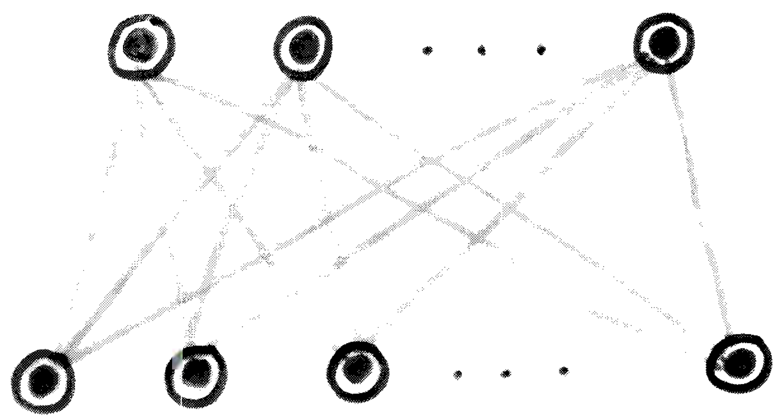
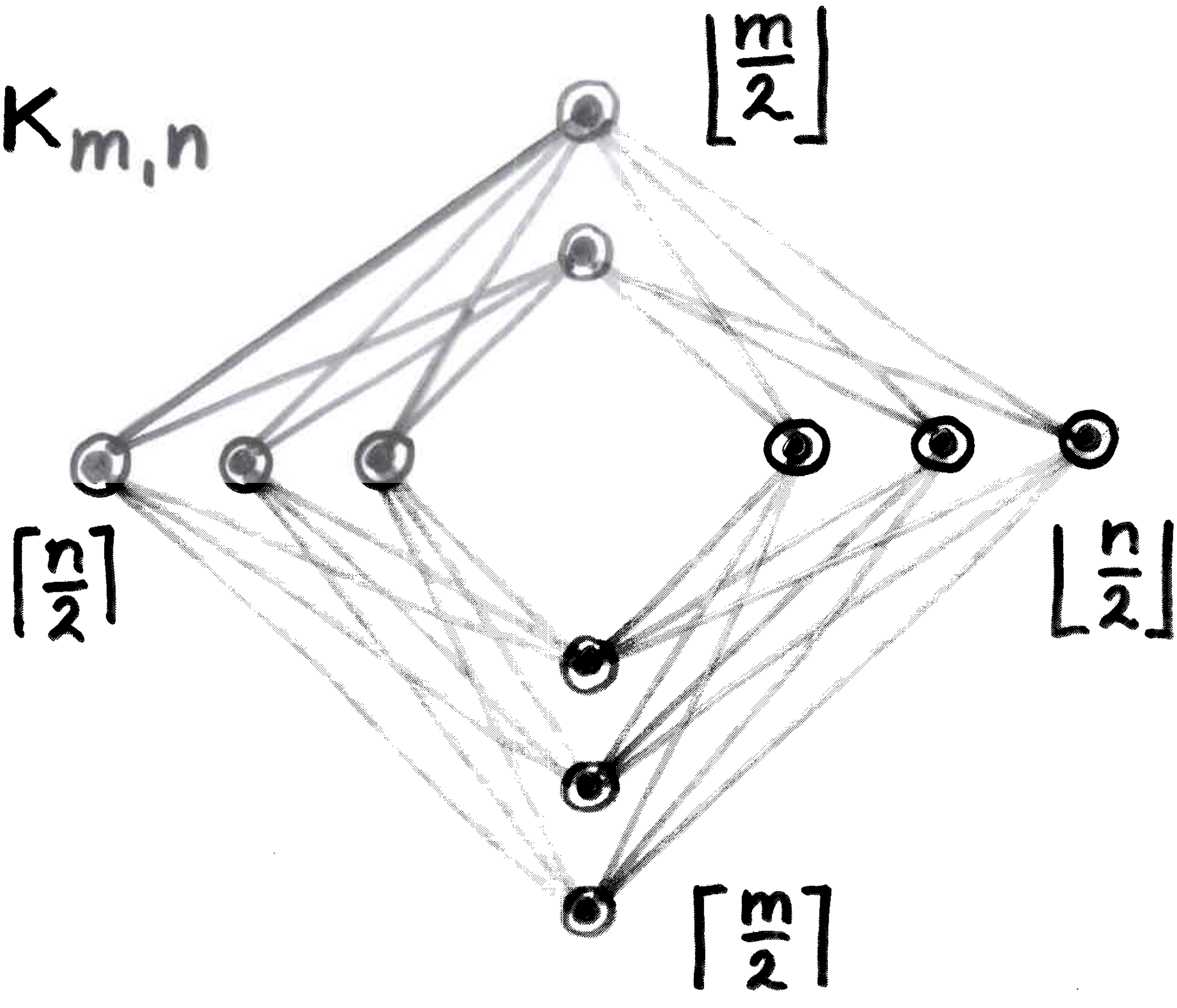
m kilns

n storage places



$K_{m,n}$



TURÁN'S BRICK FACTORY PROBLEM (1944)

Minimize the number of edge crossings in a drawing of $K_{m,n}$

CONJECTURE OF ZARANKIEWICZ

The crossing number of $K_{m,n}$

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$\Downarrow$$

$$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor$$

True for $m \leq 6$ (Kleitman 70)

$$\lim_{n\to\infty} \frac{cr(K_{n,n})}{\binom{n}{2}^2}, \quad \lim_{n\to\infty} \frac{cr(K_n)}{\binom{n}{4}} > 0$$

exist

CONWAY'S THRACKLE CONJECTURE

A *thrackle* is a graph drawn in the plane so that the edges are represented by Jordan curves, any pair of which

- either meet at a common vertex,
- or cross precisely once.

In a thrackle,

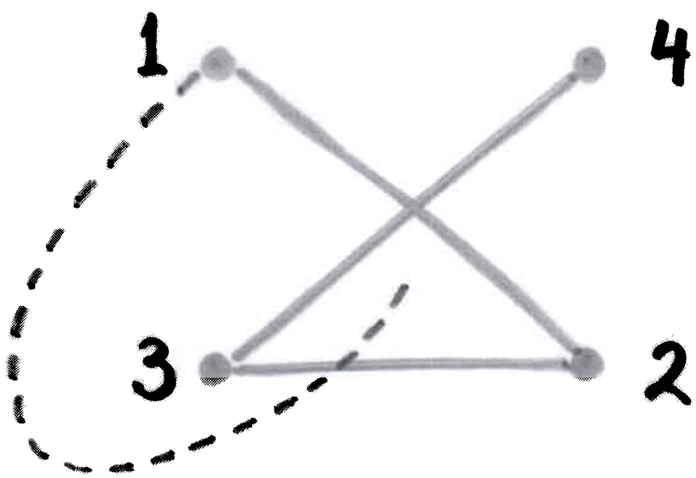
$$\#\,\text{edges} \leq \#\,\text{points}$$

FISHER INEQUALITY (1940)

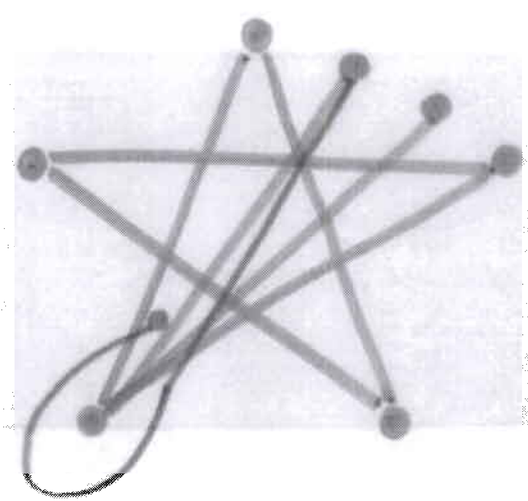
Given a system of subsets of X such that any pair have precisely one element in common,

$$\#\text{ sets} \leq \#\text{ elements of } X$$

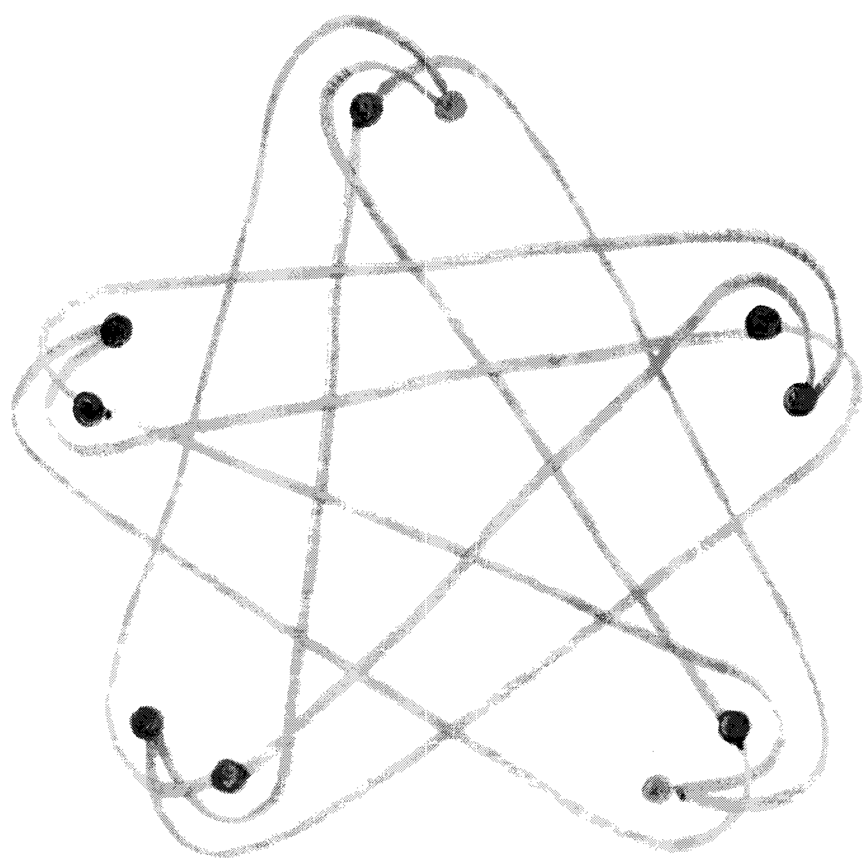
C_4 cannot be drawn as a thrackle

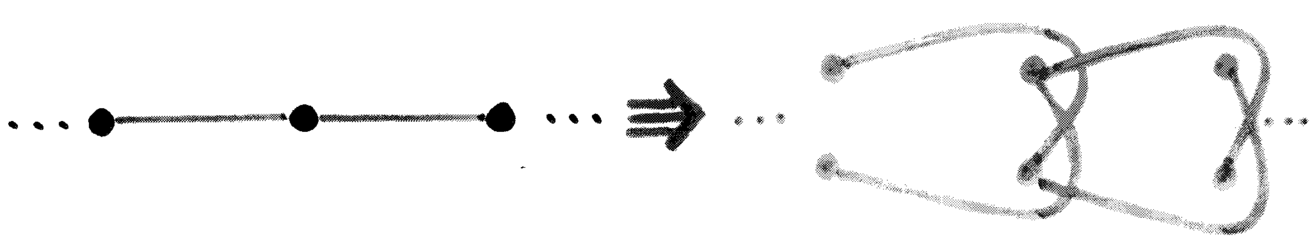


C_5 can be drawn



C_{10} drawn as a thrackle





Theorem (Lovász - P. - Szegedy 98)

In any thrackle,

$$\#\text{edges} \leq \underset{\substack{\uparrow \\ 1.5}}{2}\ \#\text{points}$$

(Cairns-Nikolayevsky 00)

Theorem

A bipartite graph can be drawn as a

mod 2 - thrackle

$\Updownarrow$

planar

The crossing number $cr(G)$ of a graph G is the minimum number of edge crossings in any drawing of G in the plane.

$cr(G) = 0 \iff G$ is planar

$cr(G) \leq 1 \iff G$ does not have $H_1, \ldots, H_{41}$ as a minor

(Robertson-Seymour 1993)

$cr(G) \leq 2 \iff$??

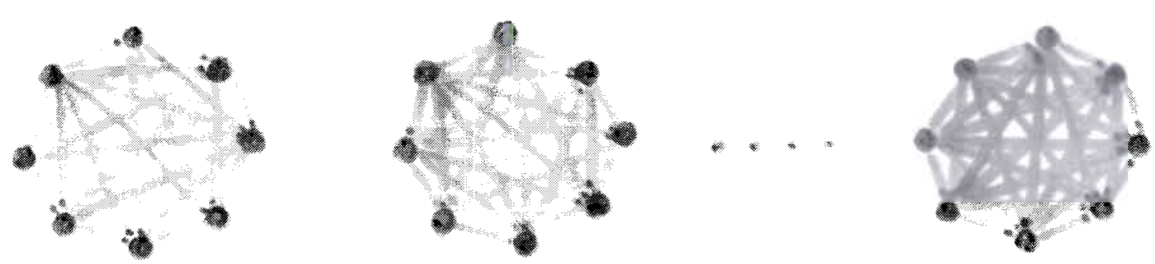
Theorem (Ajtai-Chvátal-Newborn-Szemerédi 1982, Leighton 1983)

For every simple graph with n vertices and $e \geq 4n$ edges,

$$cr(G) \geq \frac{1}{64} \frac{e^3}{n^2}$$

Construction

$1 \qquad 2 \qquad \cdots \qquad n^2/2e$



$K_{2e/n} \qquad K_{2e/n} \qquad K_{2e/n}$

Lemma

$$cr(G) \geq e-(3n-6) > e-3n$$

Proof of Theorem

Pick each $v \in V(G)$ with probabil ity p, and let $G' \subseteq G$ denote the subgraph induced by these points.

$$E[cr(G')] > E[e'] - 3E[n']$$

#edges of G' #vertices of G'

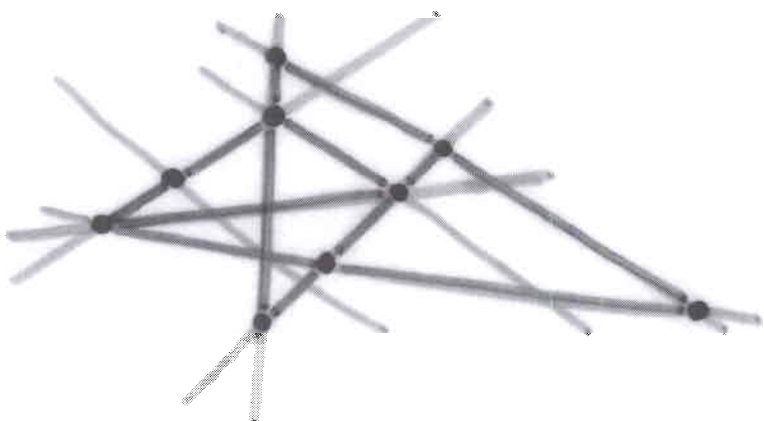
$$p^4 cr(G) > p^2 e - 3pn$$

Set $p = 4n/e$.

APPLICATIONS – "SZÉKELY'S METHOD"

I. **Theorem** (Szemerédi-Trotter 1983)
The number of incidences between n points and m lines in the plane is

$$\leq 2.57\, n^{2/3} m^{2/3} + 2.07(n+m)$$

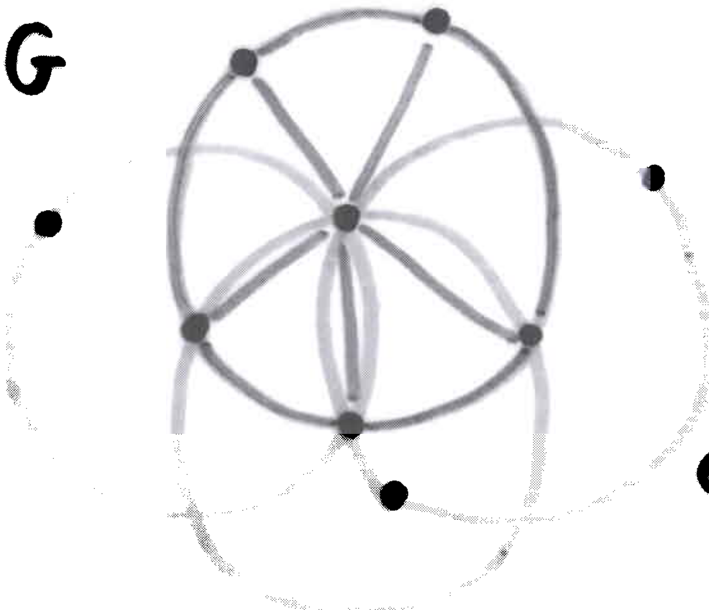
G 

I incidences
$e = I - m$ edges
n vertices

$$\binom{m}{2} \geq cr(G) \geq \frac{1}{33.75} \frac{(I-m)^3}{n^2}$$

II. **Theorem** (Spencer-Szemerédi-Trotter)
The number of unit distances determined by n points in the plane is $O(n^{4/3})$.

G



u unit distances

$2u$ incidences between circles & points

$e = 2u$ edges (?!)

n vertices

$$2\binom{n}{2} \geq cr(G) \geq \frac{1}{33.75} \frac{u^3}{n^2}$$

III. **Theorem** (Chung-Szemerédi-Trotter)
The number of distinct distances determined by n points in the plane is $\geq c n^{4/5} / \log^c n$.

Theorem (Szemerédi-Trotter 1983)
The number of incidences between n points and all lines that pass through $\geq k$ points is

$$\leq O\left(\frac{n^2}{k^2} + n\right)$$

Let A be a set of k numbers

$$S =: A + A = \{a+b \mid a, b \in A\}$$
$$P =: A \cdot A = \{a \cdot b \mid a, b \in A\}$$

Consider the k^2 lines

$$y = a(x-b) \qquad a, b \in A$$

Each passes through $\geq k$ points of $S \times P$ (namely, $(b+c, ac)$, $c \in A$)

$$k^3 \leq \#\text{incidences} \leq O\left(\frac{|S \times P|^2}{k^2} + |S \times P|\right)$$

Theorem (Elekes 1997)

For any set A of k distinct numbers, we have

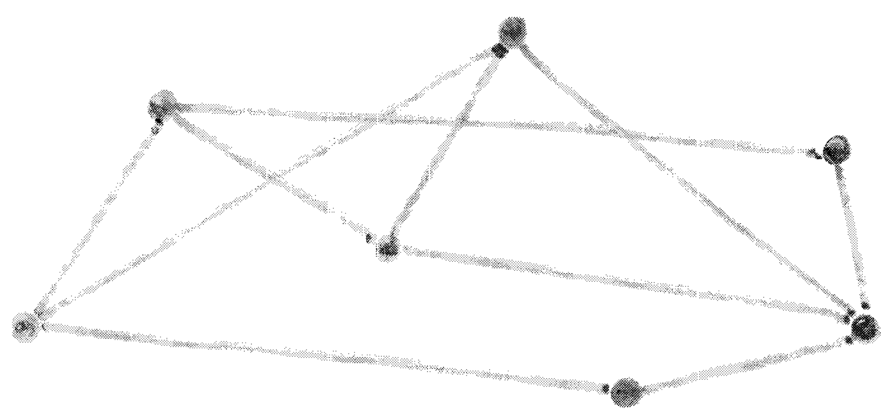
$$\max(|A+A|, |A \cdot A|) = \Omega(k^{\frac{5}{4}})$$

topological

A geometric graph is a graph whose vertices are represented by points in general position in the plane and whose edges are drawn by (possibly crossing) segments

Jordan arcs

$$G = (V(G), E(G))$$



G is a convex geometric graph if V(G) is in convex position

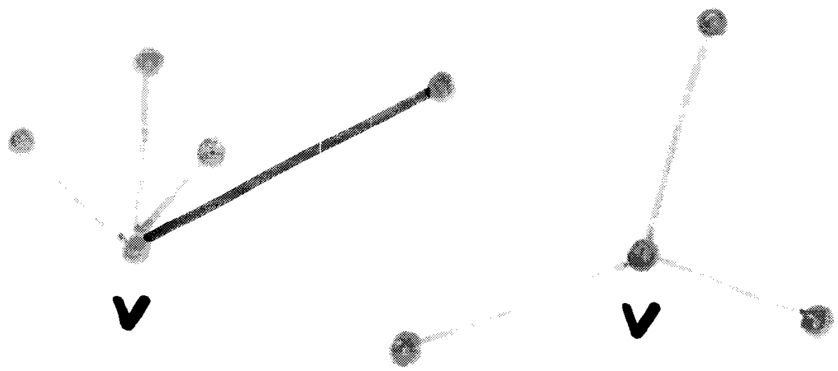
Theorem

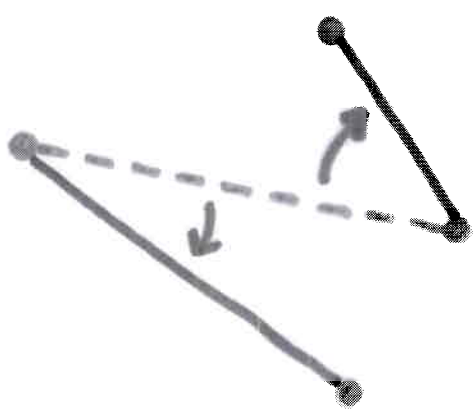
Let G be a geometric graph with no 2 disjoint edges. Then

$$|E(G)| \leq |V(G)|$$

Proof (Perles)

At each $v \in V(G)$, delete the rightmost edge.





Problem (Avital-Hanani 1966, Kupitz 1979, Perles, Erdös)

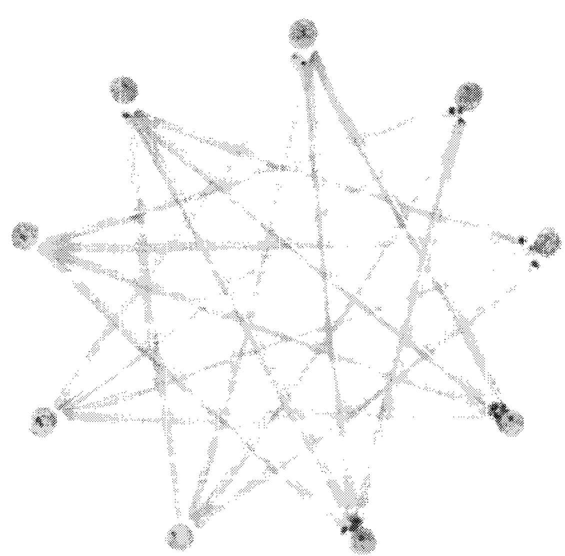
What is the maximum number of edges that a geometric graph of n vertices can have without containing k pairwise disjoint edges ?

$k=2$	n	Hopf-Pannwitz
$k=3$	$\leq 3n$	Alon-Erdös O'Donnell-Perles
$k=4$	$\leq \cancel{10}n$	Goddard-Kleitman Katchalski
	8.5	
	$\leq \cancel{(k-1)^4}n$	Pach-Töröcsik
	$2^9(k-1)^2$	Tóth

Proposition (Kupitz 1982)

For any $n \geq 2k-1$, the maximun number of edges that a convex geometric graph of n vertices can have without containing k pairwise disjoint edges is $(k-1)n$.

Proof Suppose w.l.o.g. that the vertices form a regular n-gon, and consider the parallel classes.



$k=3$

Problem (Gärtner - Welzl)

What is the maximum number of edges that a geometric graph of n vertices can have without containing k pairwise crossing edges ?

$k = 2$	$3n - 6$	Euler
$k = 3$	$O(n)$	Agarwal-Aronov P.-Pollack-Sharir
$k \geq 4$	$O(n \log^{2k-6} n)$	Shahrokhi-P.-Szegedy
	$O(n \log n)$	Valtr

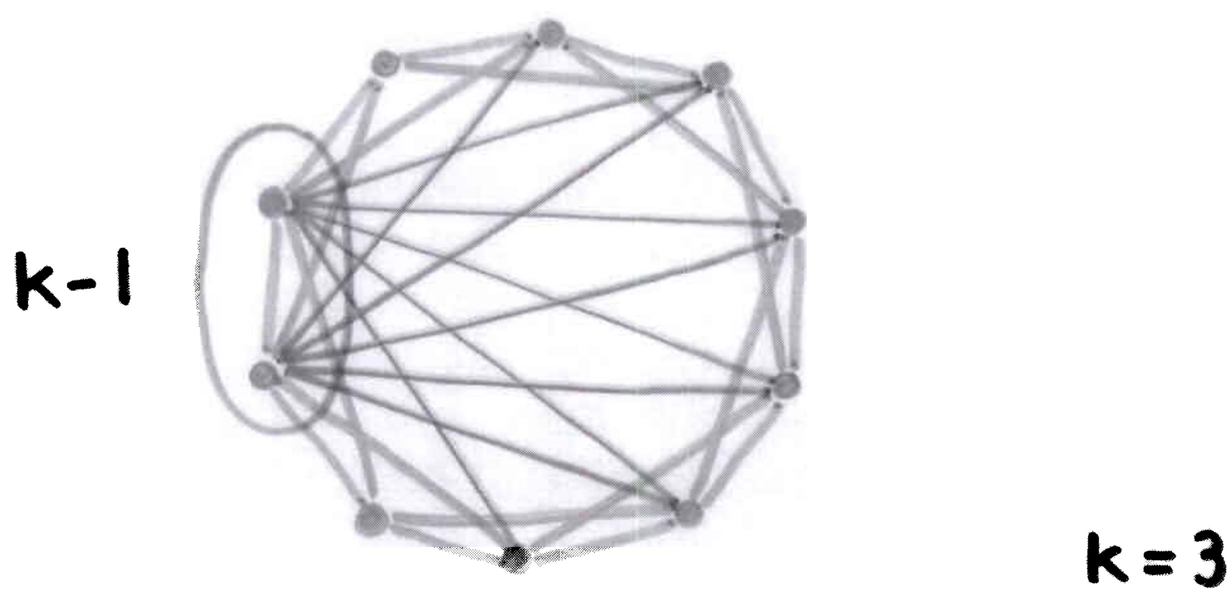
? $O(n)$?

Theorem (Capoyleas-Pach 1992)

For any $n \geq 2k-1$, the maximum number of edges that a convex geometric graph of n vertices can have without containing k pairwise crossing edges is

$$(2k-2)n - \binom{2k-1}{2}$$

Proof Define a partial order on the edges across a fixed chord.



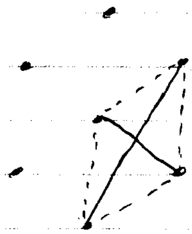
$m=n$

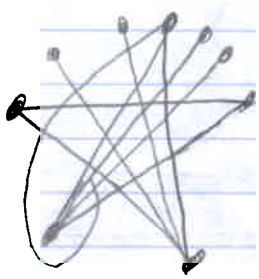
$K_{n,n}$ $\quad cr(K_{n,n}) \sim \frac{n^{4}}{16}$

$\wedge|$

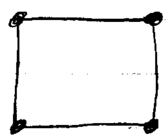
$\underline{\text{lin-cr}}(K_n) \sim c\,n^4$

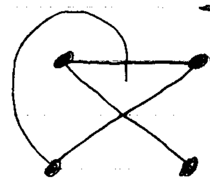
straight-line drawings





Woodall





dead lock

n vertices

$G \not\supseteq C_4$

$\Downarrow$

$$|E(G)| \le \underbrace{cn^{3/2}}_{1.5n}$$

1. $e \sim \overset{\overset{4}{\wedge\!|}}{c}n$

$cr(G) \geq c'n$

$cr(G) \geq e - (3n-6)$

$\boxed{cr(G) > e - 3n}$

2. $e \sim cn^2$

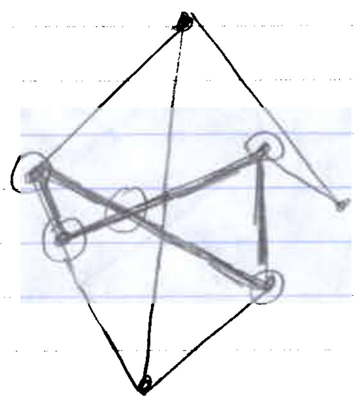
$cr(G) \geq c'n^4$

$K_{n,n}$

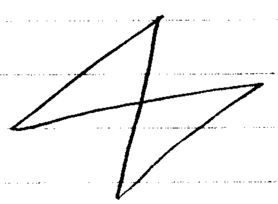
K_n

prob p

G'



$\boxed{cr(G) \geq \frac{1}{33.75} \frac{e^3}{n^2}}$



$2 \longrightarrow k$

$\max_{\substack{G \text{ has} \\ \text{no } k \\ \text{pairwise} \\ \text{disjoint} \\ \text{edges}}} |E(G)| = O(n)$

$< c k^2 n$ Toth

$\boxed{< c k n \text{ ???}}$