Before going further, we note that it is natural, when defining an inner product on the C^p to declare the simplices to be orthogonal. With this stipulation, an inner product is completely defined by the choice of a positive weight $w(\alpha)$ for each simplex α. So that, for any cochains c_1 and c_2 we set

$$\langle c_1, c_2 \rangle = \sum_{\alpha} w(\alpha) c_1(\alpha) c_2(\alpha).$$

We now present two different combinatorial analogues of the Bochner-Weitzenbock idea.

The first is (a very special case of) an approach due to Garland, as expanded by Borel. (The precise result we present is due to Wang.) [Zuk, Pansu,Ballman- Swiatkowski, Dymara-Januszkiewicz]

Let M be a connected finite m–dimensional simplicial complex with the property that the link of every vertex is connected. Assign weights by setting

$$w(\alpha^{(p)}) = m(m-1)(m-2)\dots(m-p) \quad [\#\ of\ m-simplices\ which\ contain\ \alpha].$$

These weights have the nice property that for any p–simplex α,

$$w(\alpha) = \sum_{\beta^{(p+1)}>\alpha} w(\beta).$$

For any vertex v, the weights w induce a set of weights w_v on $link(v)$, by setting for any simplex β of $link(v)$

$$w_v(\beta) = w(v * \beta).$$

Theorem: For any vertex v, let $\kappa(v)$ denote the smallest positive eigenvalue of $\Box_0(link(v), w_v)$, and $\kappa = \inf_v \kappa(v)$. Let λ_1 denote the smallest positive eigenvalue of $\Box_0(M, w)$. Then

$$\lambda_1 \geq 2 - \frac{1}{\kappa}.$$

Garland studied spaces with a lot of symmetry (Tits buildings) and the restrictions on the set of weights is unnecessary in that setting. Moreover, he proved similar theorems in every dimension. By estimating the local curvatures $\kappa(v)$ he was able to prove beautiful vanishing theorems for the cohomology of p –adic Lie groups.

For general weights

$$F_p(\alpha) =$$

$$w(\alpha)\left[\sum_{\beta^{(p+1)}>\alpha}\frac{w(\alpha)}{w(\beta)} + \sum_{\gamma^{(p-1)}<\alpha}\frac{w(\gamma)}{w(\alpha)}\right.$$

$$\left. - \sum_{\tilde{\alpha}^{(p)}\neq\alpha}\left|\sum_{\substack{\beta^{(p+1)}>\alpha \\ \beta>\tilde{\alpha}}}\frac{\sqrt{w(\alpha)w(\tilde{\alpha})}}{w(\beta)} - \sum_{\substack{\gamma^{(p-1)}<\alpha \\ \gamma<\tilde{\alpha}}}\frac{w(\gamma)}{\sqrt{w(\alpha)w(\tilde{\alpha})}}\right|\right]$$

functions of Curvature which Integrate
[redacted]
to give topological invariants

If M is a compact triangulated n-dimensional manifold, then

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \#\{i - dimensional\ simplices\}.$$

If M is a compact Riemannian n-manifold, then
Theorem: (Allendoerfer-Fenchel-Weil-Chern-Gauss-Bonnet):

$$\chi(M) = \int_M Pf(R/2\pi)\ d(vol)$$

where $Pf(R)$ denotes the Pffafian of the curvature operator

That is, fixing a point $m \in M$, and $X, Y \in T_m M$, we get a skew-symmetric map

$$R(X,Y) : T_m M \to T_m M,$$

by

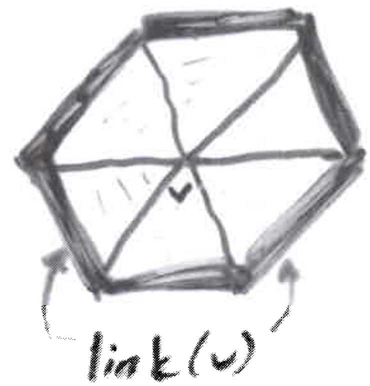
$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where ∇ denotes the Levi-Civita connection, and, in the right hand side, X and Y actually refer to any extension of the vectors X and Y to a neighborhood of m.

The upshot of all of this is:
One can think of the curvature operator as a skew-symmetric matrix of 2-forms.

$Pf(R)$ denotes the Pfaffian of this matrix.

Now let us make the combinatorial formula look more like the Riemannian formula;



$$\chi(M) = \sum_{vertices\ v} [1 + \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\#\{i - simplices\ in\ link(v)\}}{i+2}].$$

Now, for any finite simplicial complex S, define

$$E(S) = 1 + \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\#\{i - simplices\ in\ S\}}{i+2}.$$

So that for any triangulated manifold M

$$\chi(M) = \sum_{vertices\ v} E(link(v)).$$

The next development came from the differentiable side: Stieffel, Whitney, Pontryagin, Chern, Weil.

Theorem (Chern-Weil): Write

$$det(I + \frac{t}{2\pi}R) = 1 + t\sigma_1(R) + t^2\sigma_2(R) + \ldots.$$

For each i, $\sigma_i(R)$ is a $2i$-form.
(i) If i is odd, then $\sigma_i(R) = 0$.
(ii) For each k, $\sigma_{2k}(R)$ is a closed $4k$-form. Moreover, the cohomology class of $\sigma_{2k}(R)$ is independent of the Riemannian metric, i.e. depends just on the underlying differentiable structure.

The cohomology class represented by $\sigma_{2k}(R)$ is called *the k'th Pontryagin class of M* and is denoted $p_k(M)$.

So ...

Differentiable Structure ⇝ Characteristic Classes

Riemannian Metric ⇝ Canonical "local" representatives

Characteristic Numbers: Suppose that M is an oriented manifold of dimension $4n$. Let $r = r_1, r_2, r_3 \ldots$ be any sequence of nonnegative integers such

$$n = r_1 + 2r_2 + 3r_3 + \ldots.$$

Then we can consider

$$P_r(M) = \int_M p_1^{r_1}(M) \cup p_2^{r_2}(M) \cup p_3^{r_3} \cup \ldots.$$

The function P_r is called a Pontryagin number. More generally, any linear combination of the P_r's is a Pontryagin number. The Pontryagin numbers are independent of the Riemannian metric, and hence depend only on the underlying differentiable structure.

Theorem: (Hirzebruch) The signature of M is a Pontryagin number.

That is, there exist universal constants c_r so that for any oriented $4n$-dimensional manifold M,

$$signature(M) = \sum_r c_r P_r(M).$$

Note that for any Pontryagin number P, and any oriented $4n$-manifold M,

$$P(-M) = -P(M).$$

(Whereas $\chi(-M) = \chi(M)$.)

The Basics of Combinatorial Manifolds

1) Two simplicial complexes M and N are said to be *combinatorially equivalent* if there are finite subdivisions M' and N' of M and N, respectively, which are isomorphic.

2)Let Σ^n denote the boundary of the $(n+1)$-simplex. (Note that Σ^n is a topological n-sphere.)

3) A simplicial complex is a *combinatorial n-sphere* if it is combinatorially equivalent to Σ^n.

4) A simplicial complex M is a *combinatorial n-manifold* if for every vertex v of M, $link(v)$ is a combinatorial $(n-1)$-sphere. Equivalently, if for every simplex s, $link(s)$ is a combinatorial $n - dim(s) - 1$-sphere.

Theorem(Whitehead, Cairns): (i) If M is a smooth manifold, then any smooth triangulation on M results in a combinatorial manifold, and any two smooth triangulations are combinatorially equivalent.
(ii) There exist combinatorial manifolds which arise from more that one smooth manifold (i.e there exist non diffeomorphic manifolds with the same underlying combinatorial (equiv. PL) structure,
(iii)There exist combinatorial manifolds which arise do not arise in this fashion at all (i.e. there exist non-smoothable combinatorial manifolds).

Theorem(Thom, Rohlin-Švarč): Characteristic numbers can be defined for combinatorial manifolds. That is if

$$P : \{closed\ oriented\ differentiable\ n-manifolds\} \to R$$

is a characteristic number. Then there is a naturally-defined function

$$\tilde{P} : \{closed\ oriented\ combinatorial\ n-manifolds\} \to R$$

which is a combinatorial invariant, and such that if M is a smooth manifold, and $\tilde{M}$ is the result of smoothly triangulating M, then

$$P(M) = \tilde{P}(\tilde{M}).$$

In particular, the Pontryagin numbers of a smooth manifold depend only on the underlying combinatorial (equiv. piecewise linear) structure.
[In fact, much more is true. Not only can the characteristic numbers be extended to combinatorial manifolds, the rational Pontryagin classes can be extended to combinatorial manifolds.]
(And Novikov proved they are topological invariants)

Let M be a compact oriented smooth manifold, and P a Pontryagin number. We've seen that given any Riemannian metric on M, $P(M)$ is given by integrating over M a function constructed (in a universal fashion) from the curvature of M). We've also seen that given a smooth triangulation of M, $P(M)$ depends only on the combinatorial structure of the triangulation.

Big Question I: Are the Pontryagin numbers of a combinatorial manifold also locally defined?

More precisely, for any function

$$g : \{oriented\ combinatorial\ (n-1)-spheres\} \to R,$$

we can construct a function

$$F_g : \{oriented\ combinatorial\ n-manifolds\} \to R,$$

by setting, for any oriented combinatorial n-manifold M,

$$F_g(M) = \sum_{vertices\ v} g(link(v)).$$

Any function F_g arising in this way is said to be *locally-defined.*

For example, $\chi(M)$ is locally-defined, since $\chi(M) = F_E(M)$.

Answer to Question I: 1) Yes for p_1 (Gabrielov-Gelfand-Losik)
2) Yes for all characteristic numbers (Levitt, Cheeger). -Rourke

Big Problem II: Given a Pontryagin number P, find g so that $P = F_g$.

(There is an analogous problem for all rational Pontryagin classes.)
[Gelfand-MacPherson et al]

Our main result is a converse to the theorem of Levitt and Cheeger.
A function on the set of combinatorial n-manifolds is said to be a *combinatorial invariant* if it assigns the same value to combinatorially equivalent manifolds. For example, each Pontryagin number is a combinatorial invariant.

Theorem: (i) A locally-defined function

$$F_g : \{oriented\ \ combinatorial\ \ n - manifolds\} \to R$$

is a combinatorial invariant if and only if it assigned the same number to every combinatorial n-sphere.

(ii) If F_g is a locally-defined combinatorial invariant of combinatorial n-manifolds, then there is a $c \in R$ and a Pontryagin number P such that for all combinatorial n-manifolds M

$$F_g(M) = c\chi(M) + P(M).$$

In his thesis in 1973 (unpublished [redacted]) Ed Miller stated a result which implies: If two smooth manifolds have the same Euler characteristic and all of the same Pontryagin numbers, then there is no locally defined combinatorial invariant which distinguishes them. Our proof is by different, much simpler methods.

Main ideas of the Proof:

If $F : \{oriented\ \ combinatorial\ \ n - manifolds\} \to R$ is any function, then we can write, for any oriented manifold M,

$$F(M) = \frac{1}{2}[F(M) + F(-M)] + \frac{1}{2}[F(M) - F(-M)]$$

$$:= F_u(M) + F_o(M).$$

The main result is a consequence of the following results.

Proposition I: Let F_g be a locally-defined unoriented function on the set of combinatorial n manifolds which assigns the same number to every combinatorial n-sphere. Then there is a $c \in R$ such that for any combinatorial n-manifold M, $F_g(M) = c\chi(M)$.

[Moreover, if we write $g = g_u + g_o$, then $g_u = cE$.]

Proposition II: Let F_g be a locally-defined oriented function on the set of combinatorial n manifolds which assigns the same number to every combinatorial n-sphere. Then there is a Pontryagin number P so that for any oriented combinatorial n-manifold M, $F_g(M) = P(M)$.

[We can also say a few things about g.]

Proposition I: Let F_g be a locally-defined unoriented function on the set of combinatorial n manifolds which assigns the same number to every combinatorial n-sphere. Then there is a $c \in R$ such that for any combinatorial n-manifold M, $F_g(M) = c\chi(M)$.

[Moreover, if we write $g = g_u + g_o$, then $g_u = cE$.] [Lenk, F.]

Write

$$F_g = F_{g_u} + F_{g_o}.$$

Since F_g and F_{g_u} are unoriented, we see that F_{g_o} is both oriented and unoriented, and hence must be 0. Thus, without changing F_g, we may assume that g is unoriented.

Proposition: Suppose that g is an unoriented function on combinatorial $(n-1)$-spheres such that F_g assigns the same number to every combinatorial n-sphere. If n is odd, then F_g assigns the number 0 to every combinatorial n-sphere.

"Proof": Let y be the common value of F_g applied to combinatorial n-spheres.

Consider first the case that $n = 1$. Applying F_g to $M = \Sigma^1$ yields

$y = F_g($  $) = 3g($  $).$

Applying F_g to $M = suspension(\Sigma^0)$ yields

$y = F_g($  $) = 4g($  $).$

From these two equations we see that $y = 0$ (as does $g(\Sigma^0)$).

Now consider the case that $n = 2$. Applying F_g to $M = \Sigma^2$ yields

$y = F_g($  $) = 4g($  $).$

Applying F_g to $M = suspension(\Sigma^1)$ yields

$y = F_g($  $) = 2g($  $) + 3g($  $).$

Applying F_g to $M = suspension^2(\Sigma^0)$ yields

$y = F_g($  $) = 6g($  $).$

We note that in this case the equations are linearly dependent, and hence we cannot conclude that $y = 0$.

For general n, we get an equation by applying F_g to $suspension^k(\Sigma^{n-k})$, $0 \leq k \leq n$. The determinant of this $(n+1) \times (n+1)$ system of equations is

$$\frac{(n-1)!(n+1)}{4}[-1 + (-1)^n],$$

and hence, y must be 0 if n is odd.

Proposition: Suppose there is a $c \in R$ such that $F_g(M) = c\chi(M)$ for every combinatorial n-sphere. Then $g = cE$ (and hence $F_g(M) = c\chi(M)$ for every combinatorial n-manifold).

Proof: For any combinatorial $(n-1)$-sphere N, define

$$S(N) = maximal\ integer\ k\ such\ that\ N\ is\ a$$
$$k - fold\ suspension\ of\ some\ combinatorial\ (n-k-1) - sphere.$$

We will then prove that $g(N) = cE(N)$ by downward induction on $S(N)$.

Suppose that $S(N) = n-1$. Then we must have that $N = suspension^{(n-1)}(\Sigma^0)$. If we let $M = suspension(N) = suspension^n(\Sigma^0)$ then each of the $2(n+1)$ vertices of M have a link isomorphic to N. Therefore,

$$F_g(M) = \sum_v g(link(v)) = 2(n+1)g(N).$$

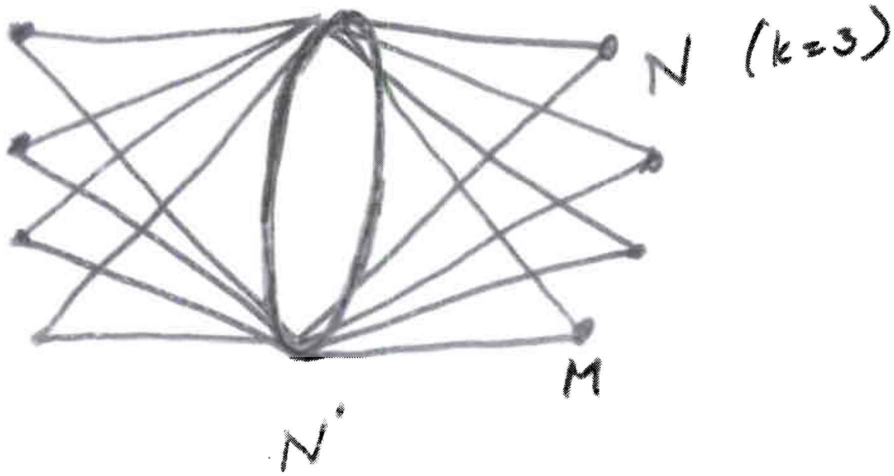
On the other hand

$$F_g(M) = c\chi(M) = 2(n+1)cE(N),$$

which implies that

$$g(N) = cE(N).$$

Now suppose that $S(N) = k$, so that $N = suspension^k(N')$ for some combinatorial $(n-k-1)$-sphere N', and consider $M = suspension(N) = suspension^{(k+1)}(N')$.



Then M has $2(k+1)$ vertices whose links are isomorphic to N, and for every other link N' we have $S(N') \geq k+1$, and hence by induction $g(N') = cE(N')$. Hence we have

$$F_g(M) = 2(k+1)g(N) + \sum_{links\ N' \neq N} g(N') = 2(k+1)g(N) + \sum_{links\ N' \neq N} cE(N'),$$

and

$$F_g(M) = c\chi(M) = 2(k+1)cE(N) + \sum_{links\ N' \neq N} cE(N'),$$

from which we can deduce that

$$g(N) = cE(N).$$

Now we move on to the oriented case. Suppose that F_g is oriented, and write

$$F_g = F_{g_u} + F_{g_o}.$$

Then F_{g_u} is both oriented and unoriented, and hence must be 0. By the previous proposition, if F_{g_u} assigns the value 0 to every combinatorial n-sphere, then we must have $g_u = 0$. This proves

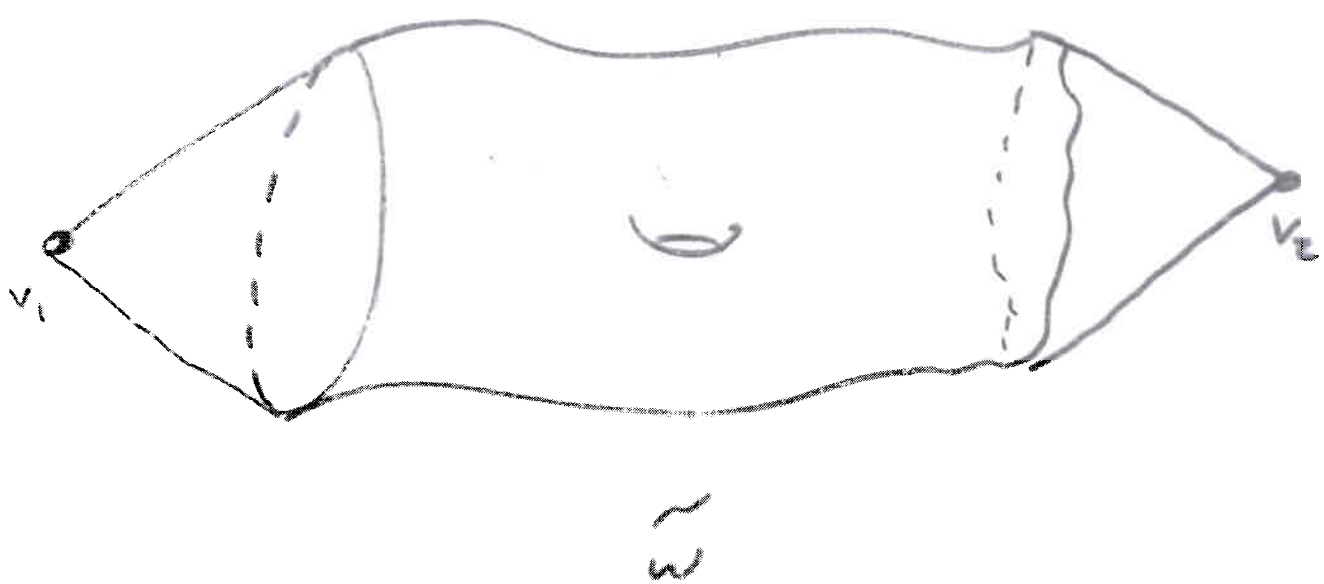
Lemma: If F_g is an oriented function which assigns the same value to every combinatorial n-sphere, then g is oriented.

Say that two oriented combinatorial n-manifolds M_1 and M_2 are *cobordant* if there is some oriented combinatorial $(n+1)$-manifold W with

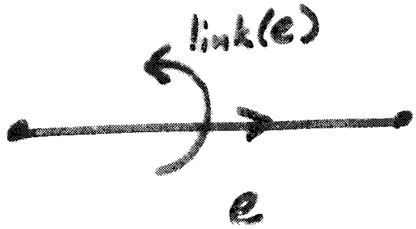
$$\partial W = M_1 \sqcup (-M_2).$$

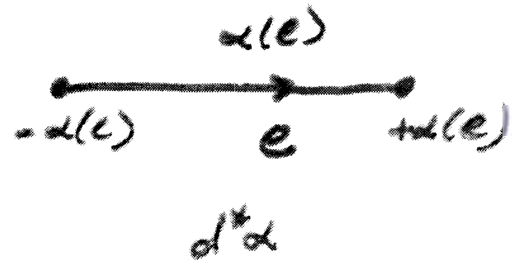
Proposition: Let F_g be an oriented locally-defined function on oriented combinatorial n-manifolds. Suppose that F_g assigns the same number (which must be 0) to every combinatorial n-sphere. If M_1 and M_2 are cobordant then $F_g(M_1) = F_g(M_2)$.

Proof: Let W be a cobordism bewteen M_1 and M_2, and let $\tilde{W}$ be the result of coning off the ends of W with vertices v_1 and v_2, resp.



Then every point of $\tilde{W}$ is a manifold point except (possibly) v_1 and v_2. Let α denote the 1-cochain which assigns to each oriented edge e of $\tilde{W}$ the value $\alpha(e) = g(link(e))$. Let $\beta = d^*\alpha$.



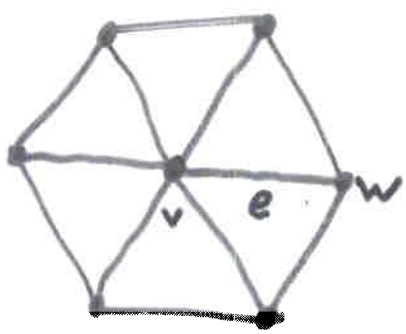


Then

$$\sum_{vertices\ v\ of\ \tilde{W}} \beta(v) = 0.$$

Now we observe that for any vertex v, if e is an edge incident to v and w is the other endpoint of v, then (ignoring important questions of orientation)

$$link(e) \cong link(w, link(v))$$



so that for any vertex v

$$\beta(v) = \sum_{e>v} \alpha(e) = \sum_{e>v} g(link(e)) = \sum_{vertices\ w \in link(v)} g(link(w, link(v)) = F_g(link(v)).$$

If v is a manifold point, then $link(v)$ is a sphere, so by hypotheses $\beta(v) = F_g(link(v)) = 0$. Therefore

$$0 = \sum_{vertices\ v\ of\ \tilde{W}} \beta(v) = \sum_{nonmanifold\ vertices\ v\ of\ \tilde{W}} F_g(link(v))$$

$$= F_g(link(v_1)) + F_g(link(v_2)) = F_g(M_1) - F_g(M_2).$$

which proves that

$$F_g(M_1) = F_g(M_2).$$

The next step is to prove

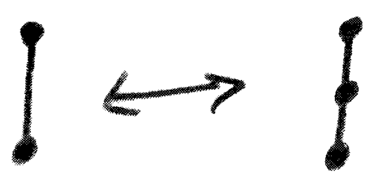
Proposition: Combinatorially equivalent combinatorial n-manifolds are cobordant.

Along with the previous result, this proves

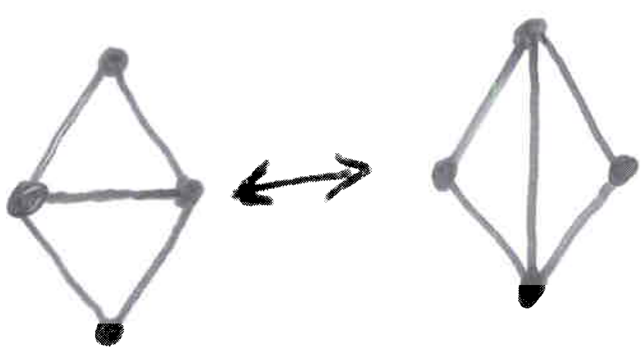
Corollary: Let F_g be an oriented locally-defined function on oriented combinatorial n-manifolds. Suppose that F_g assigns the same number (which must be 0) to every combinatorial n-manifold. Then F_g is a combinatorial invariant.

The proposition follows from the following theorem of Pachner:

Theorem: If M_1 and M_2 are combinatorially equivalent combinatorial manifolds, then it is possible to transform M_1 to M_2 by a sequence of *bistellar flips*. For example: n=1

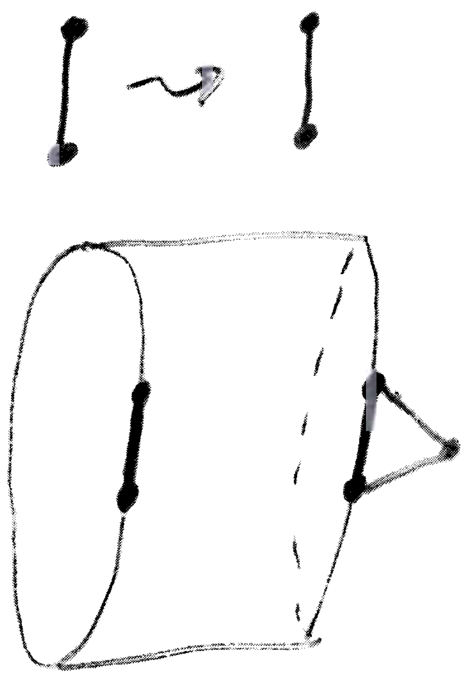


n=2



For general dimension n, a bistellar flip is any operation of teh follwoing form: Let A and B be combinatorial n-balls such that $A * B$ is an $(n+1)$-simplex, then if M is a combinatorial n manifold with a subcomplex isomorphic to $\dot{A} * B$ (this is a ball with boundary $\dot{A} * \dot{B}$), we may replace this ball with $A * \dot{B}$ (another ball with boundary $\dot{A} * \dot{B}$). To construct a cobordism between M and the resulting space, first construct a cobordism bewteen M and itself, and then attach an $(n+1)$-simplex to the boundary of the cobordism by gluing $\dot{A} * \dot{B}$ in the boundary of the simplex onto the isomorphic subcomplex in one copy of M.

e.g



Now we know that if F_g is a locally-defined oriented function of combinatorial n-manifolds which assigns the same number (which must be 0) to every combinatorial sphere, then F_g is a combinatorial invariant which is also a cobordism invariant. To show that F_g must be a Pontryagin number, we can apply the following theorem of Thom (the original proof in the differentiable category) and Wall (who extended it to the combinatorial category)

Theorem: Let $F : \{compact\ oriented\ combinatorial\ \text{n}-manifolds\} \to R$ which satisfies:
(i) $F(M_1 \sqcup M_2) = F(M_1) + F(M_2)$
and
(ii) F is a cobordism invariant
then there is a Pontryagin number P such that $F(M) = P(M)$ for every combinatorial n-manifold M.

Note that every locally-defined function satisfies (i).
[The theorem is usually not stated in this way. Rather, it is stated as a result about the *rational cobordism group*. Properties (i) and (ii) imply that F is a homomorphism from the rational cobordism group to R].