A d-Polytope - convex hull of finite set of points in $\mathbb{R}^d$ (which aff. span $\mathbb{R}^d$)

Proper face of a polytope P is the intersection of P with a supp. hyperplane.

$\emptyset, P$ - trivial faces.

Combinatorial Theory of convex polytopes is understanding face-structure* of polytopes*

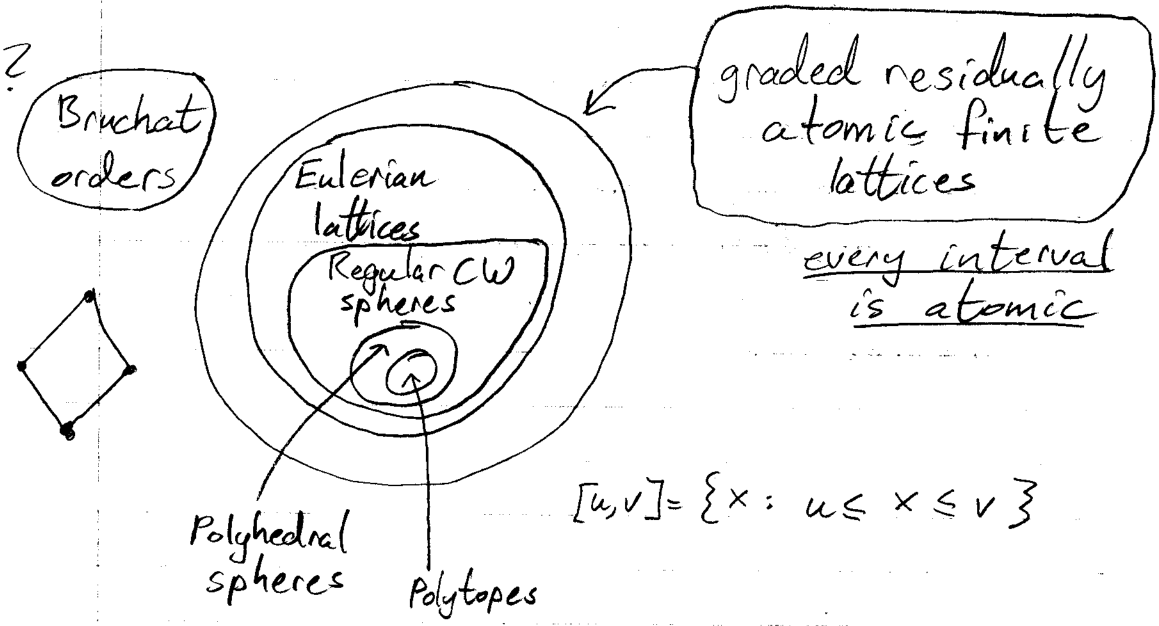
Extreme points of P, 0-faces, are called vertices.
1-faces called edges
$(d-1)$-faces called facets

Faces of polytopes are polytopes.

$G(P)$ - graph of P.

set of faces of P
finite, poset, graded, lattice
atomic

every interval in $L(P)$ is also face-lattice polytope
. . . . .
$f_k(P) = \#$ k-faces of P $\quad (f_{-1}, f_0, f_1, \ldots, f_d)^{\text{f-vec}}$



$[u,v] = \{x : u \leq x \leq v\}$

Simplicial Polytopes: all proper faces are simplices
Simple d-polytopes: every vertex belong to d-edges

1) Euler's relation

$$V - E + F = 2$$

Problem (Barany) P d-polytope
is it true $f_i \geq \min(f_0, f_{d-1})$

→ Linear Programming.
→ trying to generalize from $d=3$
→ relations w. combinatorics, geometry, recently alg. geom. comm. alg.
→ cheer imagination

1) Connectivity

2) Missing faces and skeletons of polytopes with few vertices

3) Reconstructions

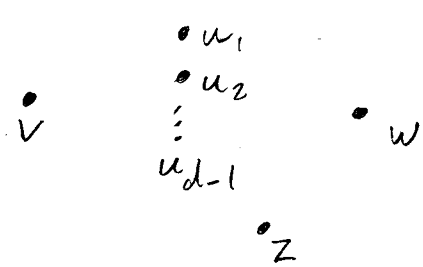
4) Diameter of graphs of polytopes

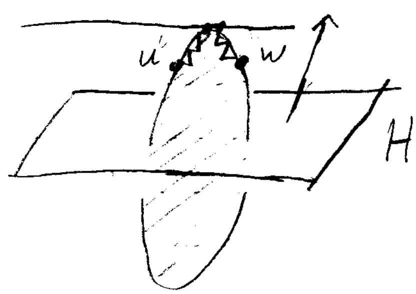
5) low dim faces of big dim polytopes

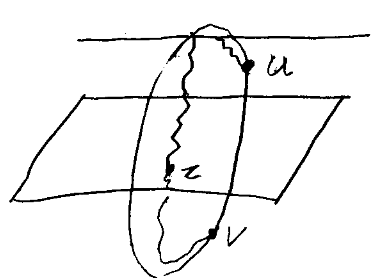
Connectivity

THM (Balinsky) Graphs of d-polytopes are d-connected

Proof:







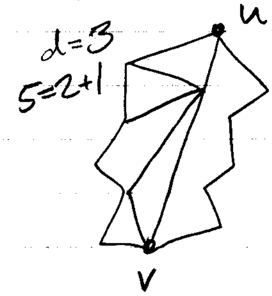
Diversion

<u>Problem 2</u> How many graph (isom types) of simple (general) d-polytopes are there w n vertices

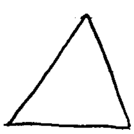
is it only exponential

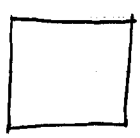
Conjecture (Luckeberg)
$d = d_1 + d_2 + \cdots + d_r$
v, u vertices
there are r distinct paths between u and v s.t. the j-th path is of type $(d_j, d_j - 1)$

a (k, l) path between u and v is a path of k-faces $F_1, F_2, \ldots, F_t$ such that $\dim(F_i \cap F_{i+1}) = l$, $v \in F_1$, $u \in F_t$



<u>Theorem</u> (Whiteley, $d=3$ Alexandrov)
~~$d \geq 3$~~ Graphs of d-polytopes in $\mathbb{R}^d$ with 2-faces triangulated are infinitesimally rigid. (Hence geometrically d-rigid)



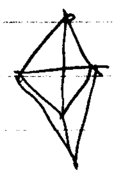


THM:

Let $f(r, d, \beta)$ be the number of combin. types of r-dimensional skeletons of d-polytopes with $d+\beta$ vertices. Then, for r and β fixed, $f(r, d, \beta)$ is bounded.

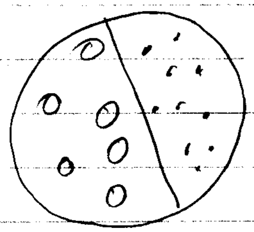
For simplicity, we consider simplicial polytopes

A missing face F is a set of vertices such that F is not a face and every proper subset is a face



Thm

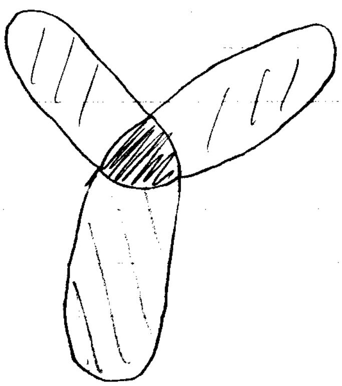
The number of missing faces of dim $\leq r$ in a simplicial d-polytope with $d+\beta$ vertices is bounded by a fn $\Psi(r, \beta)$



Lemma a d-polytope w $d+\beta$ vertices has at most $\beta-1$ <u>disjoint</u> missing faces

Proof: Ex.

Erdos-Raden Sunflower theorem. if we have large enough ($\geq O(r,\beta)$) collection of sets of size $\leq r$ then we must have sunflower of size β



$A_1, \dots, A_\beta$

$A_i \cap A_j = A_1 \cap \cdots \cap A_\beta$