Complexity of Convex Bodies in Higher Dimensions. I.

by

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- Convex body: a compact convex set B in a finite-dimensional real vector space V

- Some features:

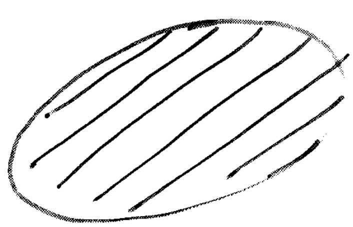
  The dimension $\dim V$ is high, though finite;

  The "**membership problem**":

  Given $v \in V$, decide whether $v \in B$ is hard

- Question:

  What B looks like?



- Why do we care?

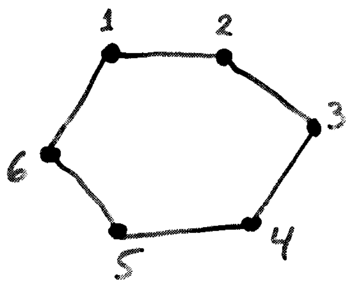
  Well, because ...

EXAMPLE (COMBINATORICS/OPTIMIZATION):

THE TRAVELING SALESMAN POLYTOPE

SYMMETRIC VERSION: ST_n

FIX A COMPLETE UNDIRECTED GRAPH WITH n VERTICES. A HAMILTONIAN CYCLE VISITS EVERY VERTEX EXACTLY ONCE. FOR EACH HAMILTONIAN CYCLE INTRODUCE AN $n \times n$ MATRIX:



$$\longleftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_{ij} = x_{ji} = 1$ IF $i - j$ IS IN THE CYCLE

ST_n IS THE CONVEX HULL OF ALL SUCH MATRICES.

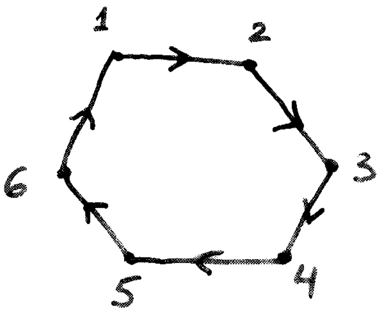
A POLYTOPE WITH $\frac{(n-1)!}{2}$ VERTICES OF DIMENSION $\frac{n^2 - 3n}{2}$.

EXAMPLE (COMBINATORICS / OPTIMIZATION)

THE TRAVELING SALESMAN POLYTOPE

ASYMMETRIC VERSION: AT_n

SAME, BUT TAKE THE DIRECTED COMPLETE GRAPH AND DIRECTED HAMILTONIAN CYCLES



$$\longleftrightarrow \quad \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{bmatrix}$$

$x_{ij} = 1$
IF $i \to j$ IS IN THE CYCLE

AT_n IS THE CONVEX HULL OF ALL SUCH MATRICES.

A POLYTOPE WITH $(n-1)!$ VERTICES OF DIMENSION $n^2 - 3n + 1$

EXAMPLE (ALGEBRA / ANALYSIS)

NON-NEGATIVE POLYNOMIALS

CHOOSE POSITIVE INTEGERS K AND n.

LET V BE THE VECTOR SPACE OF ALL HOMOGENEOUS POLYNOMIALS $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ OF DEGREE $2K$ IN n REAL VARIABLES, SO $\dim V = \binom{n+2K-1}{2K}$

LET

$$Pos_{2K,n} = \left\{ f : \quad f(x) \geq 0 \quad \forall x \in \mathbb{R}^n \text{ AND } \int_{S^{n-1}} f(x)\,dx = 1 \right\}$$

S^{n-1} IS THE UNIT SPHERE, dx IS THE ROTATION INVARIANT PROBABILITY MEASURE

HENCE $\dim Pos_{2K,n} = \binom{n+2K-1}{2K} - 1$.

$Pos_{2K,n}$ IS A CONVEX BODY.

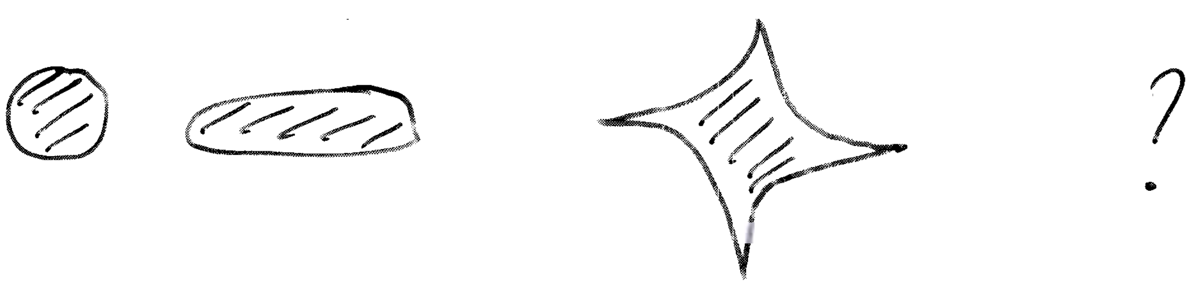
• SOME OBSERVATIONS:

1. ALL THE THREE BODIES ARE GIVEN BY "CONCISE WORD DESCRIPTIONS", SO WHILE IT IS PERFECTLY CLEAR WHAT THEY ARE, IT IS NOT CLEAR WHAT CAN WE DO WITH THEM;

2. THE TRAVELING SALESMAN POLYTOPE IS DEFINED AS THE CONVEX HULL OF MANY POINTS. IT IS INTERESTING AND NP-HARD TO OPTIMIZE ON THE ST_n, AT_n

3. THE BODY OF NON-NEGATIVE POLYNOMIALS IS DEFINED BY INFINITELY MANY LINEAR INEQUALITIES. IT IS INTERESTING AND NP-HARD TO TEST MEMBERSHIP TO $Pos_{2k,n}$ (IN A SENSE, EQUIVALENT TO SOLVING SYSTEMS OF REAL POLYNOMIAL EQUATIONS)

WHAT DO THESE BODIES LOOK LIKE?



- SOME THEORY : ELLIPSOIDS

AN ELLIPSOID IS AN AFFINE IMAGE OF A BALL :

ELLIPSOIDS ARE MORE FUNDAMENTAL THAN BALLS ; UNLIKE BALLS THEY DON'T NEED A SCALAR PRODUCT TO EXIST

THEOREM (F. JOHN) EVERY FULL-DIMENSIONAL CONVEX BODY B CONTAINS A UNIQUE ELLIPSOID E_{MAX} OF THE MAXIMUM VOLUME. IF O IS THE CENTER OF E_{MAX}, THEN

$$E_{MAX} \subset B \subset (\dim B) E_{MAX}$$

IF B IS CENTRALLY SYMMETRIC $(B = -B)$ THEN

$$E_{MAX} \subset B \subset \sqrt{\dim B} \; E_{MAX}$$

FURTHER RAMIFICATIONS POSSIBLE.

THEOREM (LÖWNER, JOHN)

EVERY FULL-DIMENSIONAL CONVEX BODY B IS CONTAINED IN A UNIQUE ELLIPSOID E_{MIN} OF THE MINIMUM VOLUME. IF 0 IS THE CENTER OF E_{MIN} THEN

$$(\dim B)^{-1} E_{MIN} \subset B \subset E_{MIN}.$$

IF B IS CENTRALLY SYMMETRIC $(B = -B)$ THEN

$$(\dim B)^{-1/2} E_{MIN} \subset B \subset E_{MIN}.$$

VARIOUS REFINEMENTS EXIST.

REMARKS:

1. THE FACTORS $\dim B$, $(\dim B)^{-1}$, $(\dim B)^{-1/2}$, $(\dim B)^{1/2}$ ARE THE BEST POSSIBLE; CONSIDER THE SIMPLEX, CUBE AND OCTAHEDRON.

2. THE ELLIPSOIDS E_{MAX}, E_{MIN} DO NOT DEPEND ON EUCLIDEAN STRUCTURE OR EVEN VOLUME FORM; THEY ARE INTRINSIC TO THE BODY AND CAPTURE SOME GENERAL INFORMATION ABOUT ITS SHAPE.

- QUESTION: HOW TO FIND THE MINIMUM - MAXIMUM VOLUME ELLIPSOIDS ?

- OBSERVATION: THE BODIES WE CARE ABOUT HAVE RICH SYMMETRY GROUPS. REASON: IT IS HARD TO BREAK SYMMETRY WITH FEW WORDS AND OUR BODIES ARE DEFINED BY SHORT WORD DESCRIPTIONS.

- ANOTHER OBSERVATION: SUPPOSE THAT A GROUP G OF ISOMETRIES ACTS IN SPACE V AND CONVEX BODY B IS G-INVARIANT:

$$g(B) = B \quad \text{FOR ALL} \quad g \in G.$$

THEN E_{MIN} AND E_{MAX} ARE G-INVARIANT:

$$g(E_{MIN}) = E_{MIN} \quad \text{AND} \quad g(E_{MAX}) = E_{MAX}.$$

- A REFINEMENT: SUPPOSE THAT THE ACTION OF G IS IRREDUCIBLE (THERE ARE NO PROPER INVARIANT SUBSPACES). THEN BOTH E_{MAX}, E_{MIN} MUST BE BALLS.

REASON: $E = \{x : q(x) \leq 1\}$, WHERE q IS A QUADRATIC FORM, $q: V \longrightarrow \mathbb{R}$

THE EIGENSPACES OF q MUST BE G-INVARIANT.

EXAMPLE: THE SYMMETRIC TRAVELING SALESMAN POLYTOPE ST_n:

IN THE SPACE OF $n \times n$ MATRICES, THERE IS AN ACTION OF THE SYMMETRIC GROUP S_n BY SIMULTANEOUS PERMUTATIONS OF ROWS AND COLUMNS: $(\sigma X)_{ij} = X_{\sigma^{-1}(i)\sigma^{-1}(j)}$. THERE IS AN INVARIANT SCALAR PRODUCT $\langle X, Y \rangle = \mathrm{Tr}(XY^T) = \sum_{ij} x_{ij} y_{ij}$. THE POLYTOPE ST_n IS INVARIANT. MOREOVER, IN THE AFFINE HULL OF ST_n (WHICH CONSISTS OF THE SYMMETRIC MATRICES WITH ZERO DIAGONAL AND ROW AND COLUMN SUMS EQUAL 2), THERE ARE NO PROPER INVARIANT SUBSPACES.

HENCE: THE ELLIPSOIDS E_{MIN}, E_{MAX} MUST BE BALLS IN THE AFFINE HULL OF ST_n CENTERED AT
$$\begin{bmatrix} 0 & & \frac{2}{n-1} \\ & 0 & \\ \frac{2}{n-1} & \ddots & 0 \end{bmatrix}$$

THE RADIUS OF E_{MIN} IS $\sqrt{2n(n-3)/(n-1)}$

QUESTION: WHAT IS THE RADIUS OF E_{MAX}? CLEARLY, AT LEAST $\sqrt{\frac{8}{n(n-1)(n-3)}}$, BUT CAN WE KNOW ANY BETTER?

• WHAT IF THE ACTION IS NOT IRREDUCIBLE (= REDUCIBLE)

SUPPOSE THAT A COMPACT GROUP G ACTS IN V. LET $\langle , \rangle$ BE A G-INVARIANT SCALAR PRODUCT IN V. LET $v \in V$ BE A POINT AND LET

$$B = \operatorname{conv}\left(gv : g \in G\right)$$

(CONVEX HULL OF THE ORBIT OF v)

SUPPOSE THAT B IS FULL-DIMENSIONAL.

THEOREM (G. BLEKHERMAN). THERE EXISTS A DECOMPOSITION $V = \bigoplus_i V_i$, WHERE V_i ARE PAIRWISE ORTHOGONAL IRREDUCIBLE COMPONENTS (SUBSPACES) SUCH THAT

$$E_{MIN} = \left\{ x : \sum_i \frac{\dim V_i}{\dim V} \frac{\langle x_i, x_i \rangle}{\langle v_i, v_i \rangle} \leq 1 \right\},$$

WHERE x_i, v_i ARE THE ORTHOGONAL PROJECTIONS OF x, v ONTO V_i.

* THIS <u>ALMOST</u> TELLS US HOW TO CONSTRUCT THE ELLIPSOID; IN THE NON-MULTIPLICITY-FREE CASE AN EXTRA CONDITION IS NEEDED; NOT RELEVANT HERE *

EXAMPLE: THE ASYMMETRIC TRAVELING SALESMAN POLYTOPE AT_n

SAME SPACE ($n \times n$ MATRICES), SAME ACTION (SIMULTANEOUS PERMUTATIONS OF ROWS AND COLUMNS)

SAME SCALAR PRODUCT ($\langle x, y \rangle = \sum_{ij} x_{ij} y_{ij}$)

THE POLYTOPE AT_n IS THE CONVEX HULL OF THE ORBIT; CHOOSE THE MATRIX REPRESENTING A PARTICULAR HAMILTONIAN CYCLE.

THE ACTION OF S_n IS REDUCIBLE; THERE ARE TWO COMPONENTS. THE ELLIPSOID E_{MIN} IN THE AFFINE HULL OF AT_n (CONSISTING OF MATRICES WITH ZERO DIAGONAL AND ROW AND COLUMN SUMS EQUAL 1) IS DEFINED BY

$$(n-1) \sum_{1 \le i \ne j \le n} \left(\frac{x_{ij} + x_{ji}}{2} - \frac{1}{n-1} \right)^2$$

$$+ \frac{n^2-3n+2}{n} \sum_{1 \le i \ne j \le n} \left(\frac{x_{ij} - x_{ji}}{2} \right)^2 \le n^2 - 3n + 1.$$

IT IS CENTERED AT $\begin{bmatrix} 0 & & \frac{1}{n-1} \\ & \ddots & \\ \frac{1}{n-1} & & 0 \end{bmatrix}$ AND

SLIGHTLY STRETCHED TOWARDS SKEW-SYMMETRIC MATRICES.

THE DUAL THEOREM IS PARTICULARLY SIMPLE.

THEOREM (A. B. + G. Blekherman).

LET G BE A COMPACT GROUP ACTING IN A FINITE-DIMENSIONAL VECTOR SPACE V. LET

$$B = \operatorname{conv}(gv : g \in G)$$

BE THE CONVEX HULL OF THE ORBIT OF A VECTOR $v \in V$. SUPPOSE THAT B IS FULL-DIMENSIONAL.

LET V^* BE THE DUAL TO V AND LET

$$B^\circ = \{\ell \in V^* : \ \ell(x) \leq 1 \text{ FOR ALL } x \in B\}$$

BE THE POLAR OF B.

THEN THE MAXIMUM VOLUME ELLIPSOID OF B° IS DEFINED BY

$$E_{MAX} = \left\{\ell \in V^* : \ \int_G \ell^2(gv)\, dg \leq \frac{1}{\dim V}\right\}$$

"dg" IS THE HAAR PROBABILITY MEASURE ON G.

EXAMPLE: NON-NEGATIVE POLYNOMIALS.

RECALL THAT $Pos_{2k,n}$ IS THE SET OF HOMOGENEOUS OF DEGREE $2k$ POLYNOMIALS $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ SUCH THAT $f(x) \geq 0$ FOR ALL $x \in \mathbb{R}^n$ AND THE AVERAGE VALUE OF f ON THE UNIT SPHERE IS 1.

THERE IS A SPECIAL POLYNOMIAL $r^{2k} \in Pos_{2k,n}$

$$r^{2k} = (x_1^2 + \dots + x_n^2)^k .$$

LET US INTRODUCE THE SCALAR PRODUCT

$$\langle f, g \rangle = \int_{S^{n-1}} f(x)\, g(x)\, dx$$

ROTATIONAL-INVARIANT PROBABILITY MEASURE.

$\overbrace{\text{THEOREM}}^{\text{G. BLEKHERMAN}}$. THE MAXIMUM VOLUME ELLIPSOID OF $Pos_{2k,n}$ IS THE BALL CENTERED AT r^{2k} OF THE RADIUS $\dfrac{1}{\sqrt{\binom{n+2k-1}{2k} - 1}}$.

QUESTION: WHAT IS THE MINIMUM VOLUME ELLIPSOID? IS IT ALSO A BALL?

WHY? BECAUSE THE SET $Pos_{2K,n}$ OF NON-NEGATIVE POLYNOMIALS IS JUST THE POLAR TO AN ORBIT.

THE GROUP: THE GROUP $O(n)$ OF ORTHOGONAL TRANSFORMATIONS OF $\mathbb{R}^n$;

SPACE V: THE TENSOR PRODUCT

$$V = \underbrace{\mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n}_{2K \text{ TIMES}} = (\mathbb{R}^n)^{\otimes 2K}$$

THE ACTION: BY THE $2K$-TH POWER OF THE ACTION IN $\mathbb{R}^n$;

THE POINT: $v = \underbrace{e \otimes \ldots \otimes e}_{2K \text{ TIMES}}$, $e = (1, 0, \ldots, 0)$

WHAT HAPPENS: THE ORBIT CONSISTS OF THE VECTORS $\underbrace{x \otimes \ldots \otimes x}_{2K} = x^{\otimes 2K}$, WHERE $x \in S^{n-1}$. A POLYNOMIAL $f(x)$ ON THE UNIT SPHERE CAN BE VIEWED AS A LINEAR FUNCTION ON THE ORBIT

$$f(x) = \ell(x^{\otimes 2K})$$

HOMOGENEOUS POLYNOMIAL OF DEGREE $2K$ ON THE UNIT SPHERE

LINEAR FUNCTION ON THE $2K$-TH TENSOR POWER

WHAT SHOULD BE THE NEXT STEP?

APPROXIMATION BY AN ELLIPSOID IS APPROXIMATION BY A QUADRATIC HYPERSURFACE. CAN WE DO ANY BETTER BY USING A HIGHER DEGREE HYPERSURFACE?

LET US CONSIDER THE CENTRALLY SYMMETRIC CASE: $B=-B$ (BECAUSE IT IS EASIER...)

ASSUMING B IS FULL-DIMENSIONAL, LET US ASSOCIATE WITH B ITS GAUGE:

$$\|x\|_B = \min\{\lambda : x \in \lambda B\}$$

$\|x\|_B = 2$.

APPROXIMATE $B \Longleftrightarrow$ APPROXIMATE $\|x\|_B$

APPROXIMATION BY AN ELLIPSOID

APPROXIMATION BY A QUADRATIC FORM

$$\sqrt{q(x)} \leq \|x\|_B \leq \sqrt{\dim B}\,\sqrt{q(x)}$$

$q: V \longrightarrow \mathbb{R}$ - POSITIVE DEFINITE QUADRATIC FORM

THEOREM [A.B.]. FOR ANY CENTRALLY SYMMETRIC CONVEX BODY $B \subset V$ AND A POSITIVE INTEGER m THERE EXISTS A HOMOGENEOUS POLYNOMIAL $q: V \longrightarrow \mathbb{R}$ OF DEGREE $2m$ SUCH THAT:

1. $q(x) \geq 0$ FOR ALL $x \in V$ (IN FACT, q IS A SUM OF SQUARES)
2. $$q^{\frac{1}{2m}}(x) \leq \|x\|_B \leq \binom{\dim B + m - 1}{m}^{1/2m} q^{1/2m}(x)$$

"FOR PRACTICAL PURPOSES" $\sqrt{\frac{\dim B}{m}}$; $m < \dim B$

ANYWAY, $\binom{\dim B + m - 1}{m}^{1/2m} \longrightarrow 1$ AS $(\dim B)/m \longrightarrow 0$.

EXAMPLE: $m = 2$, SO $\deg q = 4$.

$$q^{1/4}(x) \leq \|x\|_B \leq \binom{\dim B + 1}{2}^{1/4} q^{1/4}(x)$$

$$\approx \frac{\sqrt{\dim B}}{\sqrt[4]{2}}$$

GENERALLY, FOR $m \ll \dim B$

$$\frac{\sqrt{\dim B}}{\sqrt[2m]{m!}} \sim \frac{\sqrt{\dim B}}{\sqrt{m/e}}$$

LET q BE A POLYNOMIAL OF DEGREE 4 APPROXIMATING B THE BEST:

$$q^{1/4}(x) \leq \|x\|_B \leq c(B)\, q^{1/4}(x)$$

WITH THE SMALLEST POSSIBLE $c(B)$

LET $c(n,4) = \sup_{\dim B = n} c(B)$

WHAT I KNOW:

$$\frac{1}{\sqrt[4]{2}} \sqrt{n} \geq c(n,4) \geq c_0 \sqrt{n}$$

FOR SOME UNKNOWN $c_0 > 0$;

THERE ARE BODIES WHERE THE CONSTRUCTION OF THE THEOREM IS NOT OPTIMAL.

FOR ANY B, THE SET OF BEST APPROXIMATING POLYNOMIALS IS CONVEX; IF B HAS SYMMETRIES, THE BEST APPROXIMATING POLYNOMIALS CAN BE CHOSEN TO HAVE THE SYMMETRIES; SAME FOR THE POLYNOMIAL OF THE THEOREM

WHAT I don'T KNOW: FOR THE BEST APPROXIMATION POLYNOMIALS OR POLYNOMIALS OF THE THEOREM, DOES $q^{1/4}(x)$ HAVE TO BE CONVEX?

SKETCH OF PROOF.

LET $B^{\circ} = \{ \ell \in V^*,\ \ell(x) \le 1 \quad \forall x \in B \}$

BE THE POLAR OF B.

ASSUMING m IS ODD,

$\|x\|_B = \max\{ \ell(x) : \ell \in B^{\circ} \}$ — DUALITY;

$\|x\|_B^m = \max\{ \ell^m(x) : \ell \in B^{\circ} \}$ USED m IS ODD;

$\|x\|_B^m = \max\{ \ell^{\otimes m}(x^{\otimes m}) : \ell \in B^{\circ} \}$, WHERE

$$\ell^{\otimes m} = \underbrace{\ell \otimes \ldots \otimes \ell}_{m \text{ TIMES}} \in \underbrace{V^* \otimes \ldots \otimes V^*}_{m \text{ TIMES}} = (V^*)^{\otimes m}$$

$$x^{\otimes m} = \underbrace{x \otimes \ldots \otimes x}_{m \text{ TIMES}} \in \underbrace{V \otimes \ldots \otimes V}_{m \text{ TIMES}} = V^{\otimes m}$$

$\|x\|_B^m = \max\{ h(x^{\otimes m}) : h \in D \}$, WHERE

$$D = \operatorname{conv}\{ \ell^{\otimes m} : \ell \in B^{\circ} \} \subset (V^*)^{\otimes m}$$

D IS CENTRALLY SYMMETRIC AND CONVEX;

APPROXIMATE IT BY AN ELLIPSOID E

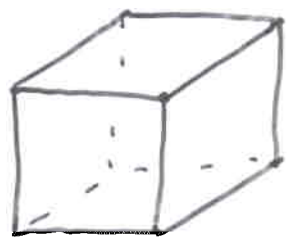
NOTE, THAT $\max\{ h(x^{\otimes m}) : h \in E \} = \sqrt{q(x)}$.

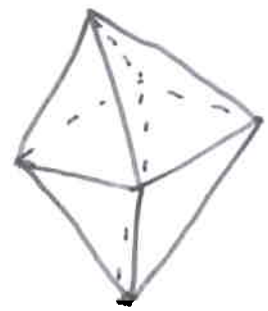
WHERE q IS THE DESIRED POLYNOMIAL;

- PUNCHLINE: $\dim D \le \binom{\dim V + m - 1}{m}$,

BECAUSE D LIES IN THE SYMMETRIC PART OF $(V^*)^{\otimes m}$

• EXAMPLE: APPROXIMATING THE n-DIMENSIONAL CUBE $-1 \le x_i \le 1$, $i=1,\dots,n$ AND THE OCTAHEDRON $\sum_{i=1}^{n} |x_i| \le 1$.





$$\|x\|_\infty = \max\{|x_i|, i=1,\dots,n\}$$

$$\sum_{i=1}^{n} |x_i| \le 1$$

APPROXIMATING BY QUADRATIC FORMS:

$$\frac{1}{\sqrt{n}}\sqrt{x_1^2+\dots+x_n^2} \le \|x\|_\infty \le \sqrt{x_1^2+\dots+x_n^2}$$

$$\sqrt{x_1^2+\dots+x_n^2} \le \|x\|_1 \le \sqrt{n}\sqrt{x_1^2+\dots+x_n^2}$$

SIMILAR BEHAVIOR

APPROXIMATING BY POLYNOMIALS OF DEGREE 4

$$n^{-1/4}\sqrt[4]{x_1^4+\dots+x_n^4} \le \|x\|_\infty \le \sqrt[4]{x_1^4+\dots+x_n^4}$$

$$\sqrt[4]{q(x)} \le \|x\|_1 \le c\sqrt{n}\sqrt[4]{q(x)}$$

DIFFERENT BEHAVIOR

- EXAMPLE (MORE COMPLICATED)

LET V BE THE SPACE OF HOMOGENEOUS POLYNOMIALS OF DEGREE d IN n REAL VARIABLES.

LET $B = \{ f \in V : \max_{x \in S^{n-1}} |f(x)| \le 1 \}$.

THE GAUGE

$$\|f\|_\infty = \max_{x \in S^{n-1}} f(x)$$

THE UNIT SPHERE

FOR INTEGER m, THEOREM GIVES

$$q(f) = \int_{S^{n-1}} f^{2m}(x)\, dx$$

AND

$$\|f\|_{2m} \le \|f\|_\infty \le \binom{n+md-1}{md}^{\frac{1}{2m}} \|f\|_{2m}$$

TIGHT

TIGHT $\Big\{$ UP TO A CONSTANT $2^{d/2}$

ROUGHLY $\dfrac{n^{d/2}}{\sqrt{m}}$ FOR $m \ll n$

TO GET A CONSTANT, CHOOSE $m = O(nd)$

- QUESTION: CAN WE DO BETTER?

FOR $d=2$, YES. WE CAN GET

$$q^{\frac{1}{2m}}(f) \le \|f\|_\infty \le n^{\frac{1}{2m}} q^{\frac{1}{2m}}(f)$$

- HINT $\|f\|_\infty = \max\{ |\lambda_i| : i = 1, \dots, n \}$

λ_i ARE THE EIGENVALUES

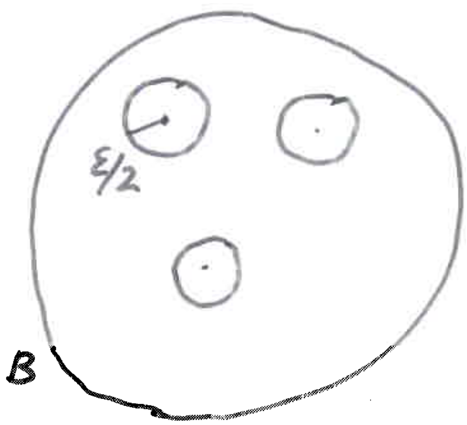
CHOOSE $q(f) = \sum_{i=1}^{n} \lambda_i^{2m}$

- OTHER INTERESTING ISSUES

- APPROXIMATIONS BY ε-NETS.

FOR A CENTRALLY SYMMETRIC CONVEX BODY B AND $\varepsilon > 0$ THERE EXISTS AN ε-NET (IN THE $\|x\|_B$ METRIC) WITH AT MOST $\left(1+\frac{2}{\varepsilon}\right)^{\dim B}$ POINTS, AND THIS IS PRETTY MUCH THE BEST POSSIBLE ...

PROOF.



CHOOSE A MAXIMAL SET OF POINTS DISTANCE ε APART. DO VOLUME ESTIMATES

- APPROXIMATIONS BY PROJECTIONS OF POLYHEDRA WITH A SMALL (REASONABLE) NUMBER OF FACETS

THEOREM [A. BEN-TAL, A. NEMIROVSKI] FOR ANY $\varepsilon > 0$ THERE IS A POLYHEDRON OF DIMENSION $\leq \text{poly}(\log \varepsilon^{-1}, n)$ AND WITH $\leq \text{poly}(\log \varepsilon^{-1}, n)$ FACETS, WHOSE PROJECTION APPROXIMATES THE EUCLIDEAN BALL WITHIN ε.

- APPROXIMATION "ON AVERAGE"

"ON AVERAGE", ANY CONVEX BODY LOOKS LIKE A BALL. FOR EXAMPLE, THE ASYMMETRIC TRAVELING SALESMAN POLYTOPE LOOKS LIKE A BALL OF RADIUS $\sqrt{2 \ln n}$

- MEASURE CONCENTRATION.

LET $S^{n-1} \subset R^n$ BE THE UNIT SPHERE, LET $dist(x,y) = \arccos \langle x,y \rangle$ BE THE GEODESIC METRIC (HENCE $0 \le dist(x,y) \le \pi$) AND LET μ BE THE ROTATION INVARIANT PROBABILITY MEASURE.

SUPPOSE $f : S^{n-1} \longrightarrow R$ IS A FUNCTION SUCH THAT $|f(x) - f(y)| \le dist(x,y)$.

LET M_f BE THE MEDIAN OF f

$\mu\{x : f(x) \le M_f\} \ge \frac{1}{2}$ AND $\mu\{x : f(x) \ge M_f\} \ge \frac{1}{2}$.

P. LEVY'S LEMMA:

FOR ANY $\varepsilon > 0$

$$\mu\{x : |f(x) - M_f| > \varepsilon\} \le \sqrt{\frac{\pi}{2}}\, e^{-\frac{\varepsilon^2 (n-2)}{2}}.$$

THUS, FOR LARGE n, FOR A TYPICAL x

$f(x)$ WITHIN $O\left(\frac{1}{\sqrt{n}}\right)$ DISTANCE FROM M_f.

"ALMOST ALWAYS ALMOST A CONSTANT".