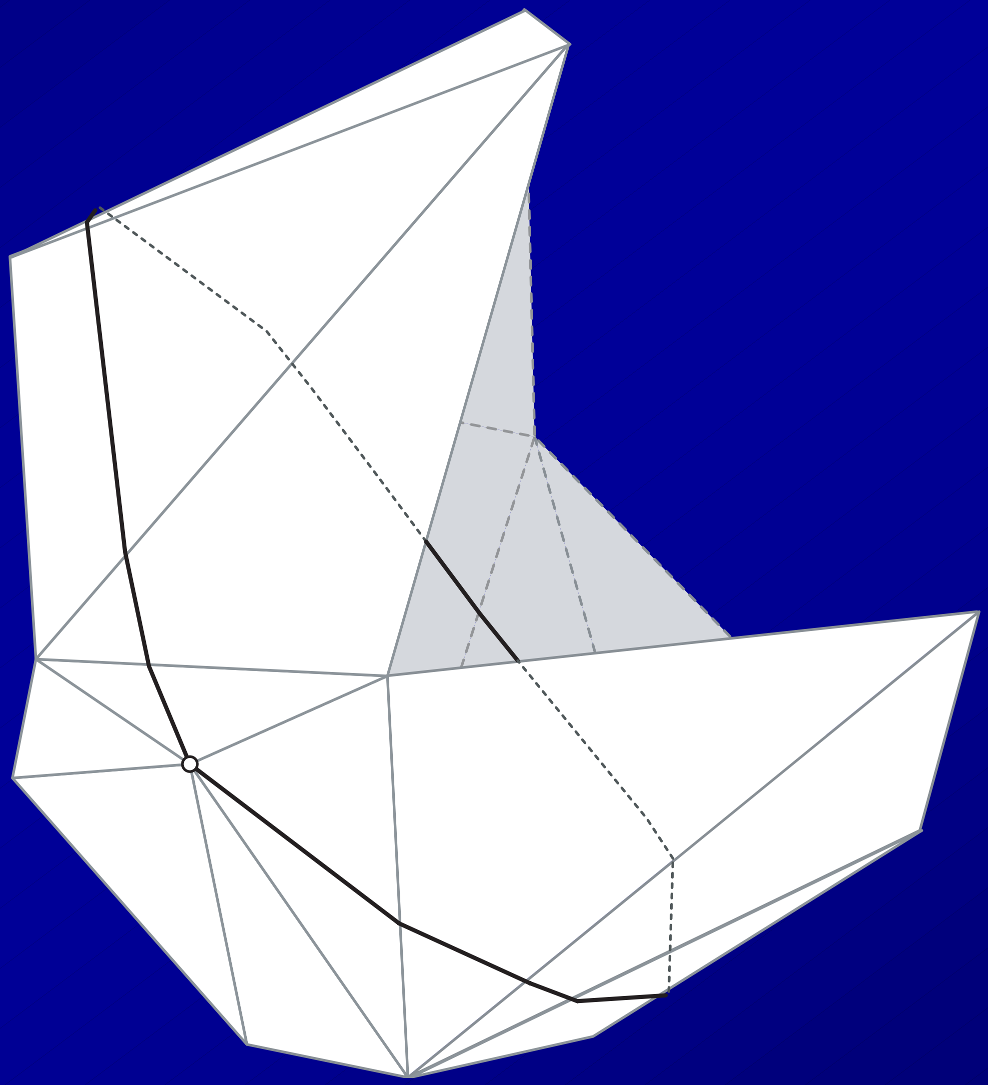


Grayson Curve Shortening

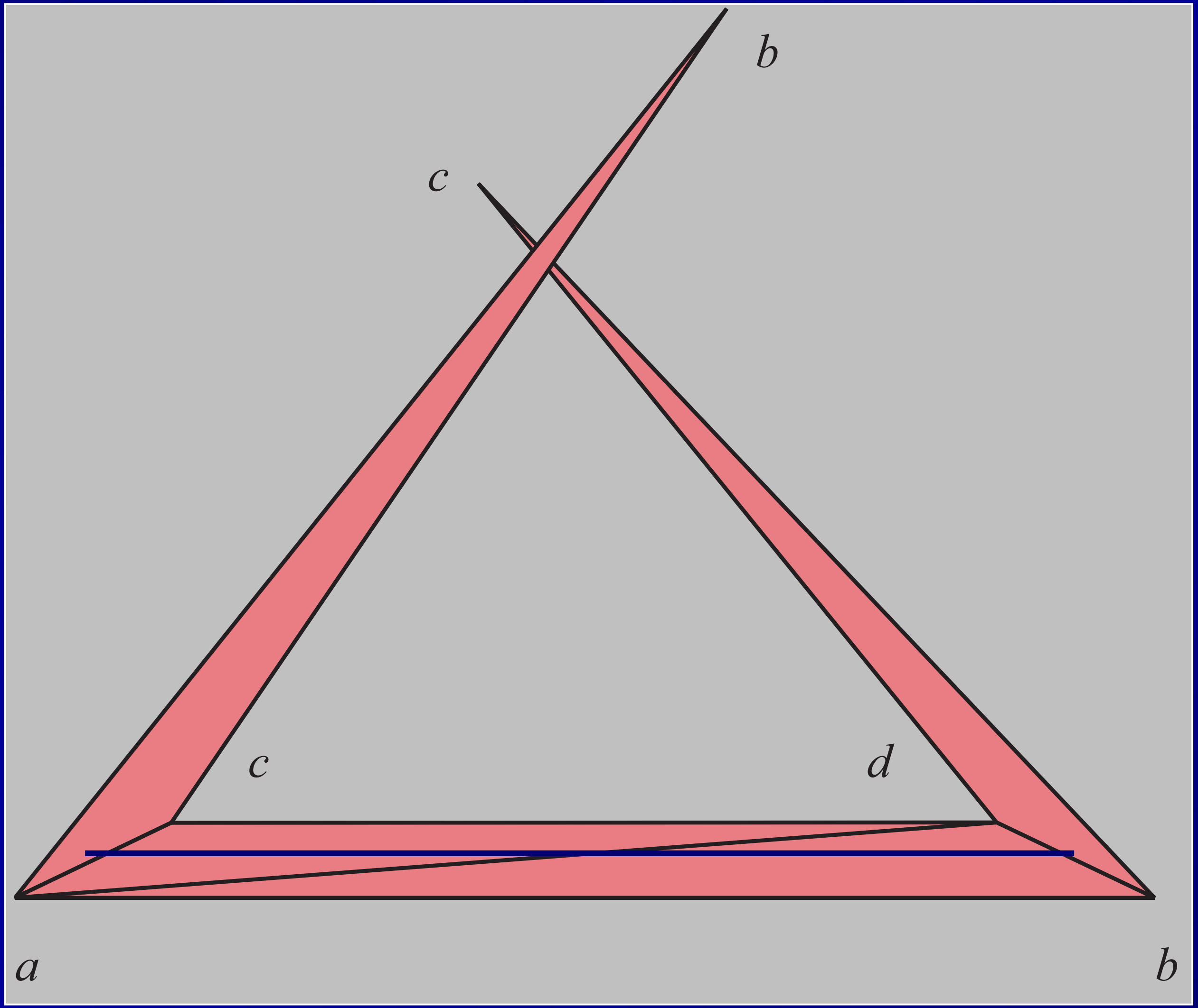


Lysyanskaya, O'Rourke 1996

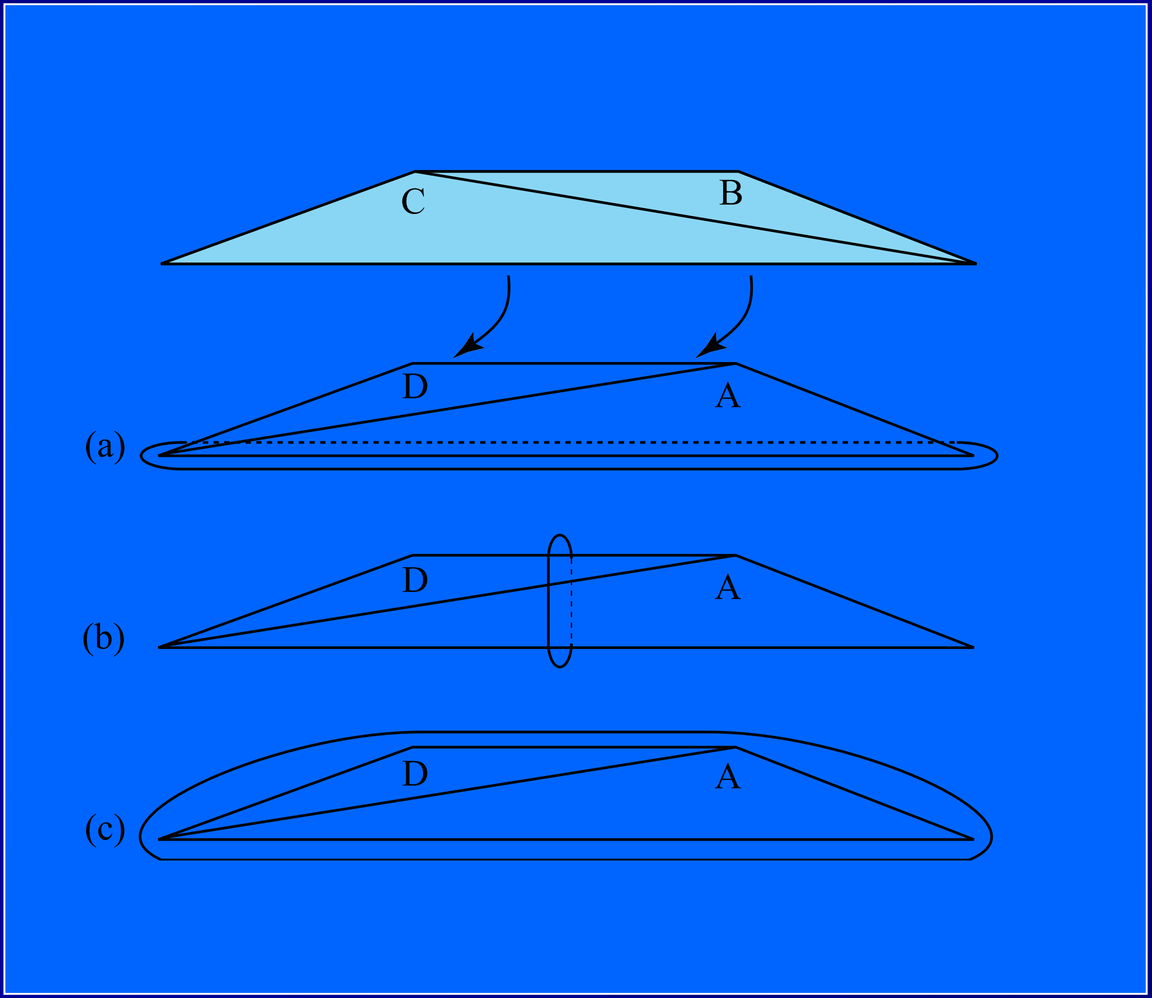
Faces Crossed by Closed Geodesic



Geodesic Overlap



Quasigeodesics



Open: Nonoverlapping Faces crossed by Closed Quasigeodesic

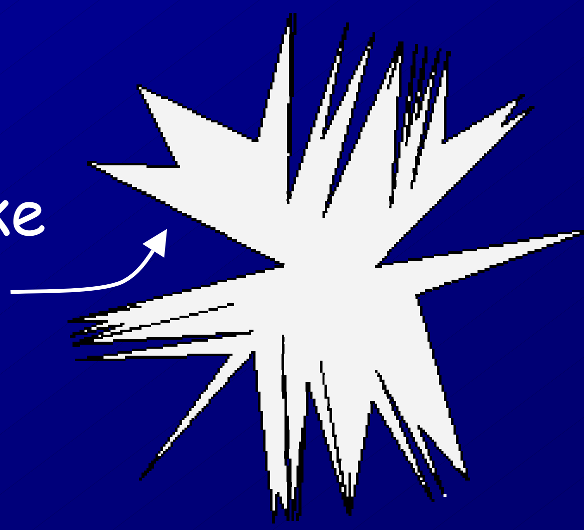
For a given closed quasigeodesic γ, is it true that the set of faces whose interior is touched by γ unfold along γ without overlap?

Outline$_3$

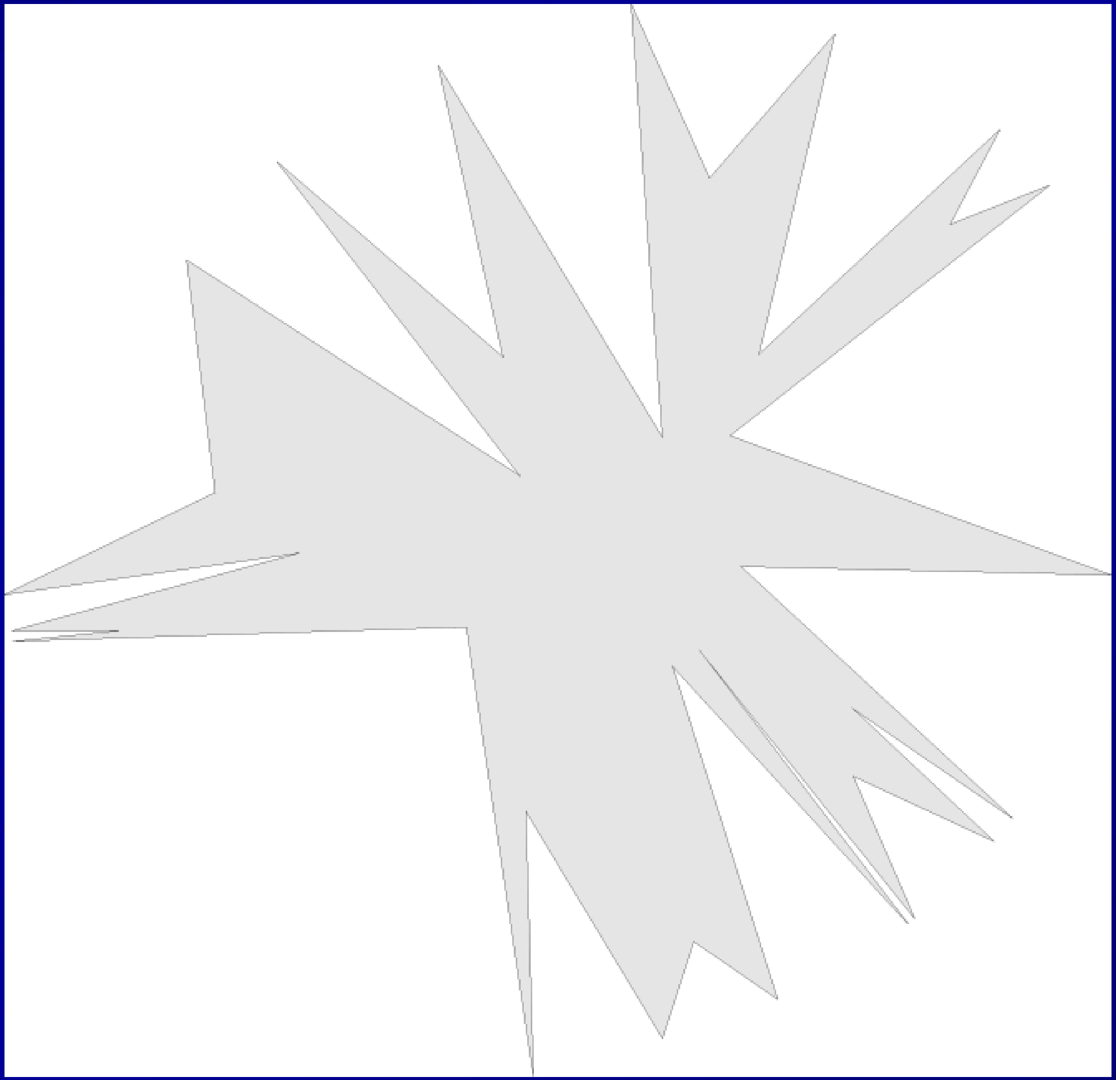
- Unrestricted Unfoldings
  - Vertex Unfolding
  - Orthogonal Polyhedra
  - Open: Nonoverlapping Unfolding for Nonconvex Polyhedra

General Unfoldings of Convex Polyhedra

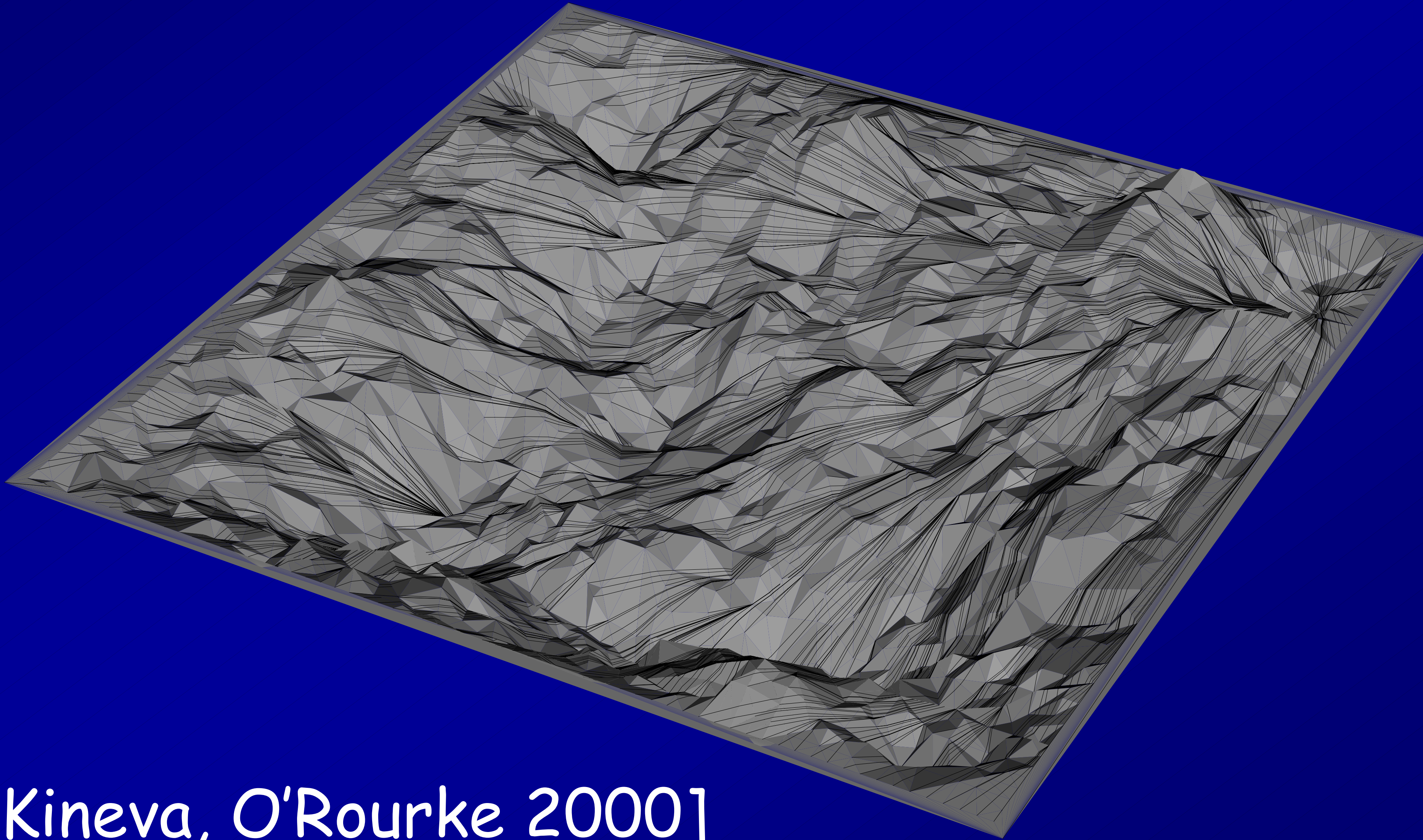
- **Theorem**: Every convex polyhedron has a *general* nonoverlapping unfolding (a net).
  - Source unfolding (Sharir & Schorr '86, Mitchell, Mount, Papadimitrou '87)
  - Star unfolding (Aronov & O'Rourke '92)



Star-unfolding of 30-vertex convex polyhedron

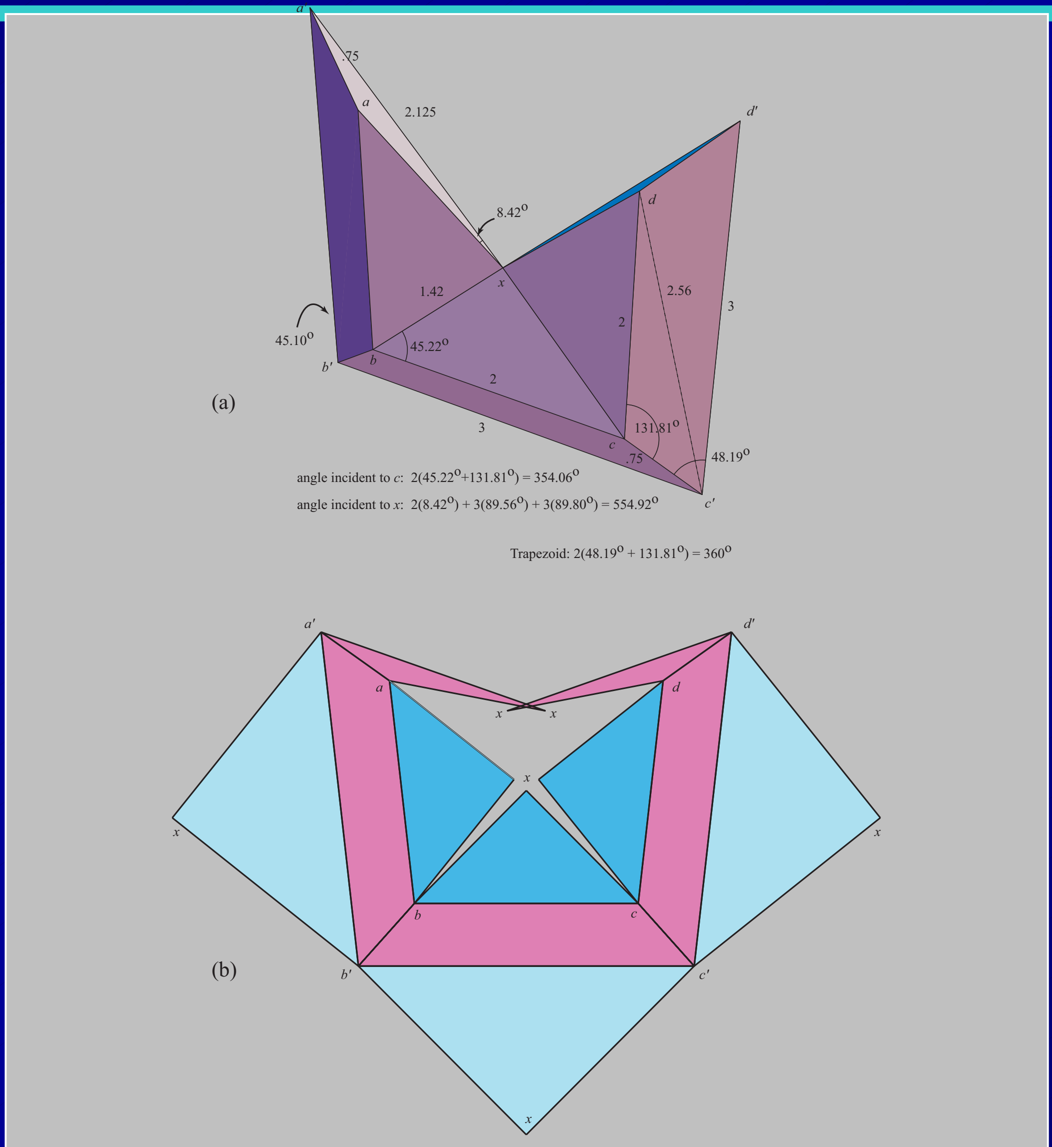


Overlapping Source Unfolding



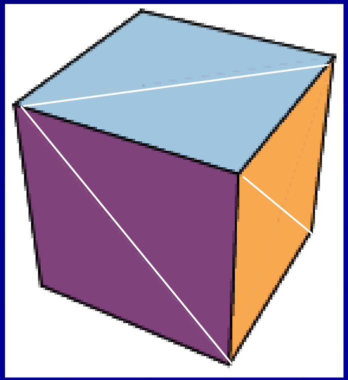
[Kineva, O'Rourke 2000]

Overlapping Star-Unfolding

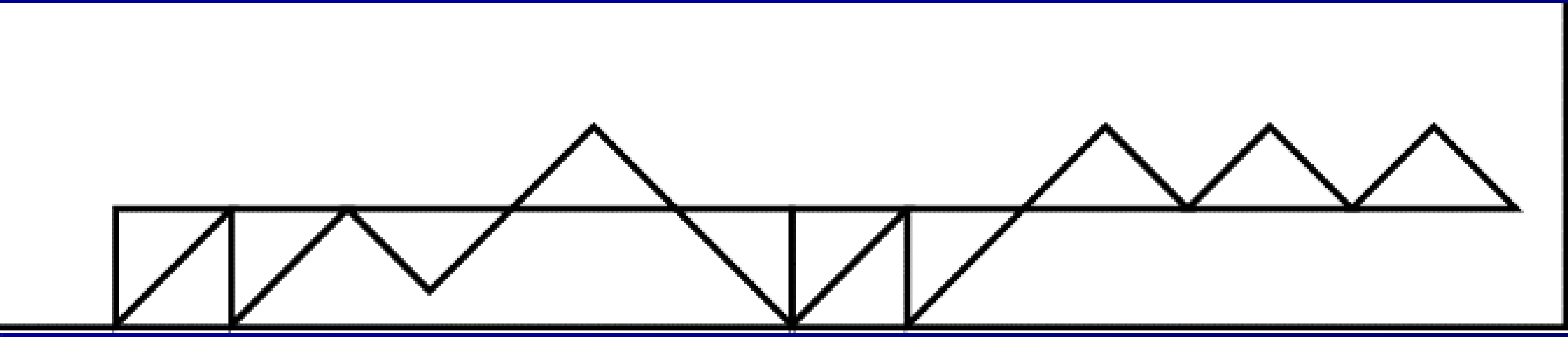


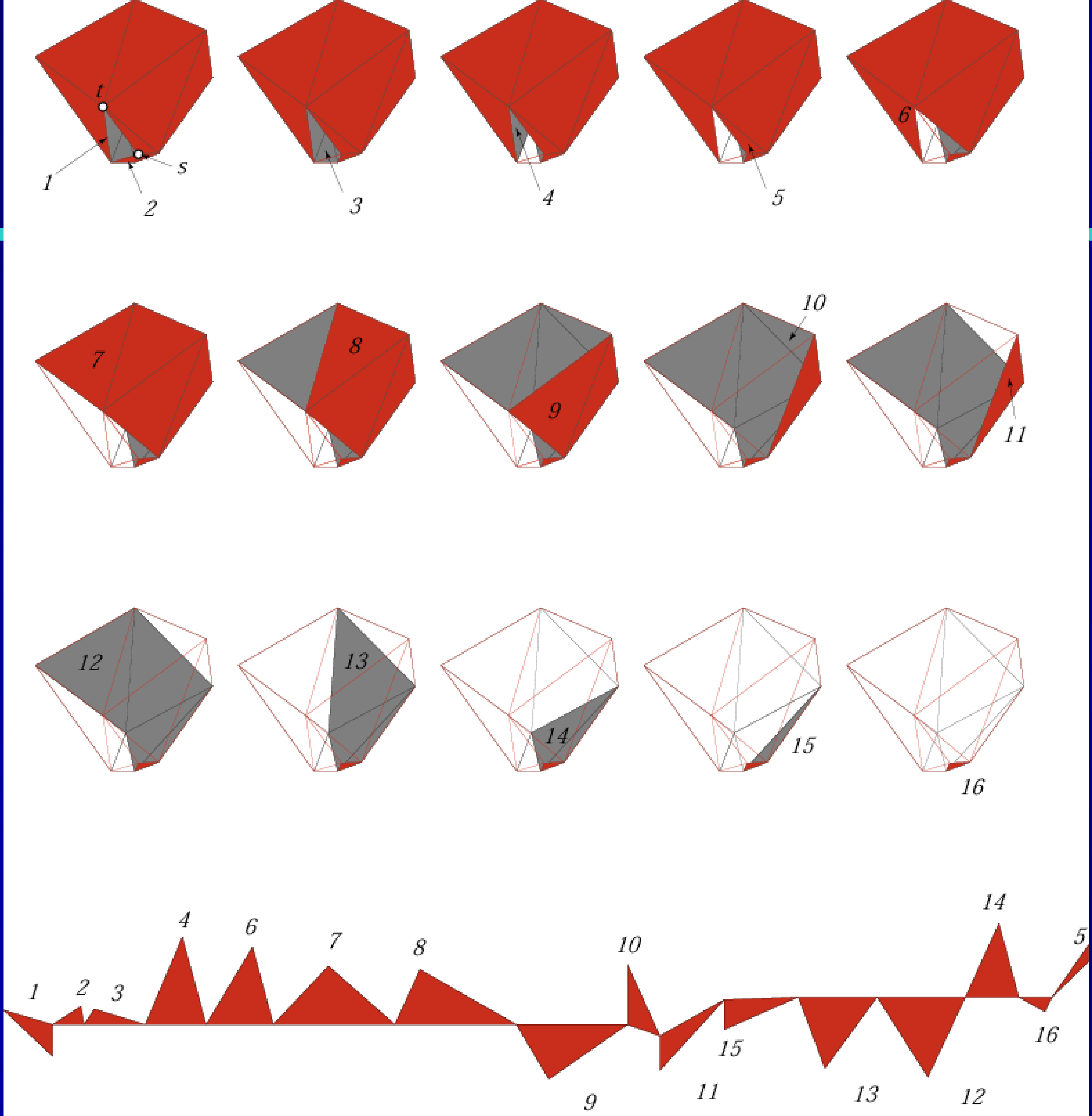
Vertex Unfolding
(Eppstein, Erickson, Hart, O'Rourke 2002)

Cube:



Vertex unfolding:

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Algorithm Overview

2-Manifold →

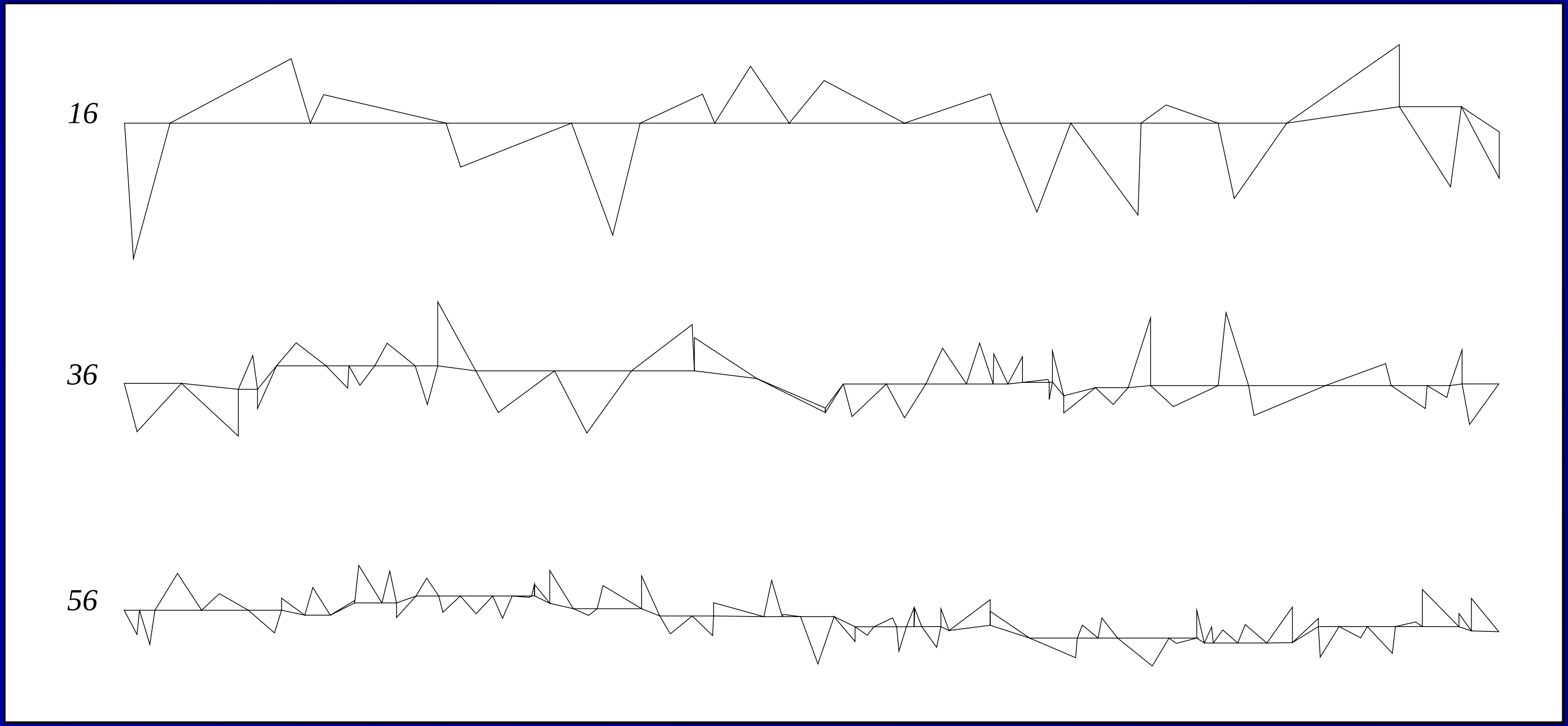
Facet-Path →

Strip Layout of Triangles :

Vertex-Unfolding

Vertex Unfolding
(Eppstein, Erickson, Hart, O'Rourke 2002)

Theorem: Every triangulated manifold has a **vertex-unfolding.**



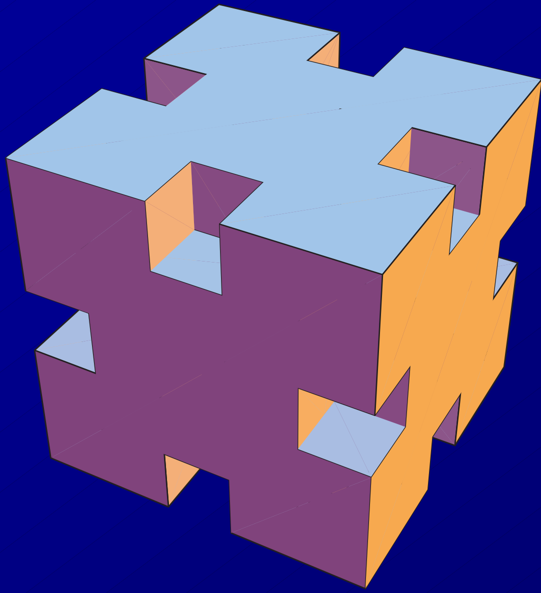
Open: Vertex-Unfolding

Does every convex polyhedron have a nonoverlapping vertex-unfolding?

Does every (perhaps nonconvex) polyhedron have a nonoverlapping vertex-unfolding?

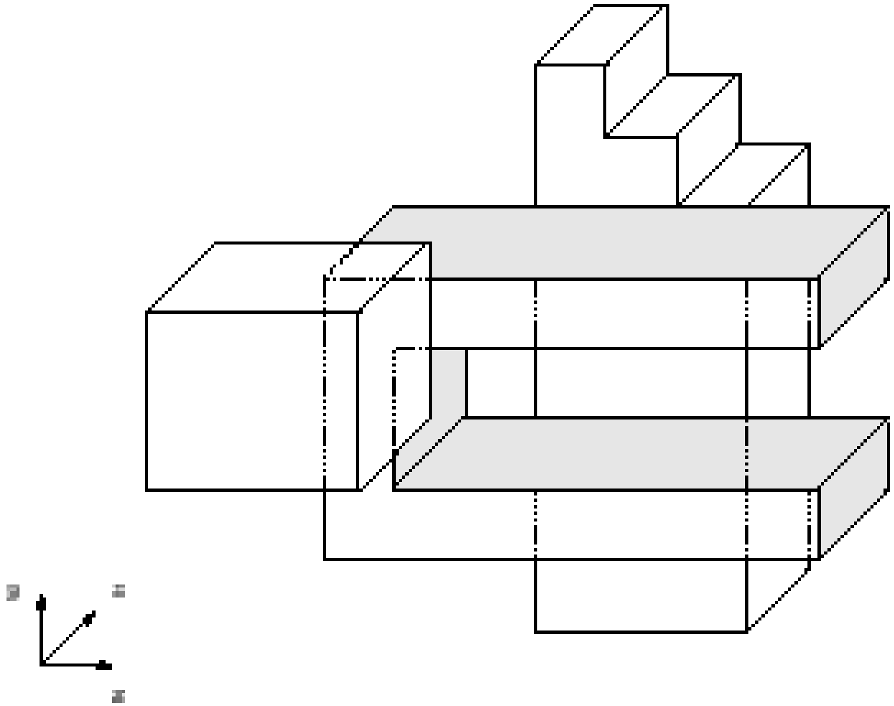
Orthogonal Polyhedra

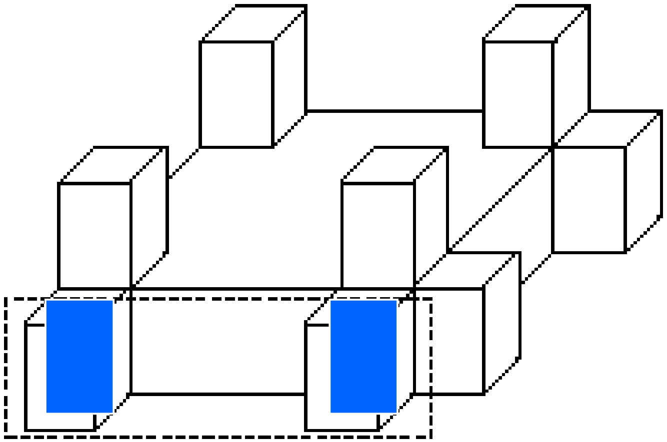
- Orthostack: stacking of extrusions of orthogonal polygons
- Orthotube: orthogonal "corkscrews" with rectangular cross-sections.



Biedl, Demaine, Demaine, Lubiw, O'Rourke, Overmars, Robbins, Whitesides CCCG98

Orthostacks

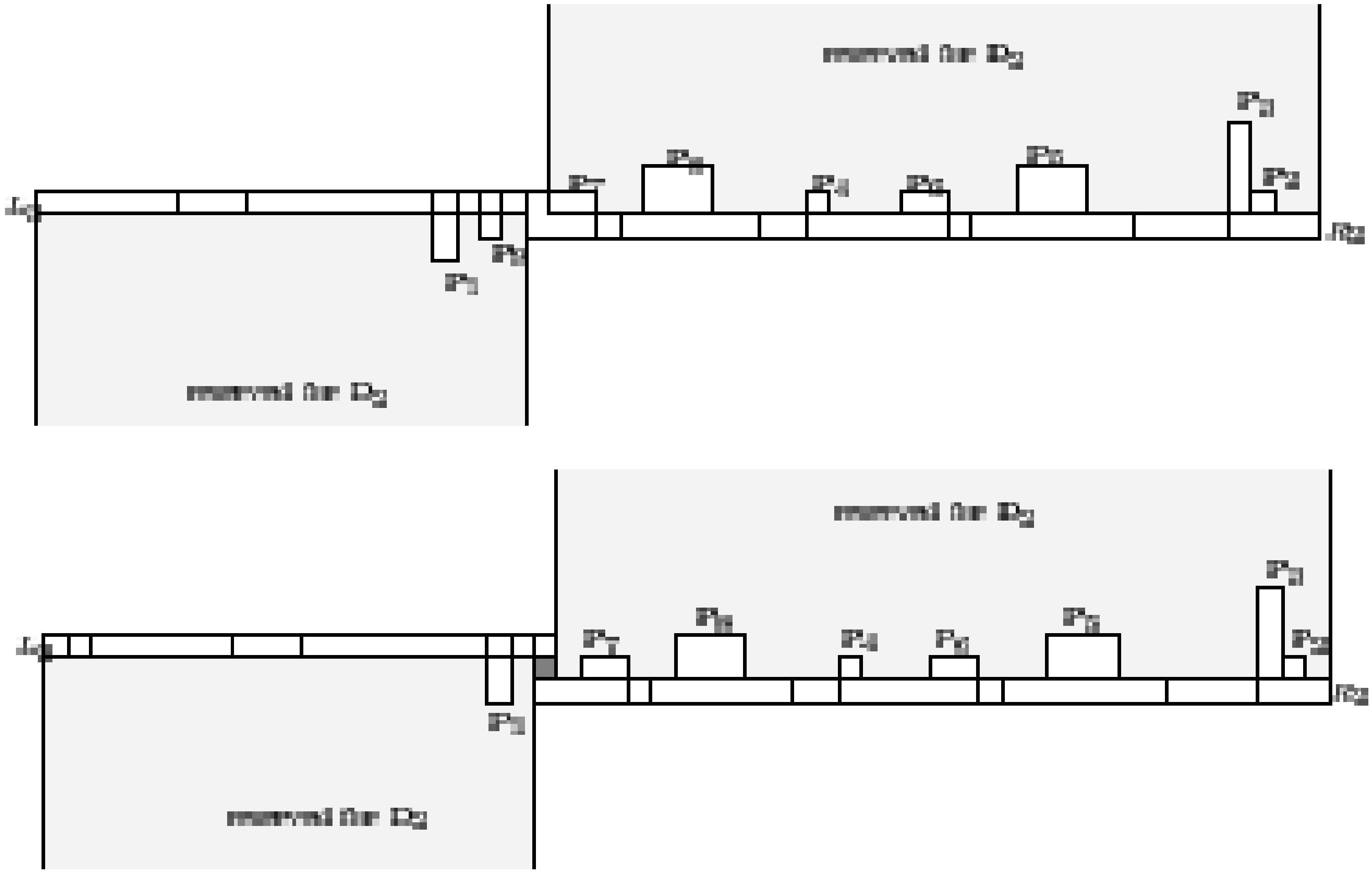




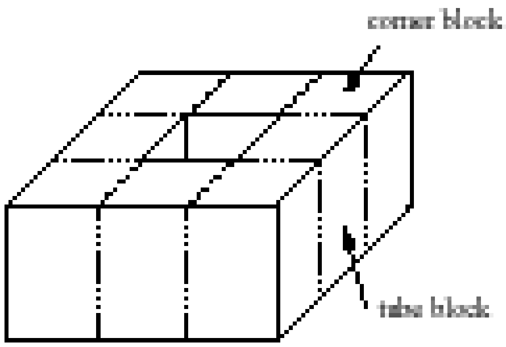
Orthostack w.r.t. z

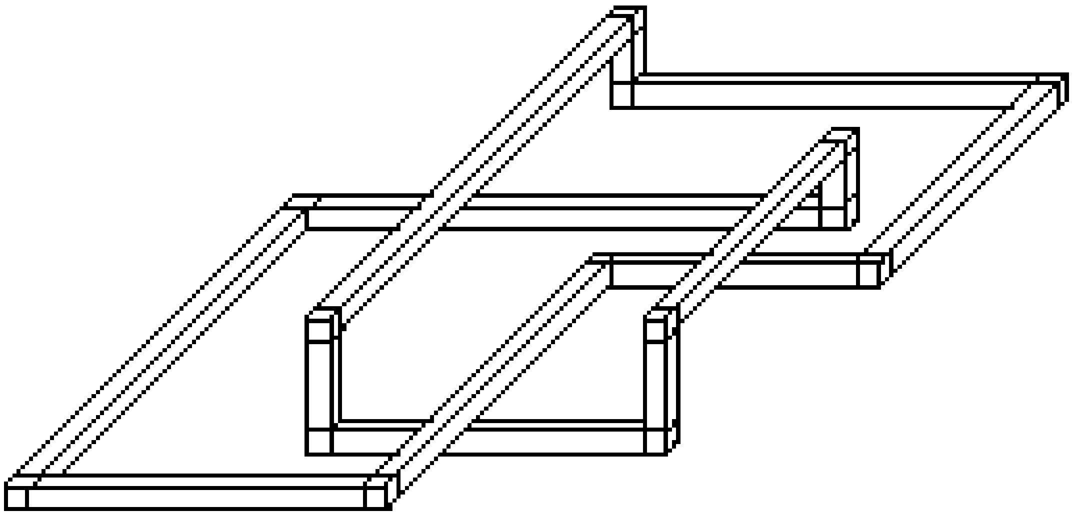
Non-Orthostack

Unfolding of Orthostack

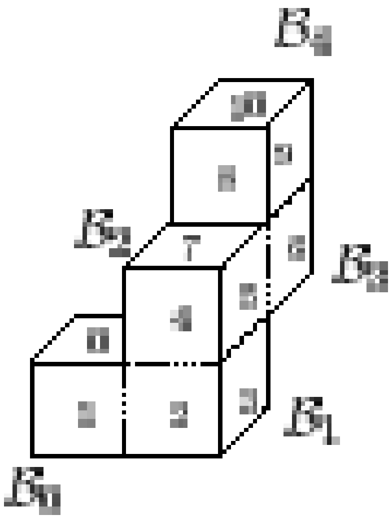


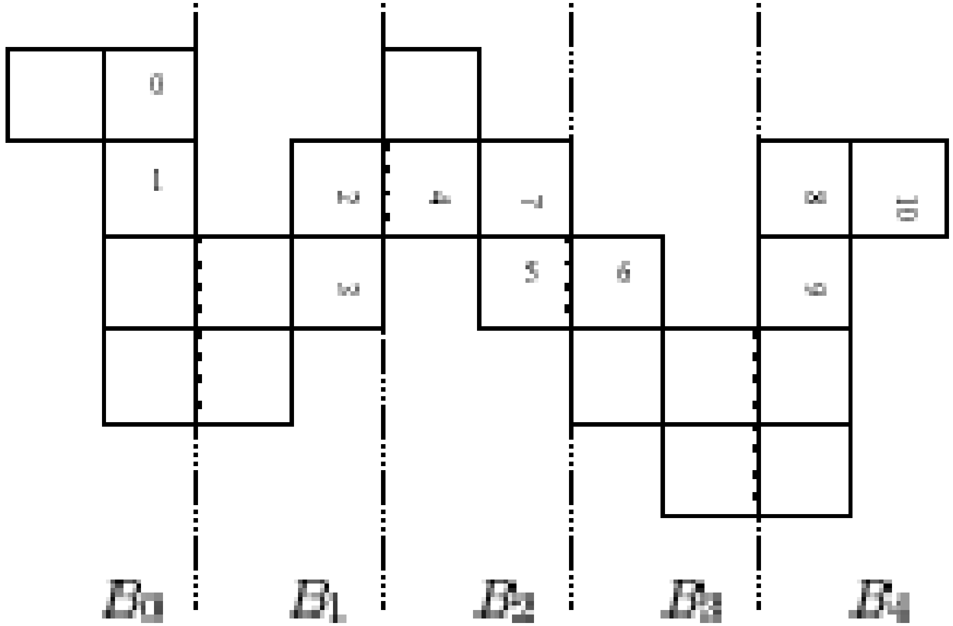
Orthotubes





Orthotube Unfolding





Open: A Net for Every Polyhedron?

Can every polyhedron (without boundary) be cut along its surface and unfolded into one piece in the plane without overlap?

Fewest Nets version: What is the fewest number of non-self-overlapping pieces into which a polyhedron may be cut, say, as a function of the number of negative-curvature vertices?