Let M be a (smooth) compact Riemannian surface. Then the Gauss-Bonnet Theorem says

$$\chi(M) = \int_M \frac{K}{2\pi} dvol,$$

where K denotes the Gauss curvature.

Key point: The left hand side is a topological invariant, while the right hand side is the integral of a function built from infinitessimal information.

Now let M be a triangulated surface. Then

$$\chi(M) = V - E + T$$

$$(= \#vertices - \#edges + \#triangles).$$

We use the relations

$$T = \frac{2}{3}E$$

and

$$E = \frac{1}{2}\sum_{v}\deg(v)$$

to write

$$\begin{aligned}\chi(M) &= V - \frac{1}{3}E \\ &= \sum_{v}(1 - \frac{1}{6}\deg(v)).\end{aligned}$$

This is already a wonderful formula, expressing a topological invariant in terms of local information.

We can also write

$$\chi(M) = \sum_{v} \frac{1}{2\pi}\left(2\pi - \frac{2\pi}{6}\deg(v)\right).$$

Comparing with Gauss-Bonnet, we can guess that maybe $2\pi - \frac{2\pi}{6}\deg(v)$ is some sort of combinatorial Gauss curvature at the vertex v.

Notes: Suppose we build a geometric model for M by declaring that each triangle of M is a flat equilateral triangle (with edge-lengths 1). Then $\frac{2\pi}{6}\deg(v)$ is the total angular measure around the vertex v. On the other hand, 2π is the angle around a point in a flat space. Therefore, we can interpret $2\pi - \frac{2\pi}{6}\deg(v)$ to be the "angle defect" around the point v in this geometric model of M.

Moreover, this geometric model of M is flat everywhere except at the vertices. In addition, one can easily make precise sense of the statement: the amount of Gauss curvature contained at the vertex is v is $2\pi - \frac{2\pi}{6}\deg(v)$. In other words, suitably interpreted, the combinatorial formula "is" the geometric Gauss-Bonnet formula applied to the piecewise Euclidean space resulting from declaring each simplex to be equilateral.

Following the "lessons" learned from the simple example, our philosphy is:

1) To define a combinatorial version of some aspect of curvature, find a relationship between the curvature and something (such as the Euler Characteristic) that is well-defined for combinatorial spaces. This relationship should continue to hold for the combinatorial curvature. In fact, the relationship can help us find the formula for the combinatorial curvature.

2) One can view the combinatorial curvature from a geometric point of view by considering the corresponding piecewise Euclidean space. The curvature can often be expressed directly in terms of the dihedral angles of this complex.

Once we start talking about piecewise Euclidean spaces, there is no need to stick to equilateral triangles. Given a triangulated surface with a positive length assigned to each edge, if the lengths satisfy the obvious (strict) triangle inequalities, then there is a unique corresponding piecewise Euclidean space with the property that each triangle is a Euclidean triangle, and each edge has the corresponding length.

For such a space we have the formula

$$\chi(M) = \sum_{v} \frac{1}{2\pi}(2\pi - (\textit{total angle at } v))$$

Some relationships between curvature and "other notions":

1) Tube formula/ kinematic formulas

2) Bochner-Weitzenbock formula

3) Functions of curvature that integrate to give topological invariants.

1. Tube Formula/Kinematic Formulas

Weyl's tube formula [Hotelling]

Let M be a smooth compact m –dimensional submanifold of n –dimensional Euclidean space E. Let $M(\epsilon)$ denote the ϵ –neighborhood (or "tube") around M in E. Then for small ϵ,

$$Vol(M(\epsilon)) = \epsilon^{n-m} \sum_{i=0}^{m} \epsilon^{i} \alpha_{n,m,i} \int_{M} R^{i} \, dvol$$

where the $\alpha_{n,m,i}$ are universal constants, and the R^i are functions on M, 0 for i odd, called the Lipschitz-Killing curvatures. $R^0 = 1$, R^2 is the scalar curvature, and, if m is even, R^m is the Chern-Gauss-Bonnet integrand - whose integral is the Euler characteristic. (The study of this formula and its implications eventually lead to the general Hopf-Allendoerfer-Fenchel-Weil-Chern-Gauss-Bonnet Theorem.)

One amazing aspect of this formula is that all coefficients in this expansion are integrals over M of polynomials in the curvature (called the Lipschitz-Killing curvatures). In particular, the integrands depend only on the intrinsic metric of M not on its embedding. And the highest order term in the expansion is a topological invariant.

What do we mean by "polynomials in the curvature"? Expressed in local orthonormal coordinates, we can think of the curvature at a point as a skew-symmetric matrix R of 2-forms. That is,

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in End(TM).$$

All of the curvature $m-forms$ we consider will be polynomials in the entries of this matrix and the standard basis 1-forms.

(A very special case of) Chern's kinematic formula [Steiner, Blaschke, Santalo, Federer]

Let $G_{k,n}$ denote the Grassmannian of affine k-planes in R^n. Then, with all restrictions on M as above

$$\int_{G_{k,n}} \chi(M \cap E^k)\ dE^k = \beta_{k,n} \int_M R^i,$$

for $i = k + m - n \geq 0$, and 0 otherwise.

(Note that the case $k = n$ is the general H-A-F-W-C-G-B Theorem.)

Here is a way to define combinatorial Lipschitz-Killing curvatures. Given a triangulated manifold M, consider the corresponding piecewise Euclidean manifold, embed it isometrically in some Euclidean space, and investigate the tube and/or kinematic formulas.

This has been done (Wintgen, Cheeger-Müller-Schrader. [Kuiper,Banchoff]).

Theorem(W, C-M-S): Let M be a PL manifold embedded in some Euclidean space. One can define a curvature r^i for each $i-simplex\ \alpha$ of M, depending only on the local intrinsic structure of the star of α, so that if one replaces

$\int_M R^i$ by $\sum_{\alpha:i} r^i(\alpha)Vol(\alpha)$ then the [redacted] kinematic formula[redacted] remain[redacted] true.

There is also a tube formula in which the volume of the tube must be modified slightly

Example; Scalar Curvature (Regge)

Let M be a triangulated m –manifold with an allowable set of edge lengths. For each $(m-2)$-dimensional simplex α, define the scalar curvature at $\alpha, S(\alpha)$ to be the angle defect

$$1-(\textit{total angular measure at } \alpha)$$

where we have normalized things so that all unit spheres have volume 1.

Regge was interested in studying Einstein's functional

$$\int_M S\, dvol,$$

and its critical points, thought of as a function on the space of metrics of fixed volume (called Einstein metrics).

In the PL case, $\int_M S\, dvol$ is replaced by

$$\sum_{\alpha^{(m-2)}} S(\alpha)Vol(\alpha).$$

The critical points of this functional (as a function on the space of allowable edge lengths) have been studied for some triangulations of $S^4, CP^2, S^2 \times S^2, S^3$ (Hartle,Hartle-Sorkin-Williams, Piran-Stromlinger)

Just to indicate the flavor of the other combinatorial Lipschitz-Killing curvatures (as defined by C-M-S) suppose $m = \dim(M)$ is even, then the Gauss-Bonnet integrand r^m is given by

$$r^m(v) = \sum (-1)^j [v, \alpha^{2d_1}][\alpha^{2d_1}, \alpha^{2d_2}] \ldots [\alpha^{2d_{j-1}}, \alpha^{2d_j}],$$

where the sum over all ascending chains

$$v < \alpha^{2d_1} < \alpha^{2d_2} < \ldots < \alpha^{2d_{j-1}} < \alpha^{2d_j}$$

of even dimensional simplices, and $[\alpha, \beta]$ denotes the angular measure of of the simplex β as seen from α.

In particular

$$\chi(M) = \sum_v r^m(v).$$

The continuum limit:

Let M be a compact Riemannian m –manifold with a smooth triangulation.Then, if the triangulation is sufficiently fat (fatness= volume of m –simplex/(length of longest side)^m one can replace M with a PL manifold with the same combinatorial structure and edge lengths.

Theorem(C-M-S): Take a sequence of smooth triangulations of M whose mesh goes to 0, and whose fatness is sufficiently bounded below by a positive constant. Then

(The total combinatorial scalar curvature of the corresponding PL manifolds, as given by Regge's definition)

converges to

(The total scalar curvature of M).

Note that this is not a statement about pointwise convergence!

Corresponding statements are true about the other Lipschitz-Killing curvature integrals.

One can also make a combinatorial analysis of extrinsic Lipschitz-Killing curvatures (e.g. mean curvature), and this was done by Steiner, and generalized by Cheeger-Müller-Schrader.

Bochner-Weitzenbock Formula

Let M be a smooth compact manifold. Then one has the deRham complex of differential forms

$$0 \to \Omega^0 \overset{d_0}{\to} \Omega^1 \overset{d_1}{\to} \Omega^2 \overset{d_2}{\to} \dots,$$

where Ω^p denotes the space of smooth p-forms on M. Then one can compute the real cohomology of M by

$$H^p(M,R) \cong \frac{Ker\ d_p}{\mathrm{Im}\ d_{p-1}}.$$

Now suppose that M is endowed with a Riemannian metric. Then each space of forms inherits an L_2 inner product. One then defines the adjoint boundary operator

$$d_p^* : \Omega^p \to \Omega^{p-1}$$

by

$$\langle d_{p-1}\alpha, \beta \rangle = \langle \alpha, d_p^* \beta \rangle$$

for all $(p-1)$-forms α and all p-forms β.

One then constructs the Laplace operator

$$\Box_p = d_{p+1}^* d_p + d_{p-1} d_p^* : \Omega^p \to \Omega^p.$$

The main theorem of Hodge theory is

Theorem(Hodge, Kodaira):

$$Ker \; \Box_p \cong H^p(M, R).$$

Weitzenbock proved that one can write

$$\Box_p = \nabla_p^* \nabla_p + F_p,$$

where ∇_p is the Levi-Civita connection, and F_p is a 0^{th}-order operator built from the curvature. These curvature operators are somewhat mysterious with the exception of

$$F_0 = F_m = 0, \quad F_1 = F_{m-1} = Ricci\ Curvature.$$

Bochner discovered this formula (later but) independently, and observed that one can immediately deduce

Theorem (Bochner) (1)Let M be a compact Riemannian manifold. If F_p is a positive operator at every point of M then $H^p(M,R) \cong 0$.

(2) Let M be a compact connected Riemannian manifold. If F_p is a nonnegative operator at every point of M, and is positive at some point of M, then $H^p(M,R) \cong 0$.

(3) Let M be a compact connected Riemannian manifold. If F_p is a nonnegative operator at every point of M, then $\dim H^p(M,R) \leq \binom{m}{p}$.

Let us try to find combinatorial analogues of the curvature operators F_p by mimicing this procedure. Let M be a simplicial complex, with cochain complex

$$0 \to C^0 \overset{d_0}{\to} C^1 \overset{d_1}{\to} C^2 \overset{d_2}{\to} \ldots,$$

where C^p denotes the space of real p-cochains on M. Then one can compute the real homology of M by

$$H^p(M,R) \cong \frac{Ker\ d_p}{\mathrm{Im}\ d_{p-1}}.$$

Now suppose that each C^p is endowed with an inner product. Then one can define the adjoint boundary operator

$$d_p^* : C^p \to C^{p-1}$$

by

$$\langle d_{p-1}\alpha, \beta\rangle = \langle \alpha, d_p^*\beta\rangle$$

for all $(p-1)$-cochains α and all p-cochains β.

One can then construct the Laplace operator

$$\Box_p = d_{p+1}^* d_p + d_{p-1} d_p^* : C^p \to C^p.$$

It follows from simple linear algebra that

Theorem (Hodge, Eckmann):

$$Ker\ \Box_p \cong H^p(M,R).$$

The next step is to find some sort of combinatorial analogue of the Bochner-Weitzenbock formula. At its essence, we would like a formula of the sort

$$\Box_p = L_p + F_p,$$

where L_p is a nonnegative operator, and F_p is a locally defined operator, that is an operator whose action on a dual-simplex σ^* is completely determined by the combinatorial structure of some local neighborhood of σ.

From such a decomposition, one can immediately deduce

Theorem: Let M be a finite simplicial complex. If F_p is a positive operator, then $H^p(M,R) \cong 0$.

We now consider a second possibilty. Consider the following decomposition of a general symmetric 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} =$$

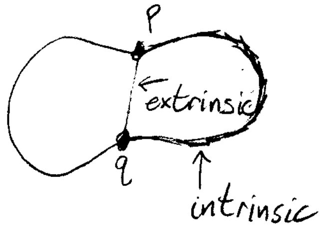
$$\begin{pmatrix} |b|+|c| & b & c \\ b & |b|+|e| & e \\ c & e & |c|+|e| \end{pmatrix} +$$

$$\begin{pmatrix} a-|b|-|c| & 0 & 0 \\ 0 & d-|b|-|e| & 0 \\ 0 & 0 & f-|c|-|e| \end{pmatrix}$$

$$= L(A) + F(A).$$

It follows from classical theorems in matrix theory that $L(A)$ is a nonnegative operator.

We apply this decomposition to the combinatorial Laplace operator to define two new operators by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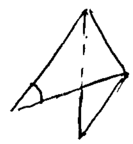

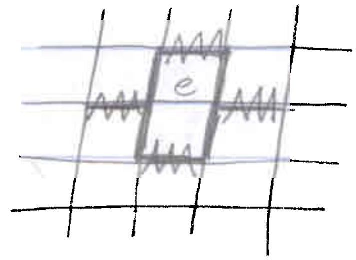
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