Algebraic topology applied in geometry

by I. Bárány

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survey by Björner
and by Zivaljević

book by Matoušek

history

examples

methods

problems

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equivariant topology

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combinatorial applications

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apologies

<u>Helly's thm.:</u> $\mathcal{F}$ is a family of convex compact sets in R^d, every $d+1$ or fewer intersect $\Rightarrow$ all intersect

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<u>Topological version:</u> $\mathcal{F}$ is a family of compact sets in R^d. Every $d+1$ or fewer have contractible non-void intersection $\Rightarrow$ all intersect

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<u>Molnár (1951)</u> $\mathcal{F}$ is a family of compact sets in R^2, each connected, any two intersect in a connected set, every 3 intersect

$\Rightarrow$ all intersect

Borsuk's thm (1933)

(1) $\forall f : S^n \to R^n$ continuous,
$\exists z \in S^n$ with $f(z) = f(-z)$

(2) $\forall f : S^n \to R^n$ antipodal, $\left(\begin{array}{l} f(x) = -f(-x) \\ \forall x \in S^n \end{array} \right)$
$\exists z \in S^n$ with $f(z) = 0$

(3) There is no $f : S^n \to S^{n-1}$ antipodal

(4) There is no $f : B^n \to S^{n-1}$
antipodal on ∂B^n

(5) Ljusternik-Shnirelman (1930)
$F_1, \dots, F_{n+1} \subset S^n$ closed sets
$S^n = \bigcup F_i \Rightarrow$ some F_i contains
an antipodal pair

(6) $f : S^n \to S^n$ antipodal $\Rightarrow$ deg f is odd

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(4) implies Brouwer's fixed pnt thm:

If $g : B^n \to B^n$ had no fixed point, then ...

A simple proof of Borsuk's thm [form (2)] :

Applications of Borsuk's thm

Ham-sandwich thm (Banach)

If $\mu_1, \dots, \mu_d$ are (nice) prob. measures on $\mathbb{R}^d$, then there is a hyperplane H with

$$\mu_i(H^+) = \mu_i(H^-) \qquad (\forall i)$$

Theorem (Gale 1961).

$C, C' \subset \mathbb{R}^d$ convex

$f : C \to C'$ Lipschitz with

$$\| f(x) - f(y) \| \le k \, \| x - y \| \quad \forall x, y$$

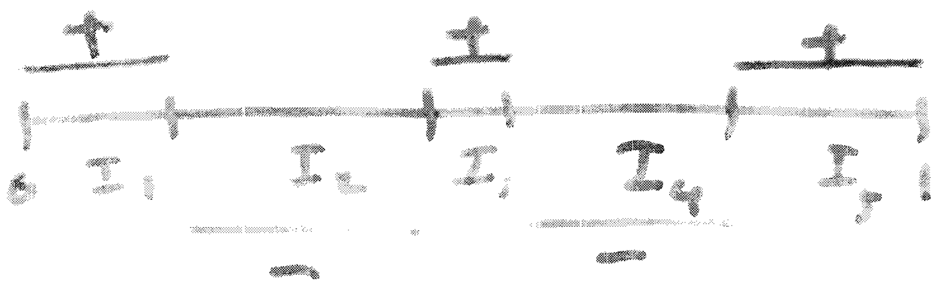
$\Rightarrow$ width of C $\le k$ width of C'

Besicowitch

Hobby, Rice (1965) $\mu_1, \ldots, \mu_d$ (nice) prob. measures on $[0,1] \Rightarrow$

$\Rightarrow \exists$ partition $[0,1] = I_1 \cup I_2 \cup \ldots \cup I_{d+1}$ (each I_j an interval) and $\varepsilon_j = \pm 1$

with
$$\sum_{j=1}^{d+1} \varepsilon_j \, \mu_i(I_j) = 0 \quad (\forall i).$$



an almost trivial proof:

another proof based on a map

$S^d \longrightarrow \{$ partitions of $[0,1]$ into $d+1$ intervals $\}$

Another form (Goldberg – West 1985, Alon – West 1986): An open necklace with $\underline{d}$ kinds of stones and two thieves $\Rightarrow$ the necklace can be cut at $\underline{d}$ points and the pieces rearranged into two equal groups.

What happens in case of $t > 2$ thieves?

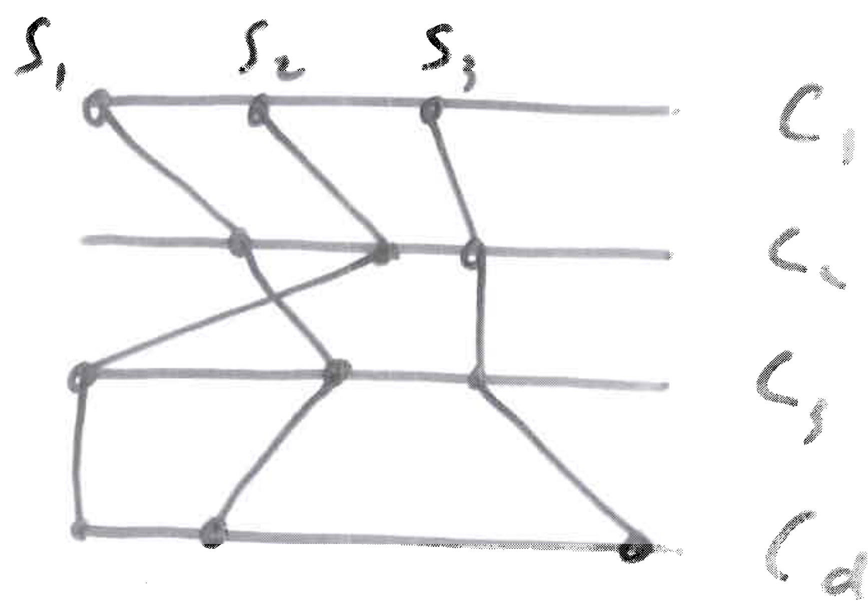
$d(t-1)$ cuts suffice (Alon 86)

Alon - Akiyama (1985):

$C_1, \dots, C_d \subset \mathbb{R}^d$, $|C_i| = n \ \forall i$ (colours) $\Longrightarrow \exists \ S_1, \dots, S_n \subset \mathbb{R}^d$

$|S_j| = d \ (\forall j)$ $|S_j \cap C_i| = 1 \ (\forall i j)$

such that the sets $\operatorname{conv} S_j$ are pairwise disjoint.



proof:

Center point thm (Rado 1947):

μ a prob. measure on $\mathbb{R}^d \Rightarrow$

$\Rightarrow \exists\, c \in \mathbb{R}^d$ (centerpoint), that is,

for each halfspace $H \ni c$

$$\mu(H) \geq \frac{1}{d+1}.$$

Helly or Brouwer imply it

Thm (Živaljević–Vrećica '90, Dol'nikov '92): $k \in \{1, \dots, d\}$, $\mu_1, \dots, \mu_k$ prob measures on $\mathbb{R}^d \Rightarrow$ $\exists$ a $(k-1)$-dim affine flat F such that

$$\mu_i(H) \geq \frac{1}{d-k+2}$$ for each halfspace H containing F.

$k=1$ centerpt. $k=d$ Ham-sandwich

Radon's theorem: $A \subset \mathbb{R}^d$, $|A| = d+2 \Rightarrow \exists$ partition $A = B \cup C$ $(B \cap C = \emptyset)$ with

$$\operatorname{conv} A \cap \operatorname{conv} B \neq \emptyset.$$

Δ^{d+1} is the $(d+1)$-dim simplex

Radon's thm: $f : \Delta^{d+1} \to \mathbb{R}^d$ linear map $\Rightarrow \exists$ disjoint faces, B and C, of Δ^{d+1} with

$$f(B) \cap f(C) \neq \emptyset.$$

Thm (Bajmóczy, B. 1979)

true for any continuous map

$$f : \Delta^{d+1} \to R^d$$

Proof: phase space

target space

test map

Tverberg's thm ('66): $N = (r-1)(d+1)$

$f : \Delta^N \to \mathbb{R}^d$ linear $\Rightarrow$

$\exists$ pairwise disjoint faces

$F_1, \dots, F_r$ of Δ^N with

$$\bigcap_{i=1}^{r} f(F_i) \neq \emptyset.$$

Thm (B. Shlosman, Szücs '81):

same holds for continuous

$f : \Delta^N \to \mathbb{R}^d$ if r is prime

Lemma. Δ^* is $[d(r-1)-1]$-conn.

Lemma. (Krasnoselśky-Zabrejko '75)

$\mathbb{Z}_p$ acts freely on S^k

$\omega : S^k \to S^k$ is its action

$g : S^k \to S^k$ is ω-equivariant

$$\begin{tikzcd} S^k \arrow[r, "g"] \arrow[d, "\omega"'] & S^k \arrow[d, "\omega"'] \\ S^k \arrow[r, "g"] & S^k \end{tikzcd}$$ commutes

Then $\deg g \equiv 1 \mod p$.

Borsuk ...

Continuous Tverberg:

r prime or prime power

Özaydin, 1987
Volovikov, 1996
Sarkaria 2000

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open for other r

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better proof by Sarkaria

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applied to k thieves
one necklace

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open: # Tverberg partitions $\geq ((r-1)!)^d$

Thm (Dold 1983)

$\mathbb{Z}_p$ acts on X and Y

X is n-connected

$\dim Y \leq n$, $\mathbb{Z}_p$ acts freely on Y

$\Longrightarrow$ there is no $X \to Y$

$\mathbb{Z}_p$-equiv. map.

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other groups as well

Coloured Tverberg.

given $d \geq 1$, $r \geq 2$ and

$C_1, \dots, C_{d+1} \subset \mathbb{R}^d$, $|C_i| = t \ (\forall i)$

? Can you find sets ?

$S_1, \dots, S_r$ with $|S_j| = d+1$ $(\forall j)$

$|S_j \cap C_i| = 1 \quad \forall i, j$

such that

$$\bigcap_{1}^{r} \operatorname{conv} S_j \neq \emptyset$$

if $t = t(r, d)$ is large enough?

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<u>Claim</u> (B. Larman) $t(r,2) = r$

(Lovász) $t(2,d) = 2$

<u>Thm</u> (Živaljević Vrećica '92)

$\forall r \geq 2$, $\forall d \geq 1$ <u>$t(r,d)$ is finite:</u>

$t(r,d) \leq 2r-1$ for r prime

implying

$t(r,d) \leq 4r-1$ $(\forall r)$

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only topological proofs

p a prime

$C_1, \dots, C_{d+1} \subset \mathbb{R}^d$ pairwise disjoint

$|C_i| = 2p-1 \quad \forall i$

Simplicial complex K:

vertex set $C_1 \cup \dots \cup C_{d+1}$

$\sigma \in K$ iff $|\sigma \cap C_i| \leq 1 \ \forall i$

Thm (Z-V) (top. col. Tverberg)

If $f: |K| \to \mathbb{R}^d$ cont., then

$\exists$ pairwise disjoint $\sigma_1, \dots, \sigma_p \in K$

with $$\bigcap_{i=1}^{p} f(|\sigma_i|) \neq \emptyset.$$

Equipartition of measures by hyperplanes

- Ham-sandwich
- $A \subset R^3$ is measurable $\Rightarrow$ $\exists$ 3 hyperplanes such that each orthant (cell) contains $\frac{1}{8}$ of A
- Given m prob measures in R^d, do there exist k hyperplanes, that each cell has measure $\frac{1}{2^k}$ in each measure?

Ramos ('96) $d \geq m(2^k - 1)/k$ is necessary

$d \geq m\,2^{k-1}$ is suffisient

$m=1$ $d=k=4$ open

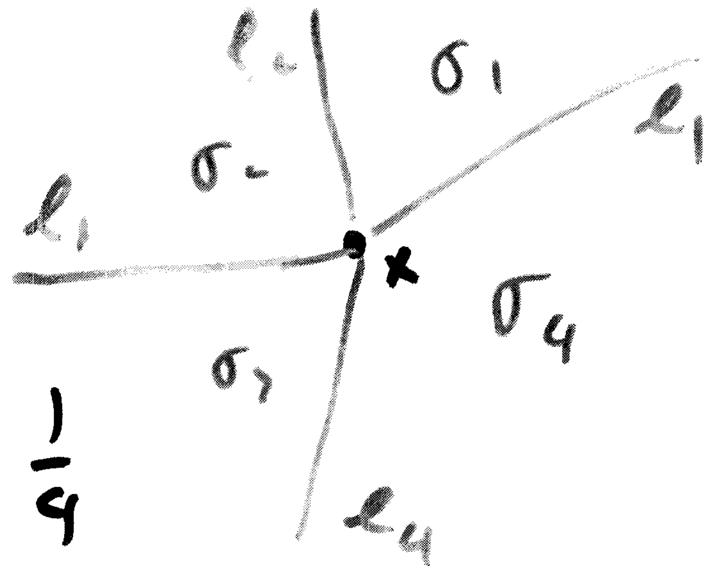
Equipartition of two measures by a 4-fan

λ, μ prob measures on R^2

find a 4-fan with

$\lambda(\sigma_i) = \mu(\sigma_i) = \frac{1}{4}$

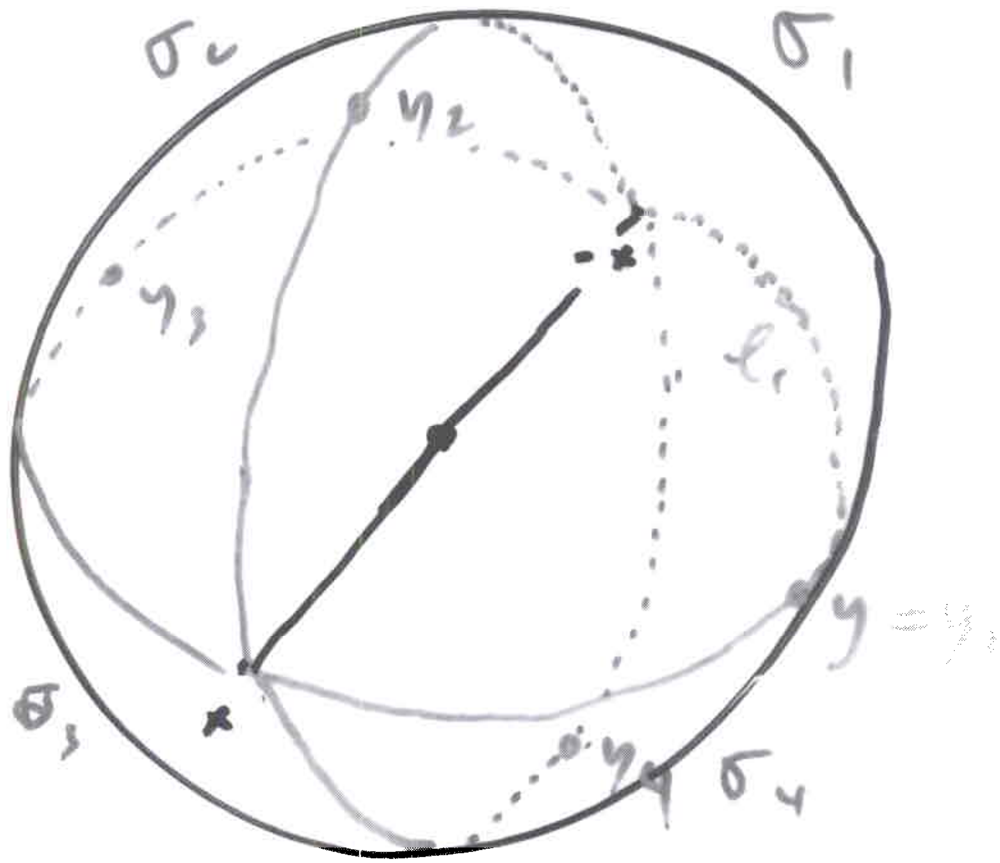
$(\forall i)$.



better to work on S^2!

λ, μ prob. measures on S^2

λ, μ on S^2



phase space

$$V = \{(x, y) \in S^2 \times S^2 : x \perp y\}$$

$$\begin{array}{ll} (x, y_1) & \\ (x, y_2) & \text{via } \lambda(\sigma_1) = \frac{1}{4} \\ (x, y_3) & \text{via } \lambda(\sigma_2) = \frac{1}{4} \\ (x, y_4) & \text{via } \lambda(\sigma_3) = \frac{1}{4} \end{array}$$

$$f : V \longrightarrow \Delta^3$$

$$f(x,y) = \big(\mu(\sigma_1), \mu(\sigma_2), \mu(\sigma_3), \mu(\sigma_4)\big) \in \Delta^3$$

Target: f takes the value

$$c = \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$$

$\mathbb{Z}_4$ acts on V: $(x,y) \xrightarrow{\omega} (x, y_2)$

— " — Δ^3: $(t_1, t_2, t_3, t_4) \xrightarrow{\omega} (t_2, t_3, t_4, t_1)$

$\mathbb{Z}_2$ acts on V: $(x,y) \xrightarrow{\nu} (-x, y)$

— " — Δ^3: $(t_1, t_2, t_3, t_4) \xrightarrow{\nu} (t_4, t_3, t_2, t_1)$

f is equivariant

$D_4 = \langle w, v \rangle$ dihedral group

$f : V \to \Delta^3$ is D_4-equiv.

$D_2 = \langle w^2, v \rangle$ $\quad Z_4 = \langle w \rangle$

Fact 1. (B. Matoušek 2001)

$\exists\, f : V \to \Delta^3 \setminus \{c\}$ D_2-equiv map

Fact 2. (B. Matoušek 2002)

such a map cannot come from two measures

(implies....)

Fact 3. (Živaljević, Vrećica 2003)

There is no $f : V \to \Delta^3 \setminus \{c\}$ D_4-equiv map.

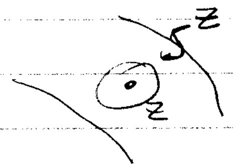
Bárány

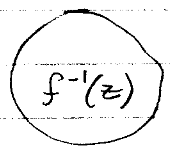
$$\begin{array}{lll} F_1, \dots, & F_{d+1} & \text{families of convex sets in } \mathbb{R}^d \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\ K_1, \dots, & K_{d+1} & \text{they intersect} \end{array}$$

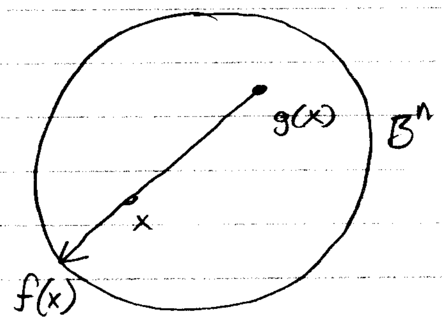
$\Rightarrow$ for some i, F_i is intersecting.

$S^n \to S^n$

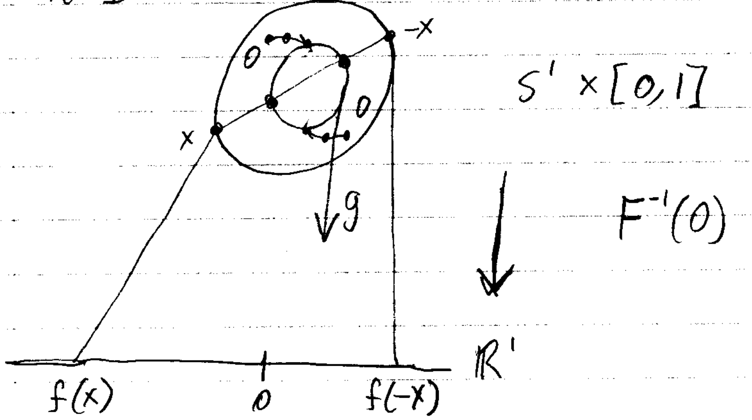
$f^{-1}(\underset{z}{\rotatebox{90}{$\in$}})$





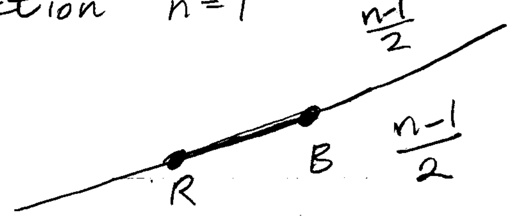


$n = 1$

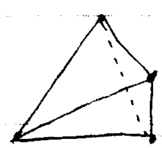


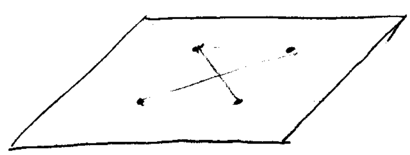
$d=2$ induction $n=1$

n odd



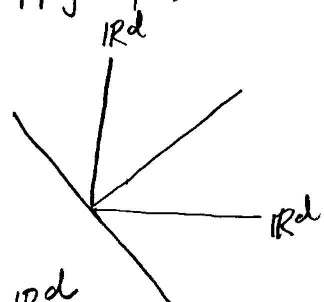
n even





$$\Delta^* = \{(x,y) \in \Delta^{d+1} \times \Delta^{d+1} : \operatorname{supp} x \cap \operatorname{supp} y = \emptyset\}$$

$$f^* : \Delta^* \longrightarrow \mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{Diag} \cong S^{d-1} \circlearrowleft$$



$$f^*(x,y) = (f(x), f(y))$$

$$\mathrm{Diag} = \{(z,z) : z \in \mathbb{R}^d\}$$

Extra $\underset{\omega}{\mathbb{Z}_2}$ acts on Δ^* and on $\mathbb{R}^d \times \mathbb{R}^d$

$\omega(x,y) = (y,x)$ $\omega(u,v) = (v,u)$

$$
\begin{tikzcd}
S^d \arrow[r, "\text{imbed}"] \arrow[d, "-"'] & \Delta^* \arrow[r, "f"] \arrow[d, "\omega"] & S^{d-1} \arrow[d, "-"] \\
S^d \arrow[r, "\text{imbed}"] & \Delta^* \arrow[r, "f"] & S^{d-1}
\end{tikzcd}
$$

$$\Delta^{d+1} - \Delta^{d+1}$$