Reconstruction of Convex Bodies from Brightness Functions*

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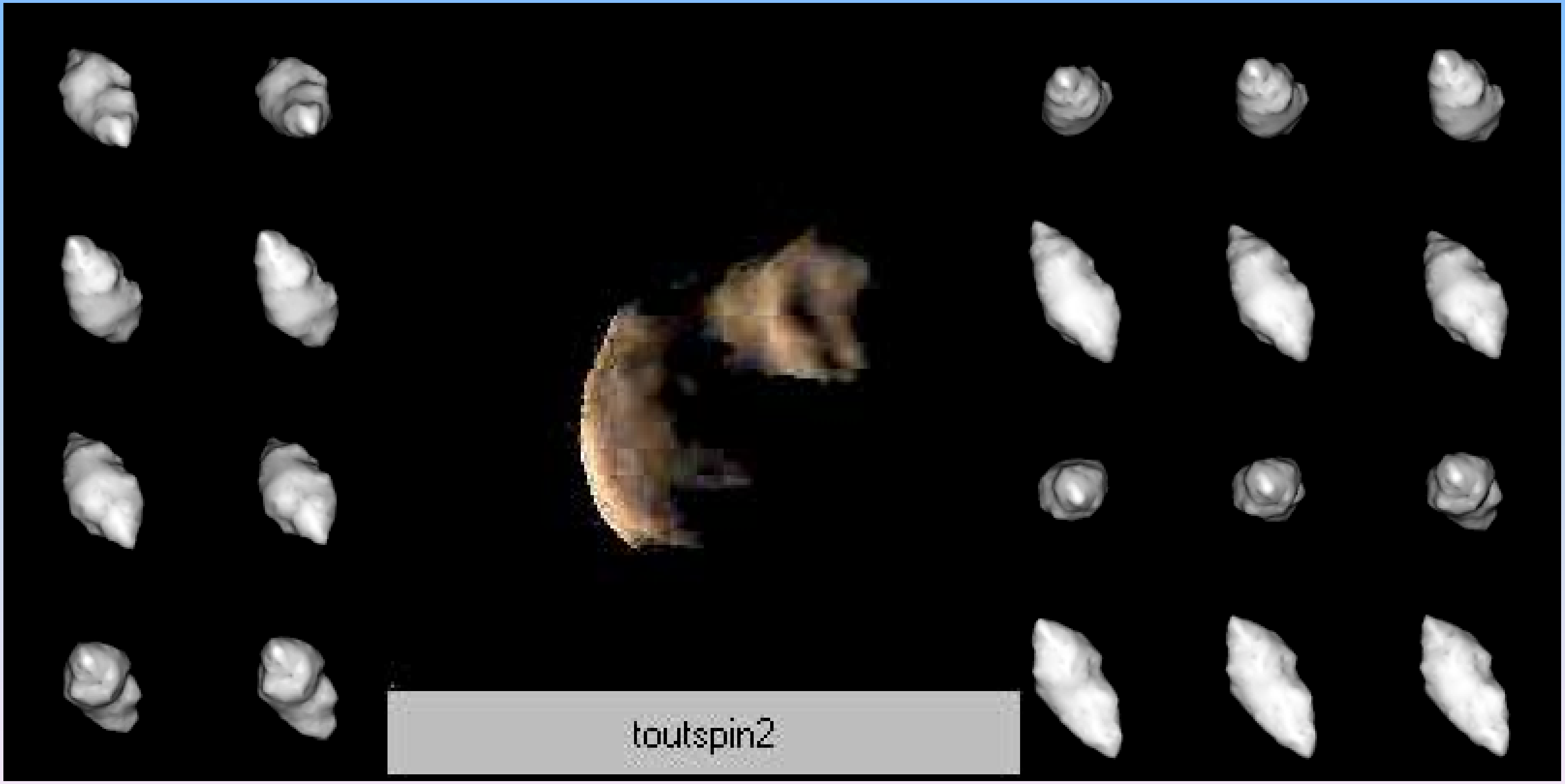
* *Disc. Comput. Geom.* **29** (2003), 279-303.

A Motivating Application: Asteroid Imaging

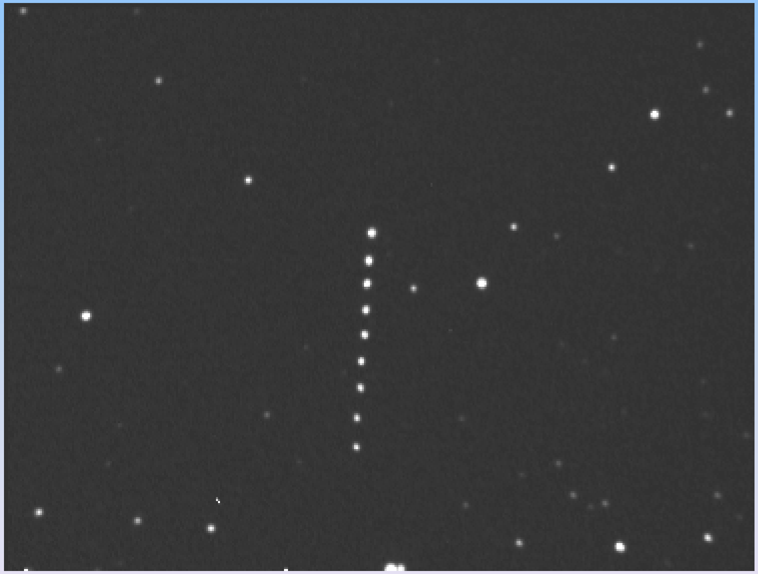
- An asteroid has an irregular shape and composition, and as it rotates, it reflects different amounts of light.



Asteroid Imaging: Lightcurves

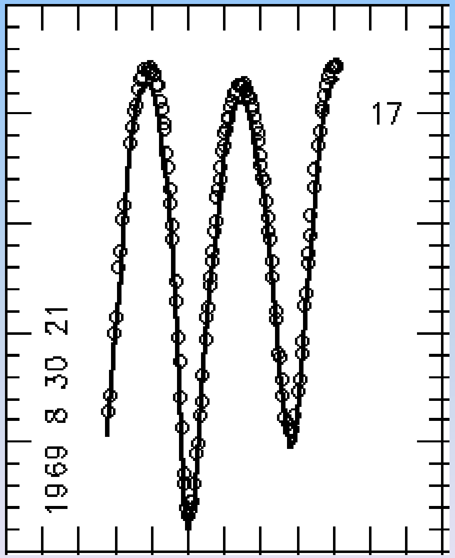
- A sufficiently far away object in relative motion is imaged with camera as a point of light with time-varying brightness.

The Asteroid Geographos



Long-exposure image

The “lightcurve”



Brightness

Time/angle

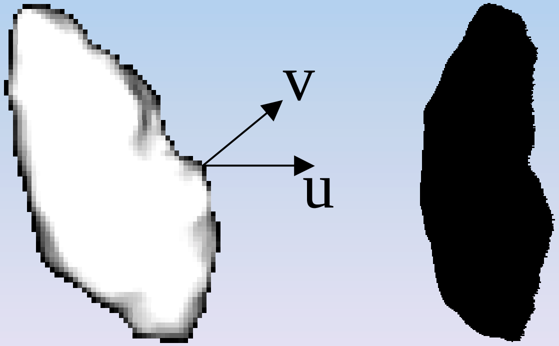
Shadows

- We can measure the total amount of light reflected from a (matte) surface at a particular position.
- In the simplest approximation, the total brightness is proportional to the area of the shadow.

$$b_K(u) = \frac{1}{2}\int |u \cdot v|\, dS(K,v)$$

$S(K,\cdot)$: Surface area measure

u : Viewing direction



Shape Reconstruction Problem

- Inverse Problem: Reconstruct (a convex approximation) to the shape from noisy measurements of brightness

Surface area measure or
EGI=extended Gaussian image

$$b_K(u_k) = \frac{1}{2}\int |u_k \cdot v|\, dS(K,v)$$

Cauchy's projection formula

$$= V(K|u_k{}^{\perp})$$

- Required Steps:
  - Construct surface area measure from b_K
  - Construct K from its surface area measure

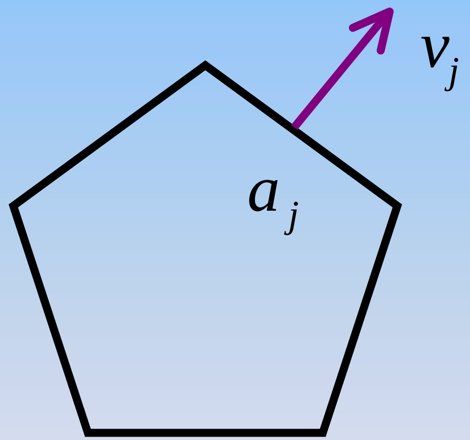
Polygons and Polyhedra

- For a convex polytope P with N facets:

$$S(P,\cdot) = \sum_{j=1}^{N} a_j \delta_{v_j}$$

Outer unit normals to facets

Areas of facets

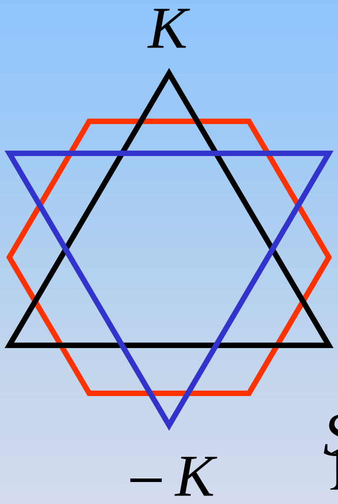


$$b_P(u_k) = \frac{1}{2} \sum_{j=1}^{N} a_j \left| u_k \cdot v_j \right|$$

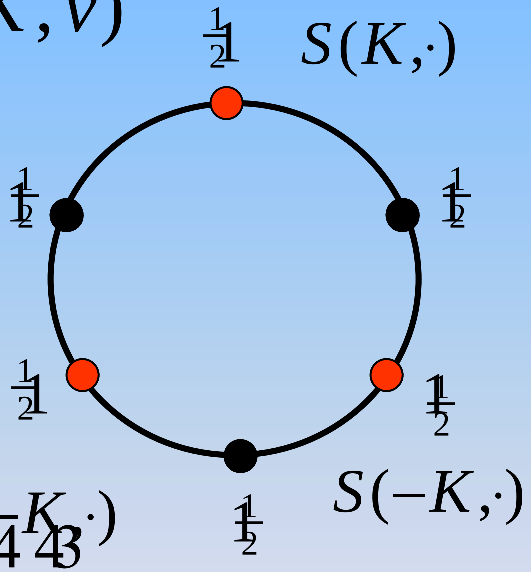
Non-uniqueness issues

- Data is so weak, we should expect non-uniqueness!
- Take a non-origin-symmetric convex body K with brightness

$$b_K(u) = \frac{1}{2}\int |u \cdot v| \, dS(K, v)$$



$0 \le t \le 1$



$$\underbrace{S(K_t,\cdot) = (1-t)S(K,\cdot) + tS(-K,\cdot)}$$

$$\Downarrow$$

$$b_{K_t} = b_K$$

Is there any hope?

- Aleksandrov's projection theorem:
  - For *origin-symmetric* convex bodies K and L

$$b_K(u) = b_L(u) \quad \forall u \;\Rightarrow\; K = L$$

- Aleksandrov's uniqueness theorem:
  - For *any* convex bodies K and L

$$S(K,\cdot) = S(L,\cdot) \;\Rightarrow\; K = L \qquad \text{(up to translation)}$$

- Plan of Action:

$$\begin{tikzcd}
\{b_K(u_k)\} \arrow[r, "\textbf{Step 1}", "\text{Origin symmetry}"'] & S(P,\cdot) \arrow[r, "\textbf{Step 2}", "\text{Convexity}"'] & P \\
 & \text{Use polytopes} \arrow[u] &
\end{tikzcd}$$

Step 1: Estimating the EGI from brightness

- Measurement Model : $\tilde{b}_K(u_k) = \frac{1}{2}\sum_{j=1}^{N} \underset{\uparrow}{a_j}\, |u_k \cdot \underset{\uparrow}{v_j}| + \underset{\uparrow}{n}(u_k)$

  unknowns noise

- Constraints

  ★ – Minkowski's existence theorem:

  A measure m is the surface area measure of some nondegenerate convex body (or polytope) iff it is not concentrated on a great sphere and

  $$\underbrace{\int v\, dm(v) = o}_{\text{arbitrary convex body}} \qquad \text{or} \qquad \underbrace{\sum_{j=1}^{N} a_j v_j = o}_{\text{convex polytope}}$$

  ★ – Positivity: $a_j \geq 0$

  ★ – Symmetry: $a_j = a_{\frac{N}{2}+j}$ and $v_j = -v_{\frac{N}{2}+j}$ for $j = 1, \Lambda, N/2$

Step 1: The Nonlinear Least-Squares Solution

$$\left(\hat{a},\hat{V}\right)=\arg\min_{(a,V)}\left\|\tilde{b}_K - C(V)a\right\|^2, \text{ such that } \quad a \geq 0 \text{ and } \quad a^T V = 0$$

- Algorithm 1: Input: N and $\tilde{b}_K(u_1),\ldots,\tilde{b}_K(u_M)$

- Use MATLAB's optimization toolbox: fmincon function uses SQP (Sequential Quadratic Programming)

- Output: surface area measure of a polytope P with at most N facets

$$\tilde{b}_K = \left[\tilde{b}_K(u_1),\ldots,\tilde{b}_K(u_M)\right]^T$$

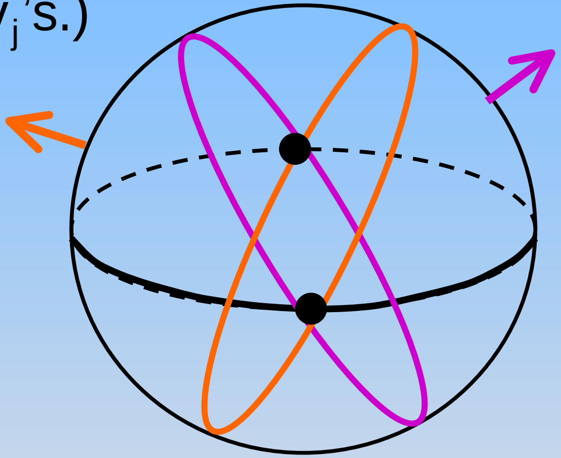
$$a = [a_1,\ldots,a_N]^T$$

$$V = [v_1,\ldots,v_N]^T$$

$$C_{jk} = \left|u_k^T v_j\right| / 2$$

The Set of Nodes

- Assume the number N of facets is not fixed.
- Given M viewing direction vectors u_k, compute the normal vectors v_j corresponding to the unique convex polytope of maximal volume with the same brightness values. (In 3-D, at most M(M-1) such v_j's.)



* S. Campi, A. Colesanti, and P. Gronchi, Convex bodies with extremal volumes having prescribed brightness in finitely many directions, *Geom. Dedicata* **57** (1995), 121-133.

Two More Algorithms

- Input: $\tilde{b}_K(u_1),\ldots,\tilde{b}_K(u_M)$
- Calculate the nodes: $V=[v_1,\ldots,v_{N/2}]^T$
- Algorithm 2: Find the linear least-squares solution

$$\hat{a}=\arg\min_{a}\left\|\tilde{b}_K-C(V)a\right\|^2,\ \text{such that}\quad a\geq 0$$

- Algorithm 2' (Kiderlen): Find the LP solution

$$\hat{a}=\arg\min_{a}\sum_{k=1}^{M}\left(\tilde{b}_K(u_k)-\sum_{j=1}^{N/2}a_j\,|u_k^T v_j|\right)$$

$$\text{such that}\quad \sum_{j=1}^{N/2}a_j\,|u_k^T v_j|\leq\tilde{b}_K(u_k),\qquad k=1,\ldots,M$$

$$\text{and}\qquad a_j\geq 0,\quad j=1,\ldots,N/2$$

Step 2: Shape from EGI

- Equivalent to shape from curvature

- Trivial for Polygons

$$w_j = w_{j-1} + a_j \, [\cos(\theta_j + \pi/2) \quad \sin(\theta_j + \pi/2)]^T \quad \text{for } j = 1, \Lambda, N, \quad \text{with } w_0 = [0,0]^T$$

- Nontrivial for Polyhedra

$$\begin{aligned} &\text{minimize} && a^T h = \sum_{j=1}^{N} a_j h_j \\ &\text{such that} && \mathrm{V}(P(h)) = 1 \\ &\text{and} && h \geq 0 \end{aligned}$$

Distance of facets from origin

duality $\longleftrightarrow$

$$\begin{aligned} &\text{maximize} && \mathrm{V}(P(h))^{1/n} \\ &\text{such that} && a^T h = 1 \\ &\text{and} && h \geq 0 \end{aligned}$$

Implementation

* J. Lemordant, P.D. Tao, and H. Zouaki, Modélisation et optimisation numérique pour la reconstruction d'un polyèdre à partir de son image gaussienne généralisée, *RAIRO Modél. Math. Anal. Numér.* **27** (1993), 349-374.

- Use MATLAB's fmincon function to solve optimization problem (convex objective function, linear constraints)
- Use free C++ program Vinci to compute V(P(h))
- Use free program qhull to convert H-representation of optimal P(h) to its V-representation and to compute the convex hull
- qhull outputs a Mathematica graphics object for display

- Thanks to ex-WWU student Chris Street

Convergence results

Th. 1 Let $n \geq 2$ and let K be an origin-symmetric convex body in R^n. Let u_k, $k = 1,2,\ldots$ be a sequence of mutually nonparallel unit vectors whose union is dense in $S^{n\text{-}1}$. For either Algorithm 2 or 2' with input as stated there, let P_M, $M \geq n$ be an output convex polytope. Then P_M converges to K in the Hausdorff metric as $M \to \infty$.

Th. 2 Similar result holds for Algorithm 1.

A Stability Result

Proposition

Let $n \geq 3, 0 < r_0 < R_0$, and let K and L be origin-symmetric convex bodies in R^n such that $r_0 B \subset K, L \subset R_0 B$. There is a constant $c(n, r_0, R_0)$ such that

$$\delta(K,L) \leq c\,\delta(\Pi K, \Pi L)^{1/(n(n+4))}.$$

- Recall that $h_{\Pi K}(u) = b_K(u) \quad \forall u.$

* S. Campi, Recovering a centred convex body from the areas of its shadows: a stability estimate, *Ann. Mat. Pura Appl. (4)* **151** (1988), 289-302.

* J. Bourgain and J. Lindenstrauss, Projection bodies, in: *Geometric Aspects of Functional Analysis (1986/7), Lecture Notes in Math. 1317* , Springer, Berlin, 1988, pp. 250-270.

A Technical Lemma

Lemma Let $n \geq 2, 0 < r < R, 0 < \varepsilon < r^{n-1}/(5R^{n-1})$, and let U be an ε-net in S^{n-1}. Let K and L be origin-symmetric convex bodies in R^n such that $rB \subset K \subset RB$ and $b_K(u) = b_L(u)$ for each $u \in U$. Then $r_0 B \subset L \subset R_0 B$,

$$\text{where } R_0 = \frac{3n\kappa_n}{\kappa_{n-1}}\left(\frac{3}{2}\right)^{1/(n-1)} \frac{R^n}{r^{n-1}} \text{ and } r_0 = \frac{\kappa_{n-1} r^{n-1}}{2^n R_0^{\ n-2}}.$$

Proof uses: projection bodies, Cauchy's surface area formula, the isoperimetric inequality, mixed volumes.

A complexity result

Th. 3 There is an oracle-polynomial time algorithm for reconstructing an approximation to an unknown convex body K in R^n that is accessible only via its brightness function.

Proof uses: Kiderlen's LP Algorithm 2', a refined estimate for the constant in the Campi-Bourgain-Lindenstrauss theorem, and a polynomial-time algorithm for constructing an approximation to a rational convex polytope from its surface area measure due to

* P. Gritzmann and A. Hufnagel, On the algorithmic complexity of Minkowski's reconstruction problem, *J. London Math. Soc. (2)* **59** (1999), 1081-1100.

Ongoing and Future Work

- Use of clustering or decimation in alternative approach to Algorithm 1.
- Systematic study of the effect of noise. To include a proof of convergence and estimates of rates of convergence with noise using empirical process theory. Joint works with Amyn Poonawala and with Markus Kiderlen.
- Develop reconstruction algorithms using different types of data.