COMPLEXITY OF CONVEX BODIES IN HIGHER DIMENSIONS. II

by

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- RECALL: $V_{2k,n}$ IS THE SPACE OF HOMOGENEOUS POLYNOMIALS $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ OF DEGREE $2k$.

THUS $\dim V_{2k,n} = \binom{n+2k-1}{2k}$. WE MAKE $V_{2k,n}$ EUCLIDEAN SPACE BY

$$\langle f, g \rangle = \int_{\mathbb{S}^{n-1}} f(x)g(x)\, dx .$$

THE INVARIANT PROBABILITY MEASURE ON THE UNIT SPHERE

THE SET:

$$\mathrm{Pos}_{2k,n} = \left\{ f: \quad f(x) \geq 0 \text{ FOR ALL } x \in \mathbb{R}^n \text{ AND } \int_{\mathbb{S}^{n-1}} f(x)\,dx = 1 \right\}$$

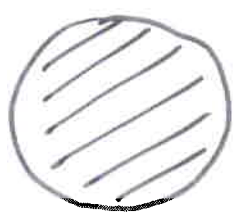
- WHY? WELL, BECAUSE...

- IMPORTANT REMARK: $Pos_{2K,n}$ HAS MANY SYMMETRIES: IF $f \in Pos_{2K,n}$ AND U IS AN ORTHOGONAL TRANSFORMATION OF $\mathbb{R}^n$, THEN $g = Uf$, $g(x) = f(U^{-1}x)$, IS ALSO A POLYNOMIAL FROM $Pos_{2K,n}$.

- THE CASE (WE THINK) WE UNDERSTAND PERFECTLY WELL: $K=1$.

THEN $Pos_{2,n}$ IS THE SET OF POSITIVE SEMIDEFINITE QUADRATIC FORMS $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ OF TRACE n. WE HAVE $\dim Pos_{2,n} = \frac{n(n+1)}{2} - 1$.

IF $n=2$, $Pos_{2,n}$ LOOKS LIKE



WHY?

- THE FACIAL STRUCTURE OF $Pos_{2,n}$.

THE FACES OF $Pos_{2,n}$ ARE PARAMETERIZED BY THE SUBSPACES OF $\mathbb{R}^n$.

$$\text{SUBSPACE } L \longleftrightarrow \text{FACE } F_L = \{ f : f|_L \equiv 0 \}$$

INCLUSION - REVERSING

$$\operatorname{codim} L = n - \dim L \qquad \dim F_L = \binom{\operatorname{codim} L + 1}{2} - 1$$

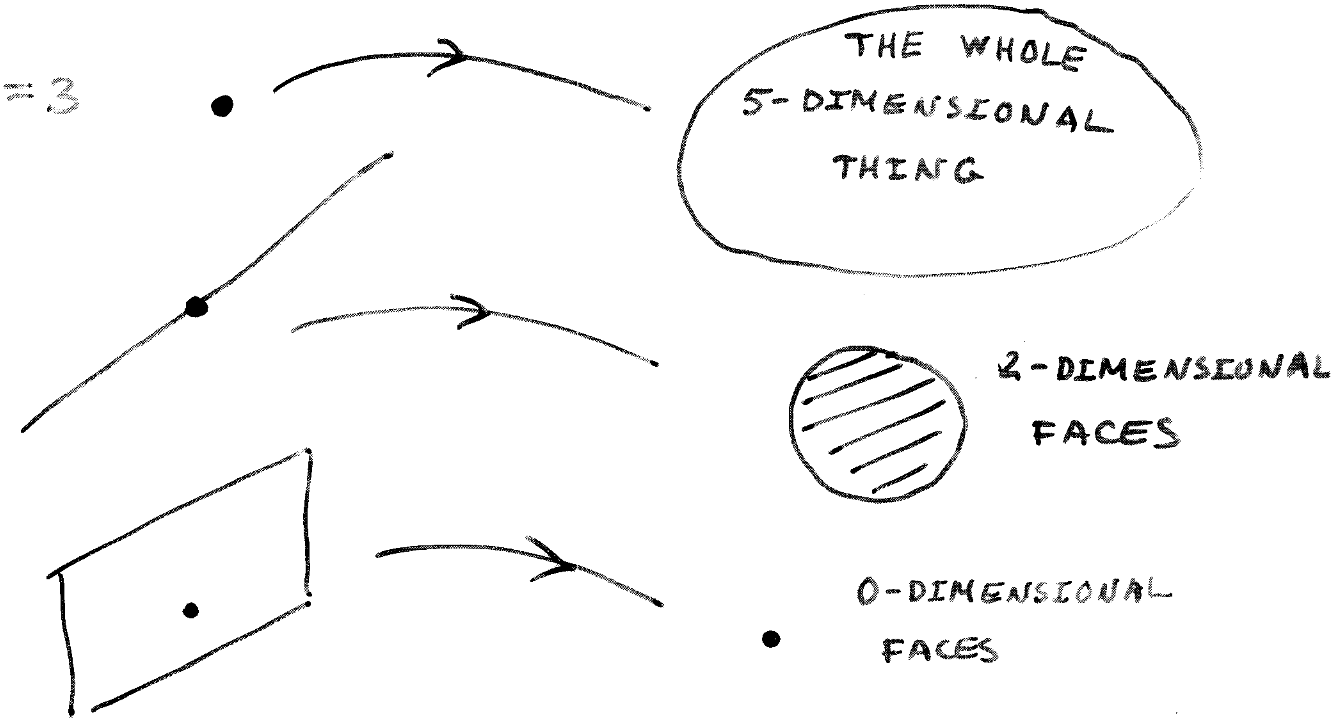
FACE F_L "LOOKS LIKE" $Pos_{2,m}$

FOR $m = \operatorname{codim} L$

THE DIMENSIONS OF THE FACES ARE

$0, 2, 5, 9, 14 \ldots$ MOST DIMENSIONS ARE MISSING.

IF $n = 3$



- SOME SURPRISES: FOR LARGE n, A COUNTEREXAMPLE TO BORSUK'S CONJECTURE [J. KAHN, G. KALAI]

- NOTE: THE EXTREME POINTS ARE FORMS OF RANK 1 ≡ SQUARES OF LINEAR FORMS

HILBERT: WHEN EVERY $f \in Pos_{2K,n}$ IS A SUM OF SQUARES OF POLYNOMIALS OF DEGREE K?

IF AND ONLY IF:

a.) $K=1$: QUADRATIC FORMS

b) $n=2$: BIVARIATE POLYNOMIALS

c.) $K=2$, $n=3$: TERNARY QUARTICS

EXAMPLES:

$$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \quad \text{[T. MOTZKIN]}$$

$$w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \quad \text{[M.-D. CHOI AND T.-Y. LAM]}$$

NON-NEGATIVE POLYNOMIALS WHICH ARE NOT SUMS OF SQUARES

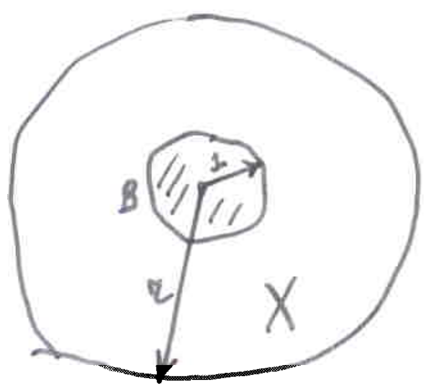
- QUESTION: HOW MANY ARE SUMS OF SQUARES?

- A USEFUL NOTION: THE VOLUME RATIO

LET V BE A d-DIMENSIONAL EUCLIDEAN SPACE, LET $B \subset V$ BE A UNIT BALL, AND LET X BE A SET.

THE VOLUME RATIO FOR X:

$$\left(\frac{\text{VOL } X}{\text{VOL } B} \right)^{\frac{1}{d}}$$



VOLUME RATIO IS r

THREE SETS

$Pos_{2k,n}$ NON-NEGATIVE POLYNOMIALS OF DEGREE $2k$ IN n VARIABLES;

$\Sigma_{2k,n}$ SUMS OF SQUARES OF POLYNOMIALS OF DEGREE k;

$Q_{2k,n}$ SUMS OF $2k$-POWERS OF LINEAR FUNCTIONS

NORMALIZED SO THAT THE AVERAGE VALUE ON THE UNIT SPHERE $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ IS 1.

WE HAVE

$$Q_{2k,n} \subset \Sigma_{2k,n} \subset Pos_{2k,n}$$

DIMENSION $d(2k,n) = \binom{n+2k-1}{2k} - 1$.

FOR $k=1$ ALL THREE COINCIDE.

$\Sigma_{2k,n}$ IS "SIMPLE": TESTING MEMBERSHIP TO $\Sigma_{2k,n}$ IS "POLYNOMIAL TIME" (FIXED k, n GROWS). " ... " BECAUSE SEMIDEFINITE PROGRAMMING ALLOWS US TO TEST IN $\mathrm{poly}(\log \varepsilon^{-1}; n)$ TIME WHETHER A POLYNOMIAL IS ε-CLOSE TO $\Sigma_{2k,n}$

CAUTION: TYPICALLY, TO BE SURE, WE NEED ε DOUBLY EXPONENTIAL IN n.

$Q_{2k,n}$ AND $Pos_{2k,n}$ ARE "HARD"; THEY ARE (ESSENTIALLY) POLARS OF EACH OTHER.

THEOREM [G. BLEKHERMAN] LET US FIX $K \geq 1$.

LET $B_{2K,n}$ BE THE UNIT BALL IN THE HYPERPLANE $\int_{S^{n-1}} f(x)\,dx = 1$. THEN

① $$\left(\frac{\operatorname{vol} Pos_{2K,n}}{\operatorname{vol} B_{2K,n}}\right)^{\frac{1}{d(2K,n)}} \geq c_0(K)\, n^{-1/2}$$

FOR SOME $c_0(K) > 0$

② $$c_1(K)\, n^{-K/2} \geq \left(\frac{\operatorname{vol} \Sigma_{2K,n}}{\operatorname{vol} B_{2K,n}}\right)^{\frac{1}{d(2K,n)}} \geq c_2(K)\, n^{-K/2}$$

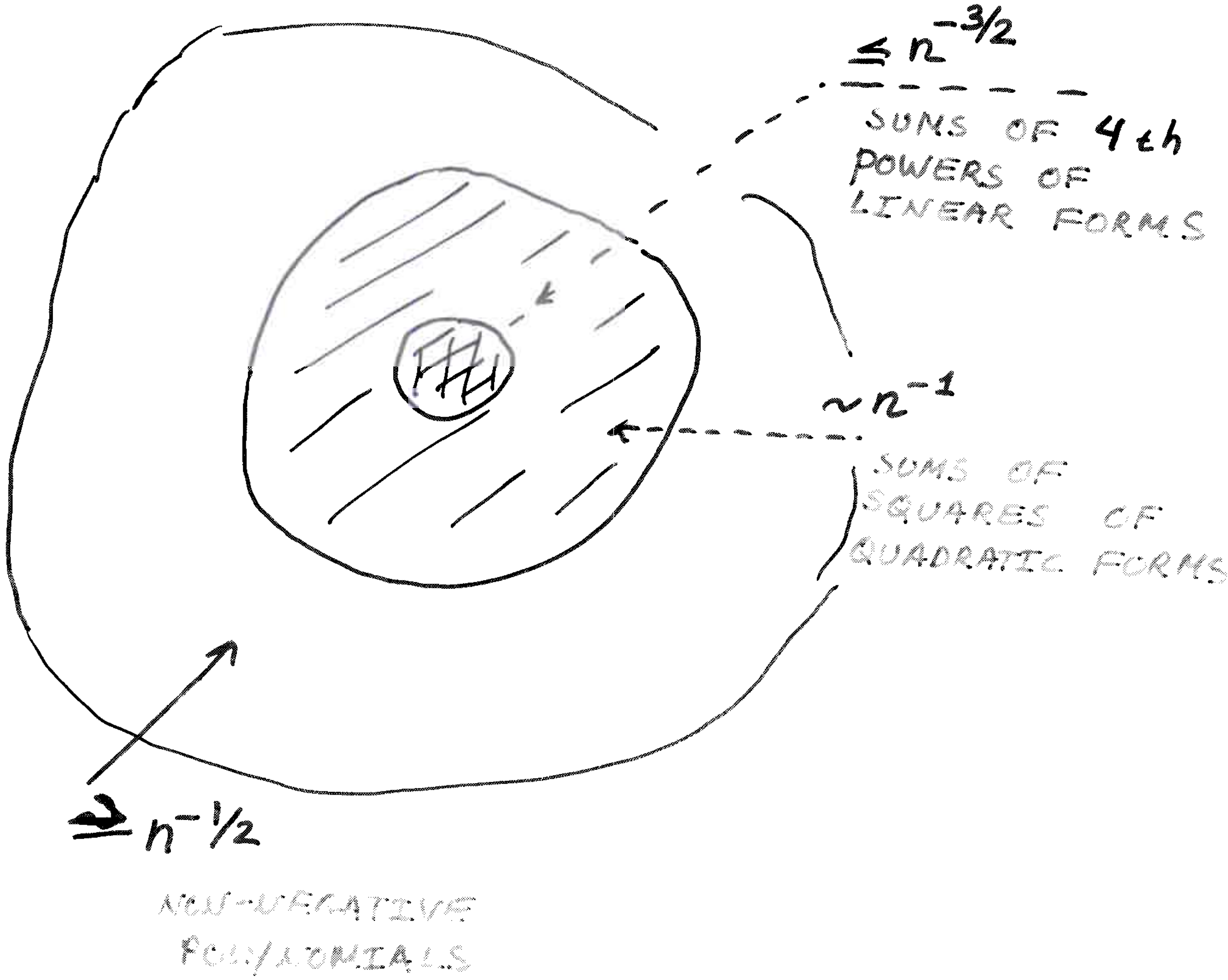
FOR SOME $c_1(K), c_2(K) > 0$

③ $$\left(\frac{\operatorname{vol} Q_{2K,n}}{\operatorname{vol} B_{2K,n}}\right)^{\frac{1}{d(2K,n)}} \leq c_3(K)\, n^{\frac{1}{2}-K}$$

FOR SOME $c_3(K) > 0$

A PICTURE'S WORTH A THOUSAND WORDS...

LET $K=2$, SO WE ARE TALKING ABOUT POLYNOMIALS OF DEGREE 4.



IT WOULD BE INTERESTING TO RECONCILE THIS PICTURE WITH THE SUCCESS OF SEMIDEFINITE PROGRAMMING METHODS IN POLYNOMIAL OPTIMIZATION. THE EXPLANATION MAY LIE WITH THE CONSTANTS $C_0(K)$, $C_2(K)$

THEOREM. [B. REZNICK] LET $f \in Pos_{2k,n}$ BE A POSITIVE POLYNOMIAL, THAT IS, $f(x) > 0$ FOR ALL $x \neq 0$. THEN THERE EXISTS $m \geq k$ SUCH THAT

$$(x_1^2 + \ldots + x_n^2)^{m-k} f(x) = \underbrace{\sum_i \langle c_i, x \rangle^{2m}}_{\text{SUM OF POWERS OF LINEAR FORMS}}.$$

FOR SOME $c_i \in \mathbb{R}^n$.

- NOTE: THE RESULT IS CLEARLY FALSE FOR NON-NEGATIVE f (LOOK AT THE SET OF ZEROS). THEREFORE, m WILL GROW AS f APPROACHES THE BOUNDARY OF $Pos_{2k,n}$.

- GOAL: TO GIVE A GEOMETRIC EXPLANATION OF THE THEOREM AND TO ESTIMATE m WHICH SERVES "ALMOST ALL" f IN THE VOLUME RATIO SENSE.

- EXERCISE: DEDUCE POLYA'S THEOREM:

  IF $f(x) > 0$ FOR ALL $x_1 > 0, \ldots, x_n > 0$ AND $\deg f = k$ THEN FOR SOME $m \geq k$ $(x_1 + \ldots + x_n)^{m-k} f$ HAS NON-NEGATIVE COEFFICIENTS.

POWERS AND DELTA-FUNCTIONS:

LET US CHOOSE $c \in S^{n-1}$ AND LET

$$q_c(x) = q_c(x; m) = \frac{\sqrt{\pi}\,\Gamma(m+0.5n)}{\Gamma(0.5n)\,\Gamma(m+0.5)} \langle c, x\rangle^{2m}$$

NORMALIZED IN SUCH A WAY THAT $\int_{S^{n-1}} q_c(x)\,dx = 1$

INVARIANT PROBABILITY MEASURE

- OBSERVATION. FOR LARGE m, $q_c(x;m)$ ARE STARTING TO RESEMBLE δ-FUNCTIONS.

$$\lim_{m \to +\infty} \int_{S^{n-1}} f(x)\, q_c(x;m)\, dx = f(c)$$

FOR A CONTINUOUS f

- THE (FIRST) OPERATOR

$$T_m : \quad f \longmapsto \int_{S^{n-1}} f(c)\, q_c(x;m)\, dc$$

- PROPERTIES:

1. IF f IS NON-NEGATIVE, $T_m(f)$ IS A NON-NEGATIVE COMBINATION OF POWERS OF LINEAR FORMS;
2. IF $q_c(x;m)$ WERE DELTA-FUNCTIONS, T_m WOULD HAVE BEEN THE IDENTITY
3. T_m COMMUTES WITH THE ACTION OF THE ORTHOGONAL GROUP

SOME REPRESENTATION THEORY

LET $V_{2k,n}$ BE THE SPACE OF HOMOGENEOUS POLYNOMIAL OF DEGREE $2k$ IN n VARIABLES. LET US INTRODUCE THE SCALAR PRODUCT

$$\langle f, g \rangle = \int_{S^{n-1}} f(x) g(x)\, dx$$

THE ORTHOGONAL GROUP $O(n)$ ACTS IN $V_{2k,n}$:

$(Uf)(x) = f(U^{-1}x)$. THE REPRESENTATION IS REDUCIBLE. HERE IS HOW:

LET $H_{2\ell,n}$ BE THE SPACE OF HOMOGENEOUS HARMONIC POLYNOMIALS OF DEGREE 2ℓ:

f IS HARMONIC IF $\left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right) f \equiv 0$.

$$\dim H_{2\ell,n} = \frac{(4\ell+n-2)(2\ell+n-3)!}{(2\ell)!\,(n-2)!}$$

LET $r^{2m} = (x_1^2 + \dots + x_n^2)^m$.

THEN

$$V_{2k,n} = \bigoplus_{\ell=0}^{k} r^{2k-2\ell} H_{2\ell,n}.$$

PAIRWISE NON-ISOMORPHIC IRREDUCIBLE COMPONENTS.

LET US CONSIDER THE OPERATOR T_m APPLIED TO POLYNOMIALS $f \in V_{2k,n}$:

$$T_m(f) = \int_{S^{n-1}} f(c)\, q_c(x;m)\, dc$$

WE HAVE $\underbrace{T_m(f)}_{\substack{\text{A POLYNOMIAL} \\ \text{FROM } V_{2m,n}}} = T_m(\underbrace{r^{2m-2k} f}_{\substack{\text{A POLYNOMIAL FROM} \\ r^{2m-2k} V_{2k,n}}})$

- BUT T_m COMMUTES WITH THE ACTION OF THE ORTHOGONAL GROUP, HENCE IT MUST RESPECT THE IRREDUCIBLE DECOMPOSITION

  HENCE WE GET

- THE SECOND AND THE LAST OPERATOR:

  WE MUST HAVE

$$T_m(f) = r^{2m-2k}\, h \quad \text{FOR}$$

SOME $h \in V_{2k,n}$.

THUS WE GET :

$$T_{m,k} : V_{2k,n} \longrightarrow V_{2k,n}$$

$$T_{m,k}(f) = h \, .$$

SOME OBVIOUS PROPERTIES OF $T_{m,K}$

① $T_{m,K} : \underbrace{V_{2K,n} \longrightarrow V_{2K,n}}$

POLYNOMIALS OF DEGREE $2K$
INTO POLYNOMIALS OF DEGREE $2K$

ALSO, IT PRESERVES THE AVERAGE VALUE ON S^{n-1}

② $T_{m,K}$ COMMUTES WITH THE ACTION OF THE ORTHOGONAL GROUP

③ If $f(x) \geq 0$ FOR ALL $x \in \mathbb{R}^n$
THEN $r^{2m-2K} T_{m,K}(f)$ IS A NON-NEGATIVE LINEAR COMBINATION OF $2m$-POWERS OF LINEAR FORMS.

④ $T_{m,K} \longrightarrow$ IDENTITY AS $m \to +\infty$,
K FIXED

CONCLUSION: If f IS POSITIVE
($f(x) > 0$ FOR ALL $x \neq 0$), THEN

$(x_1^2 + \dots + x_n^2)^{m-K} f$ IS A SUM OF $2m$-POWERS OF LINEAR FORMS FOR A SUFFICIENTLY LARGE m.

REASON: THE SET OF POSITIVE POLYNOMIALS IS OPEN + ③ + ④

SOME NON-OBVIOUS PROPERTIES OF $T_{m,k}$.

G. BLEKHERMAN COMPUTED THE EIGENVALUES OF $T_{m,k}$ AND DEDUCED THE FOLLOWING

COROLLARY. LET $Q_m \subset Pos_{2k,n}$ BE THE SET OF POLYNOMIALS p SUCH THAT $r^{2m-2k} p$ IS A NON-NEGATIVE LINEAR COMBINATION OF $2m$ POWERS OF LINEAR FORMS. THEN, FOR ANY $\varepsilon > 0$ AND ANY $m \geq (2k^2 + kn)/\varepsilon$ WE HAVE

$$\left(\frac{\mathrm{vol}\ Q_m}{\mathrm{vol}\ Pos_{2k,n}} \right)^{\frac{1}{d(2k,n)}} \geq 1 - \varepsilon ,$$

WHERE $d(2k,n) = \binom{n+2k-1}{2k} - 1$ IS THE DIMENSION OF $Pos_{2k,n}$.

QUESTION. WHAT ABOUT A SIMILAR ESTIMATES FOR POLYNOMIALS, WHICH, AFTER BEING MULTIPLIED BY r^{2m-2k} BECOME SUMS OF SQUARES?