Conflict-free coloring problems
http://www.conflictfree.com

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Some of which is joint work with

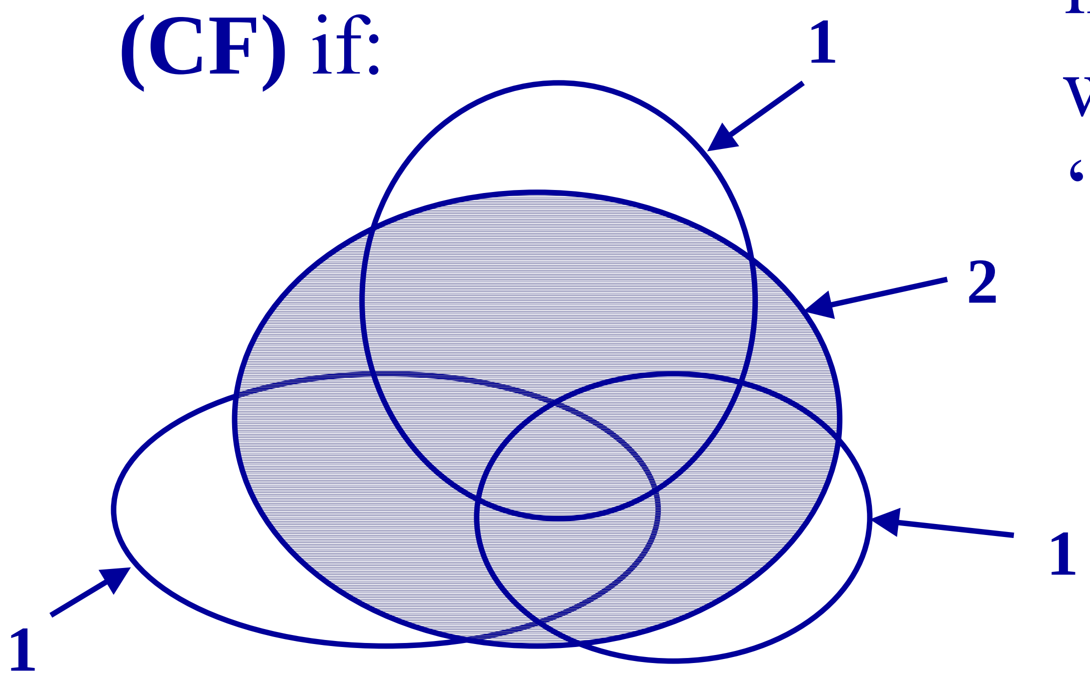
G. Even, D. Ron, Z. Lotker

Some of which is joint work with

S. Har-Peled

What the …. is Conflict-Free Coloring of Regions ?

A **Coloring** of ***n*** **regions**
is **Conflict Free**
(CF) if:



Any point in the union is contained in at least one region whose color is 'unique'

Problems Statement for discs

combinatorics

What is the minimum number ***f*(*n*)** s.t. any ***n*** **discs** can be CF-colored with only ***f*(*n*)** colors?

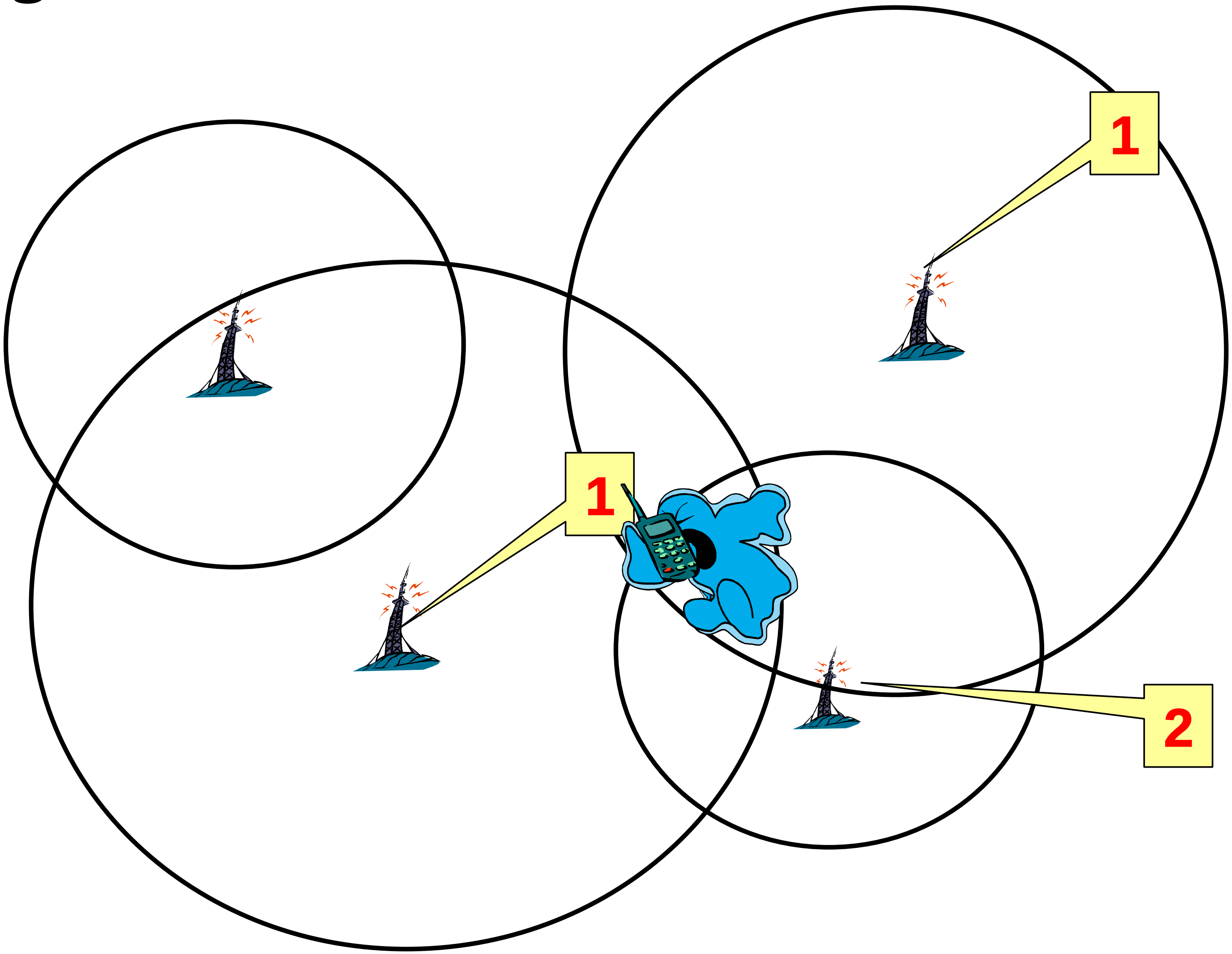
algorithmic

Given a set ***S*** of ***n*** **discs**, find a CF-coloring using minimum # of colors

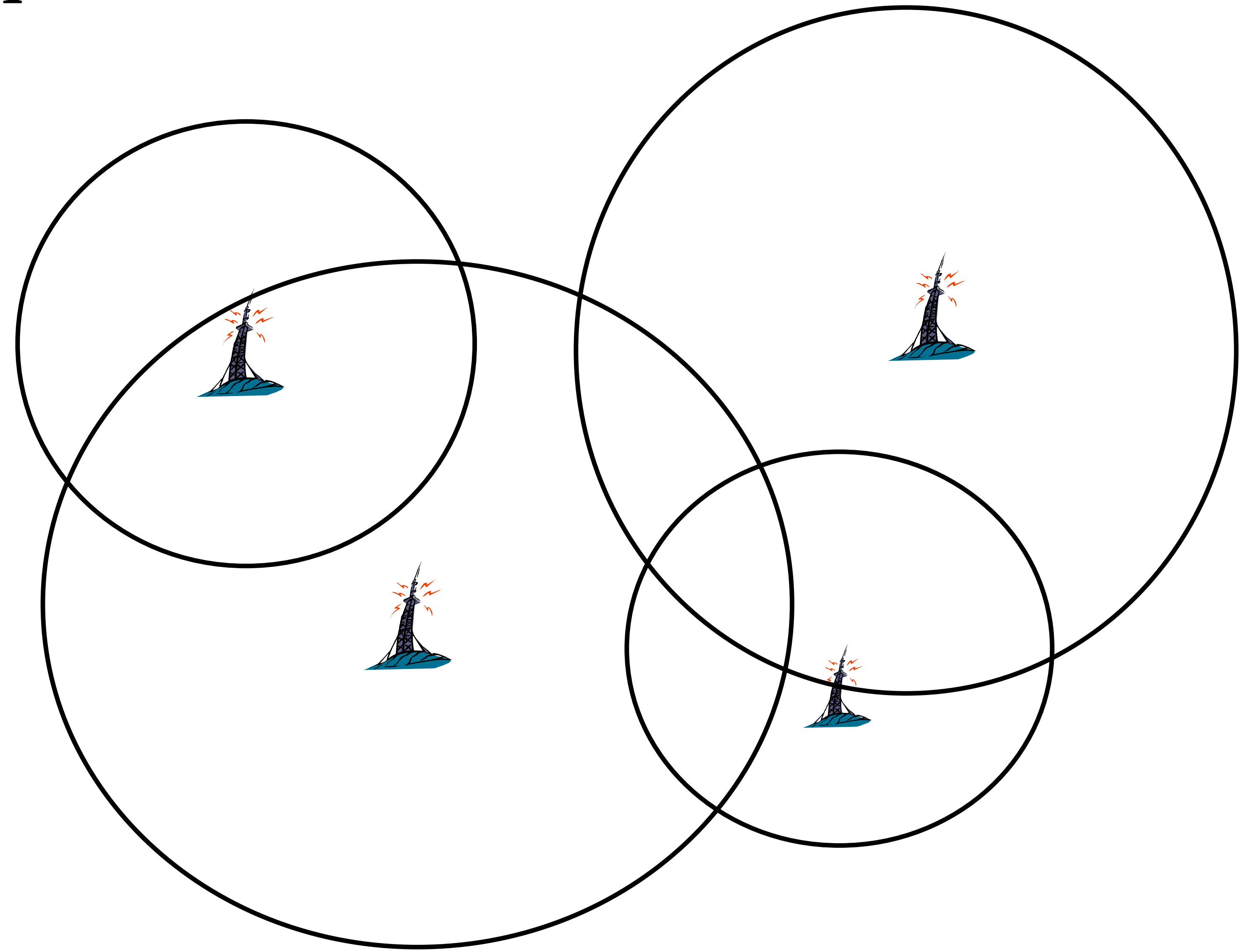
NP-HARD even for congruent discs

[Even, Lotker, Ron, S 02]

Motivation [Even et al.]: From Frequency Assignment in cellular networks



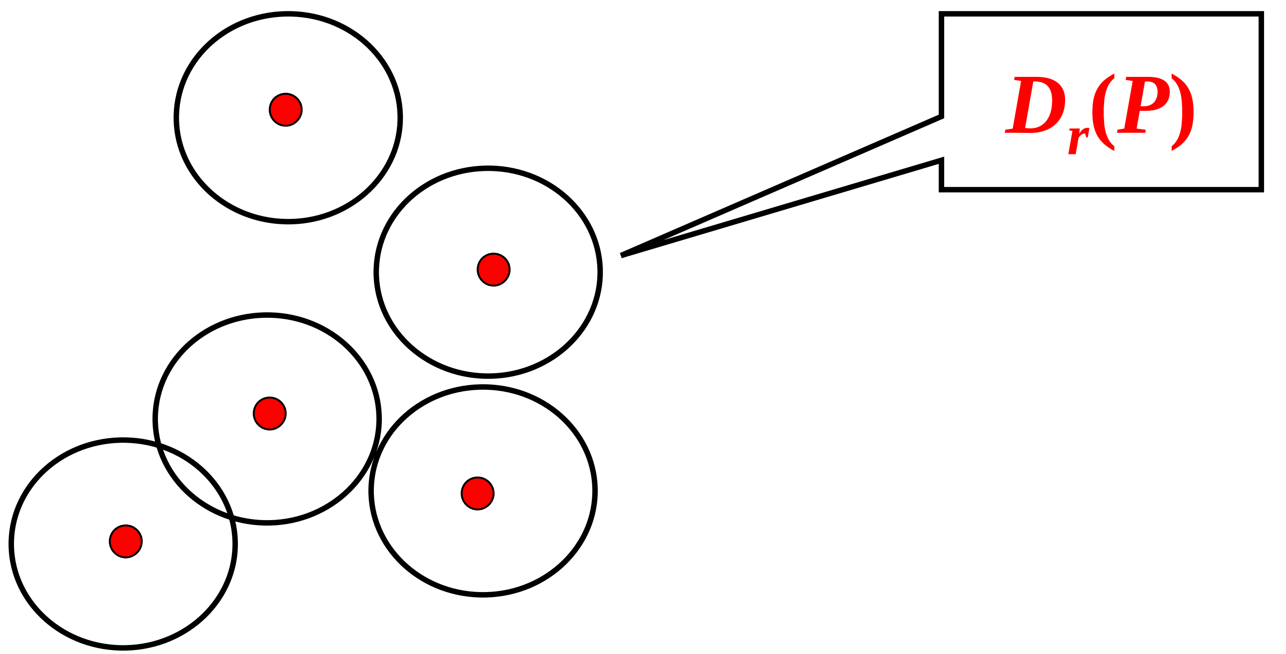
Goal: <u>Minimize</u> the total number of frequencies



CF-coloring discs (cont)

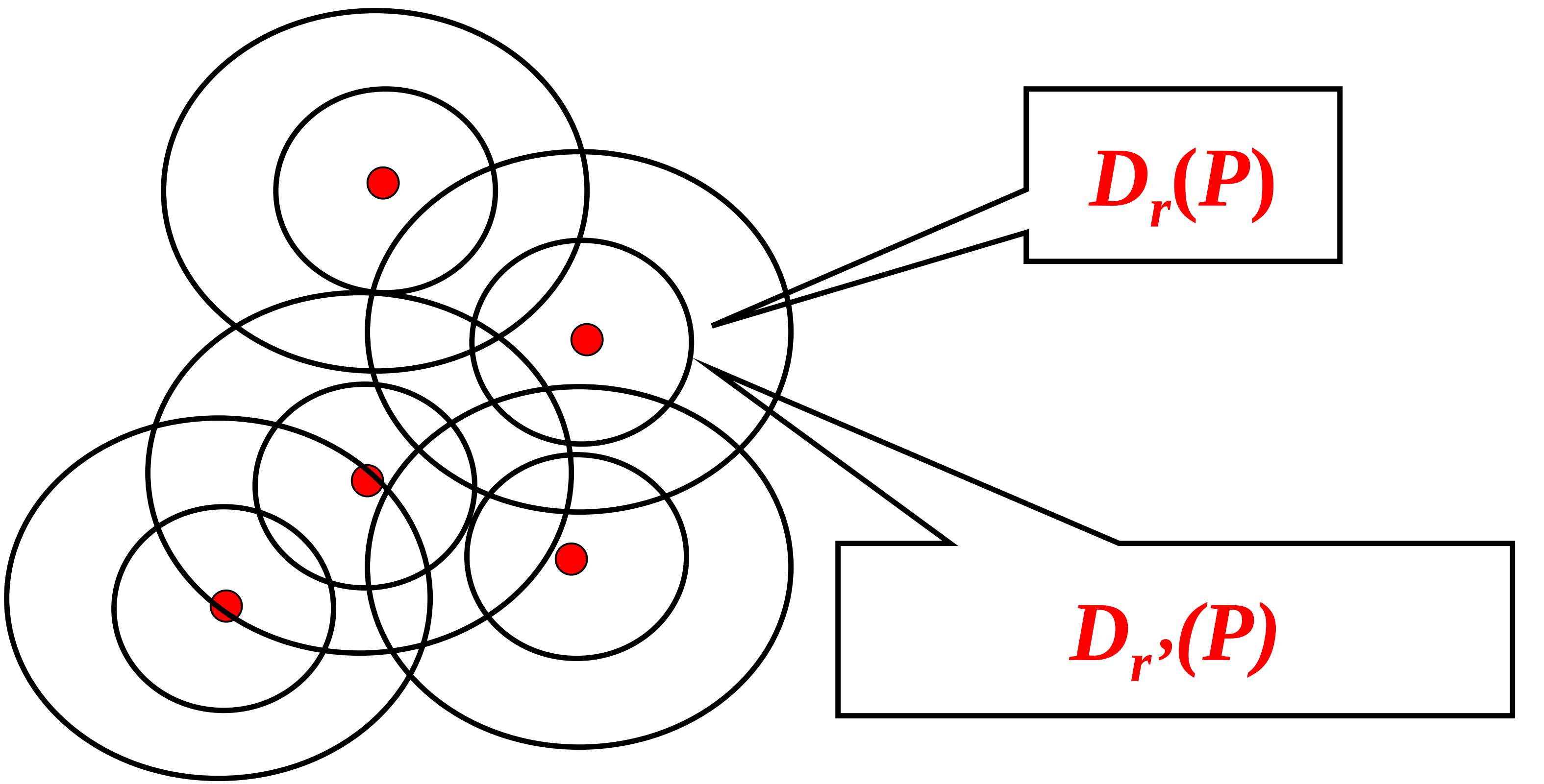
Let ***P*** be a planar set of ***n*** pts

Let $D_r(P)$ be the set of discs with radius ***r*** centered at pts of ***P***



CF-coloring discs (cont)

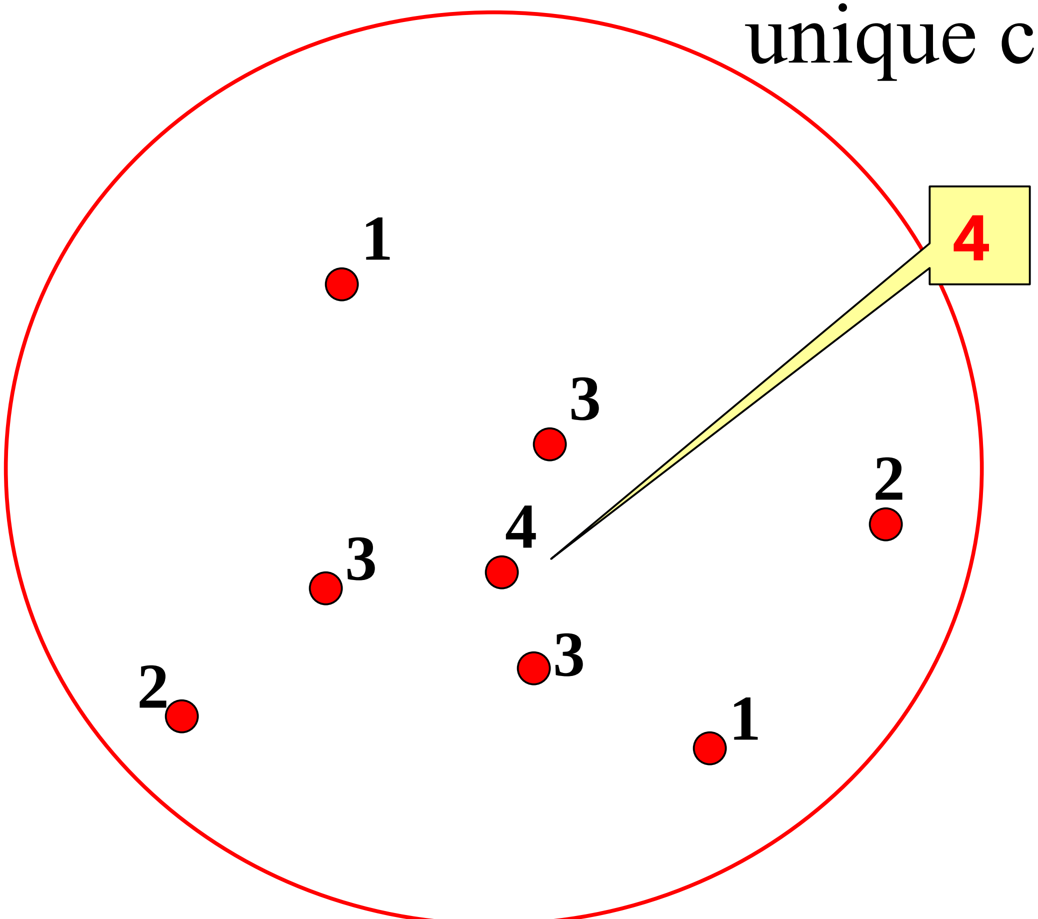
Suppose we want to color P s.t. it would be a CF-coloring of $D_r(P)$ for any $r > 0$



Equivalent problem: Conflict-Free Coloring of **Points w.r.t Discs**

A **Coloring** of pts is **Conflict Free** if:

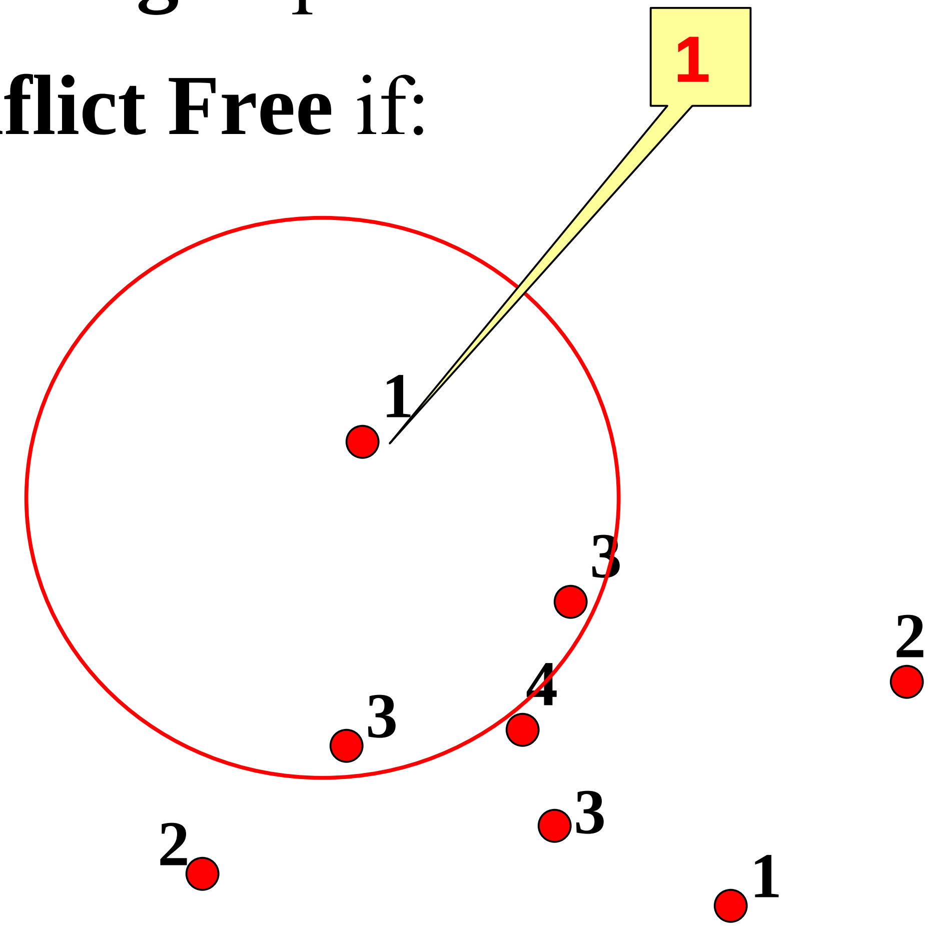
Any (non-empty) disc contains a unique color



What is Conflict-Free Coloring of <u>pts w.r.t Discs</u>?

A **Coloring** of pts

is **Conflict Free** if:



Problems Statement for Points w.r.t Ranges

1. Points (w.r.t ranges):

What is the smallest number ***f*(*n*)** s.t.

any ***n*** **points** can be CF-colored with only ***f*(*n*)** colors?

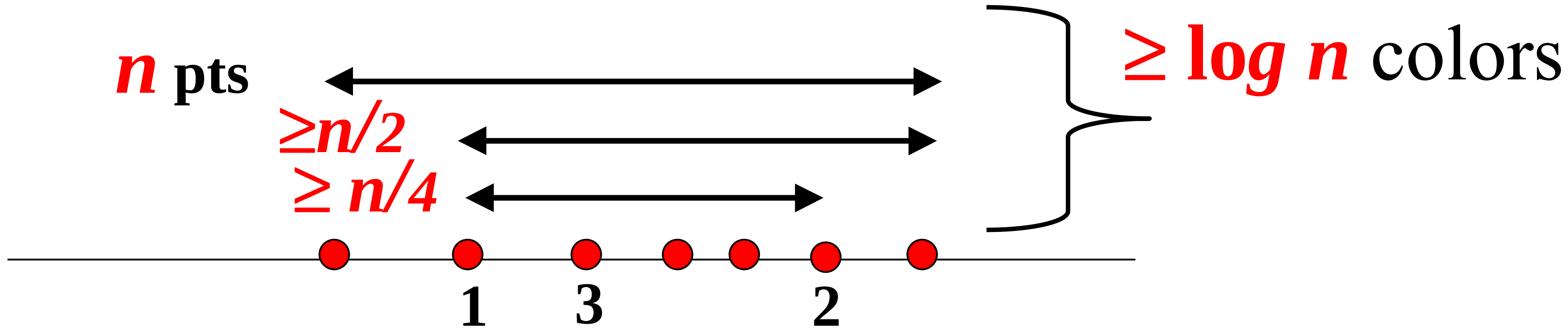
Problem Statement for points (w.r.t discs)

What is the minimum number ***f*(*n*)** s.t. any ***n*** points can be CF-colored (w.r.t discs) with ***f*(*n*)** colors?

Lower Bound ***f*(*n*)**
> log *n*

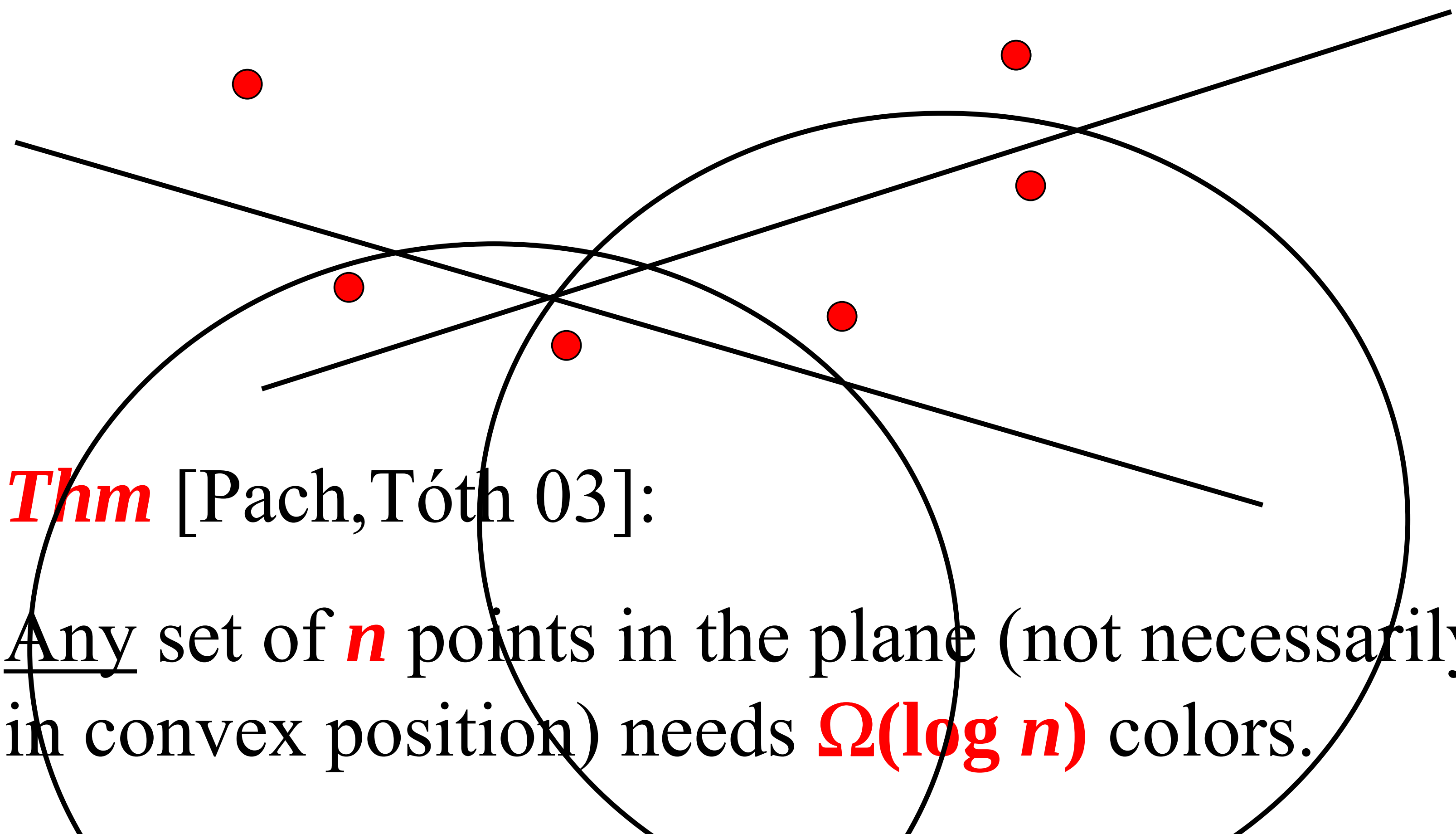
Easy:

n pts on a line! Discs => Intervals



CF-coloring points w.r.t discs (cont)

Remark: Same works for any ***n*** pts in convex position



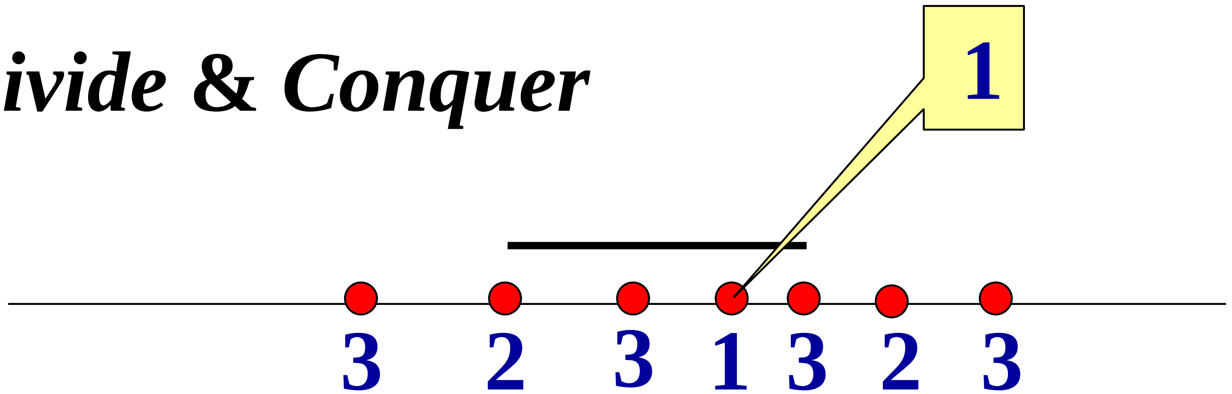
Thm [Pach,Tóth 03]:

Any set of ***n*** points in the plane (not necessarily in convex position) needs $\Omega(\log n)$ colors.

Points on a line: <u>Upper</u> Bound (cont)

log n colors suffice (when pts colinear)

Divide & Conquer



Color median with **1**

Recurse on right and left

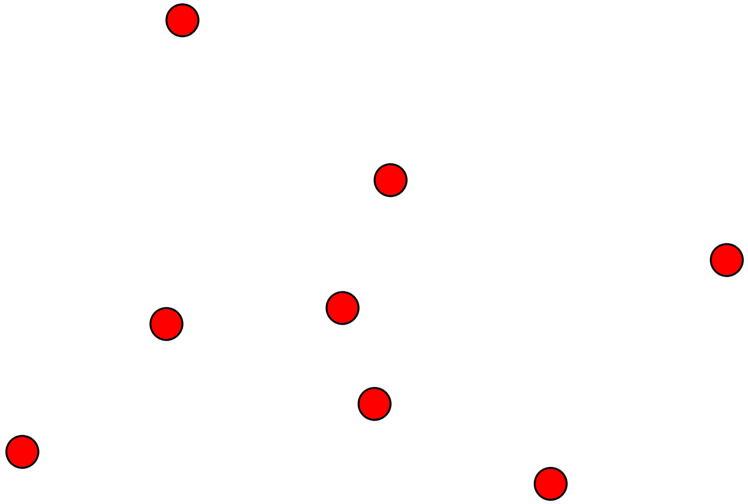
Reusing colors!

CF-coloring in <u>general case</u>: <u>Upper Bound</u>

Thm: **Divide & Conquer doesn't work!**

[Even, Lotker, Ron, S]

O(log *n*) suffice!



n pts

Proof (from the … NOTEBOOK) of Upper Bound: $f(n) = O(\log n)$

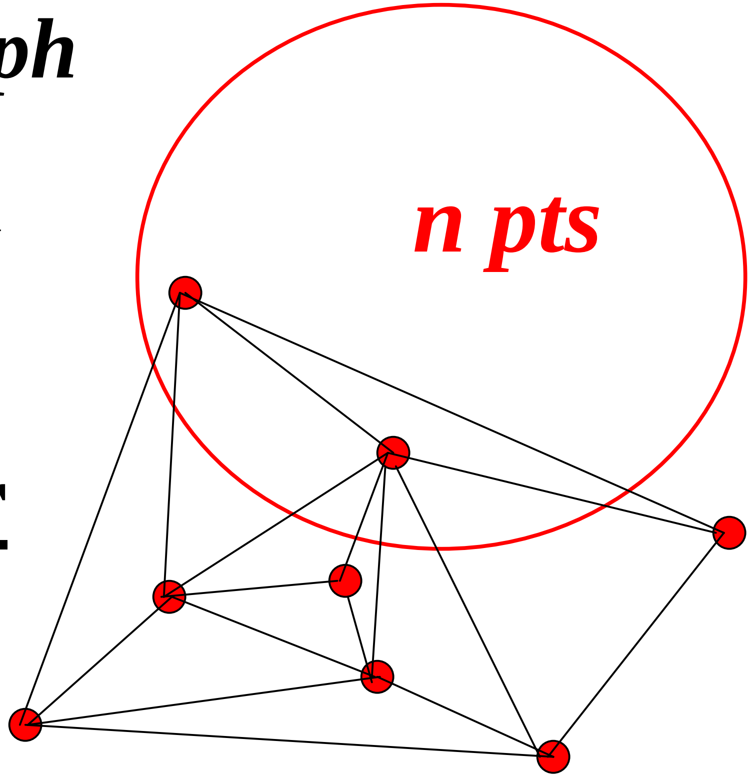
Consider the ***Delauney Graph***

i.e., the “empty pairs” graph

It is **<u>planar</u>**.

Hence, by the **<u>four color</u>** Thm

∃ “large” independent set



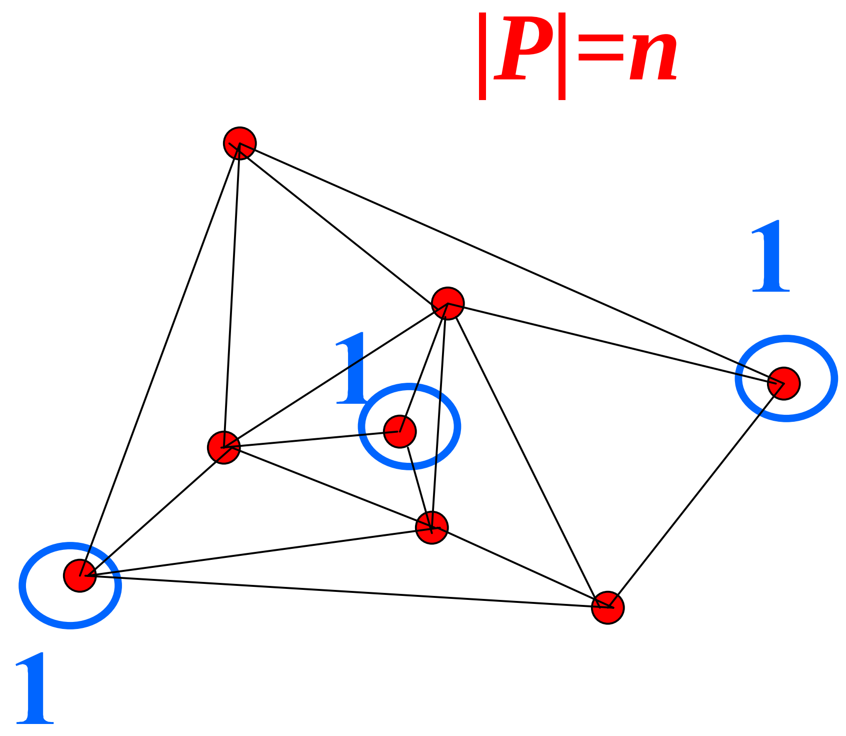
Proof of: $f(n) = O(\log n)$ (cont)

$\exists$ ***IS*** $\subset$ ***P*** ***s.t.*** $|IS| \geq n/4$ and

IS is ***independent***

1. Color ***IS*** with **1**

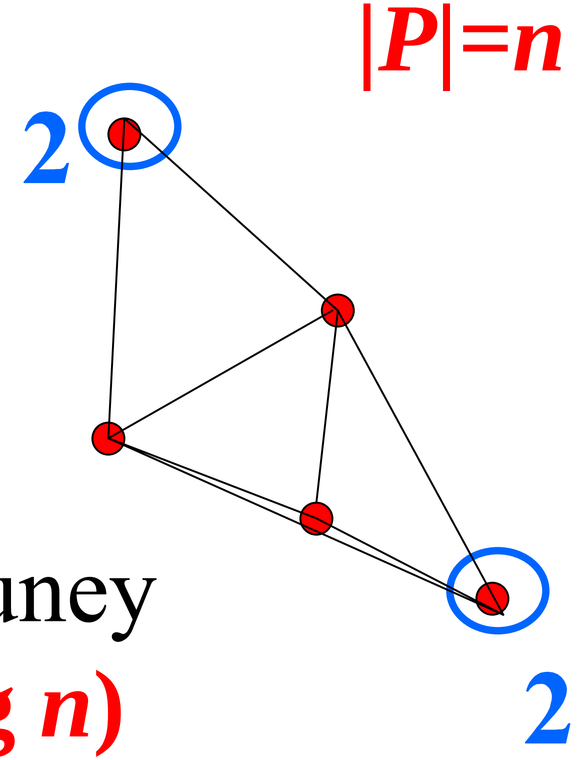
2. Remove ***IS***



Proof of: $f(n) = O(\log n)$ (cont)

$\exists$ $IS \subset P$ ***s.t.*** $|IS| \geq n/4$ and

IS is ***independent!***



1. Color ***IS*** with **1**

2. remove ***IS***

3. Construct the <u>**new**</u> Delauney graph … and iterate (**O(log *n*)** times) on remaining pts

Proof of: $f(n) = O(\log n)$ (cont)

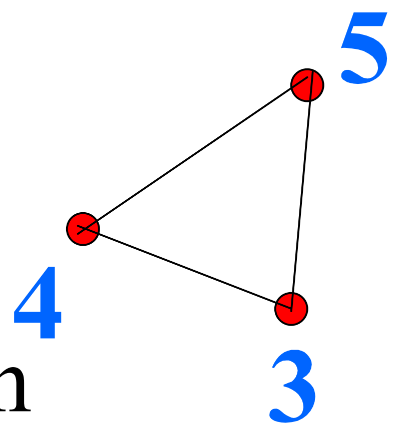
$\exists$ ***IS*** $\subset$ ***P*** ***s.t.*** $|IS| \geq n/4$ and

IS is ***independent!***

$|P|=n$

1.Color ***IS*** with **1**

2. remove ***IS***



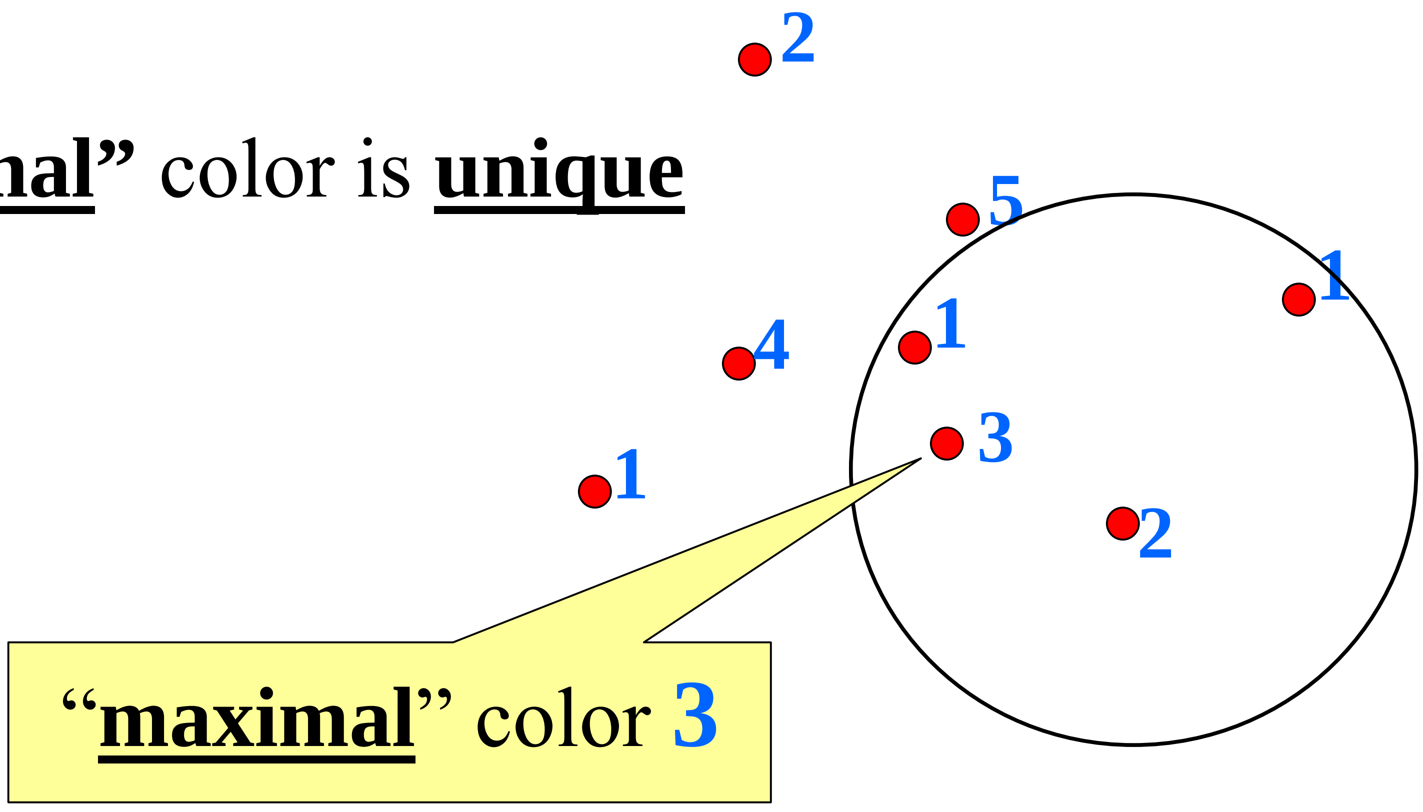
3. Iterate (**O(log *n*)** times) on remaining pts

Proof of: $f(n) = O(\log n)$ (cont)

Algorithm is correct

Consider a non-empty disc

"**maximal**" color is **unique**



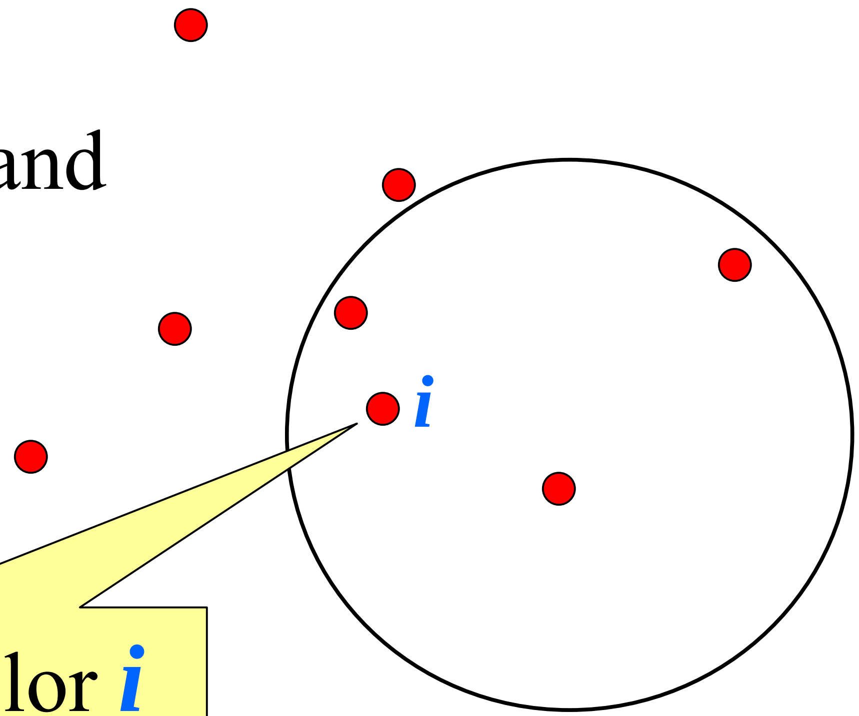
Proof of: ***f(n)* = O(log *n*)** (cont)

"**<u>maximal</u>**" color ***i*** is **<u>unique</u>**

Proof:

Assume ***i*** is not unique and

ignore colors < ***i***



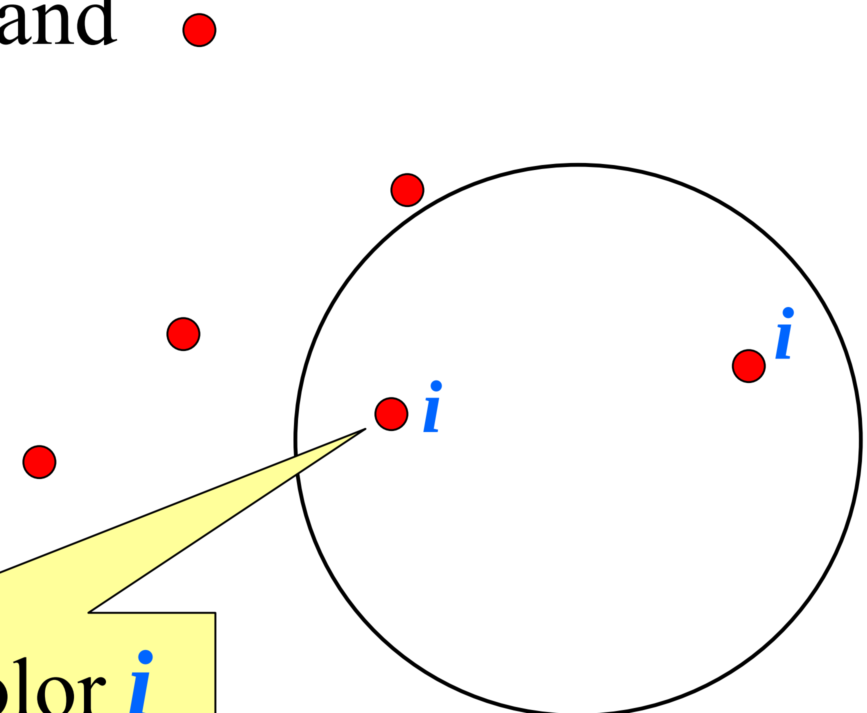
"**<u>maximal</u>**" color ***i***

Proof of: $f(n) = O(\log n)$ (cont)

“**maximal**” color i is **unique**

Assume i is not unique and

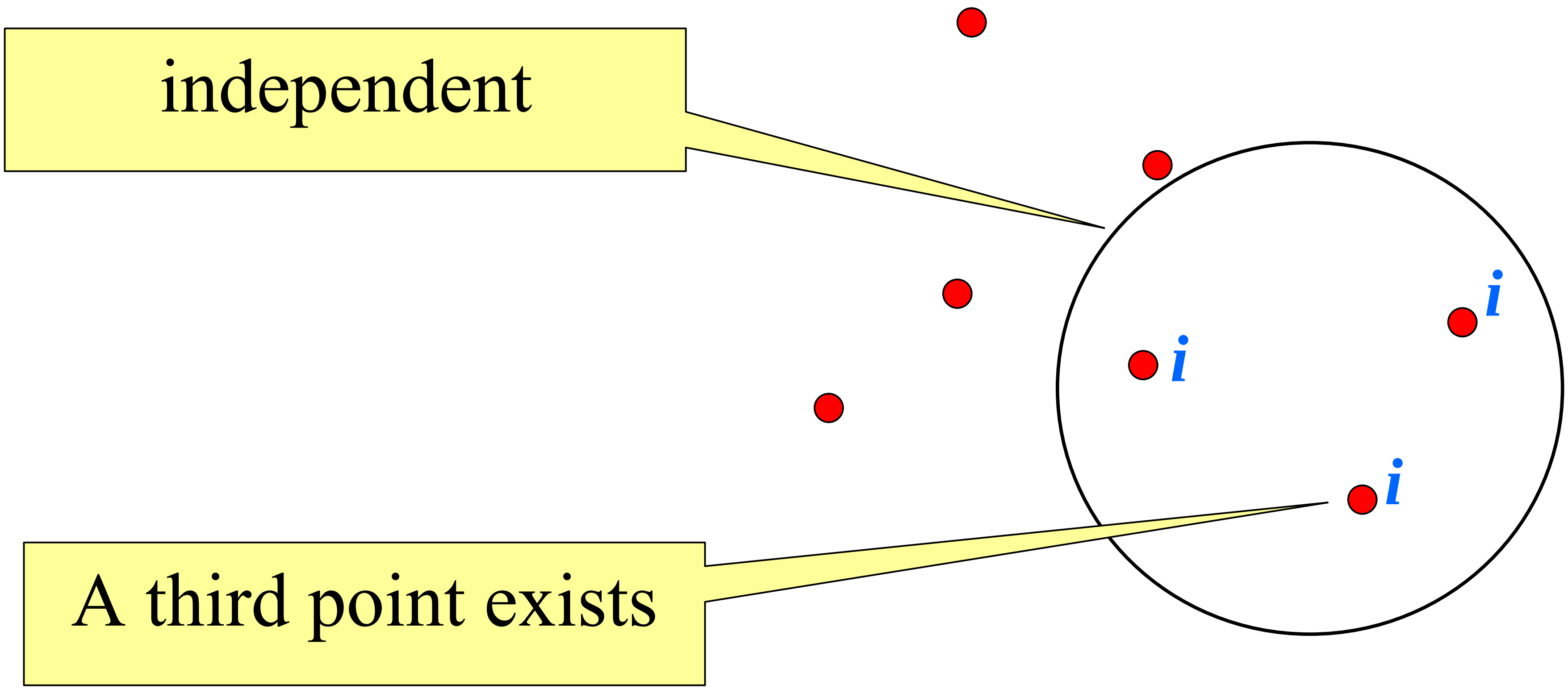
ignore colors < i



“**maximal**” color i

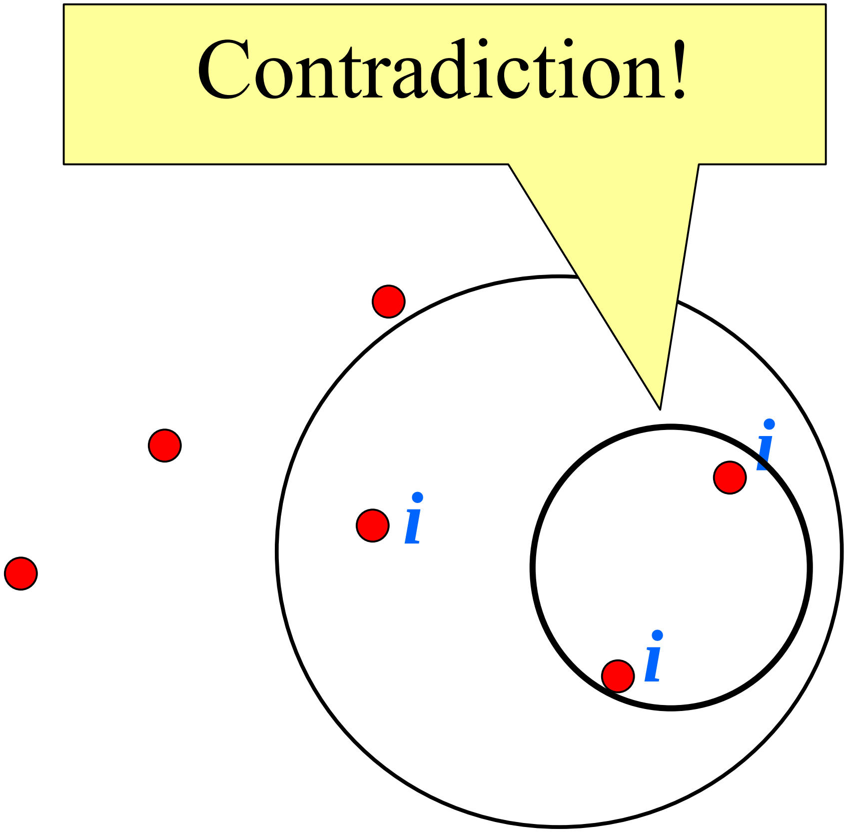
Proof: maximal color i is unique

Consider the i'th iteration



Proof: maximal color *i* is unique

Consider the *i*'th iteration

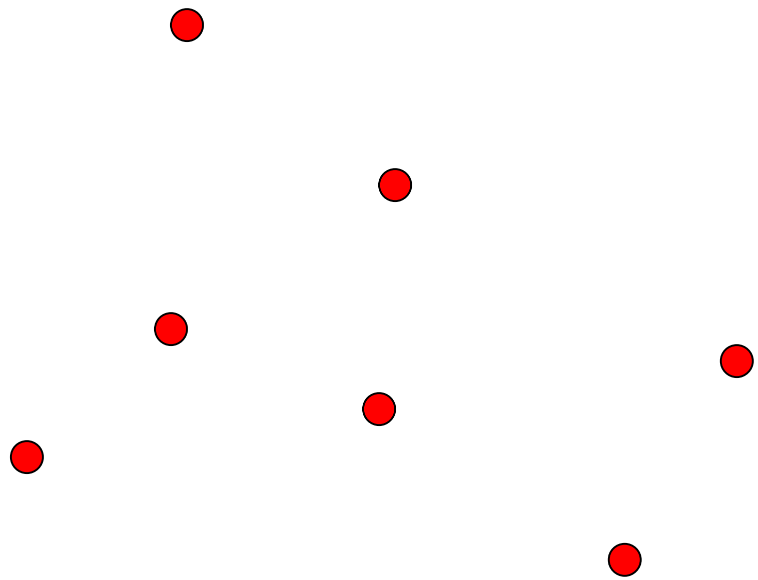


General Framework for CF-coloring a range space *(P,R)* :

∃ Find "large" *IS* in "Delauney graph"

2. Color *IS* with i; i=i+1

3. Iterate on *P\IS*

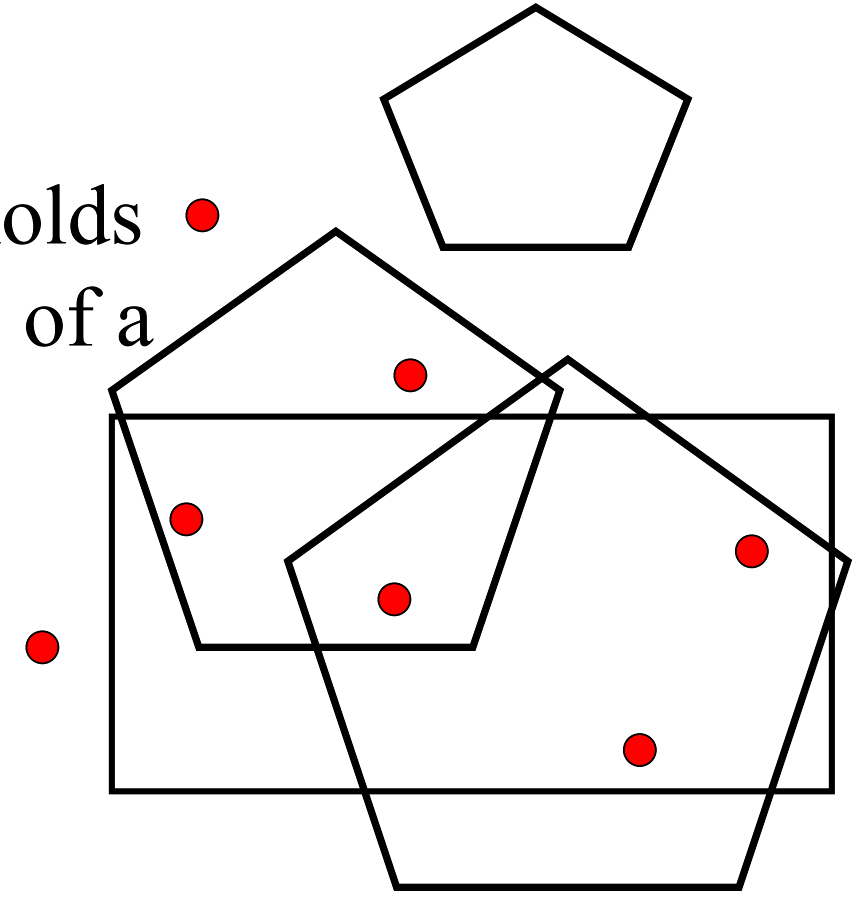


What about other ranges?

CF-coloring pts w.r.t to other ranges?

Upper bound of **O(log *n*)** holds also for **homothetic** copies of a convex body

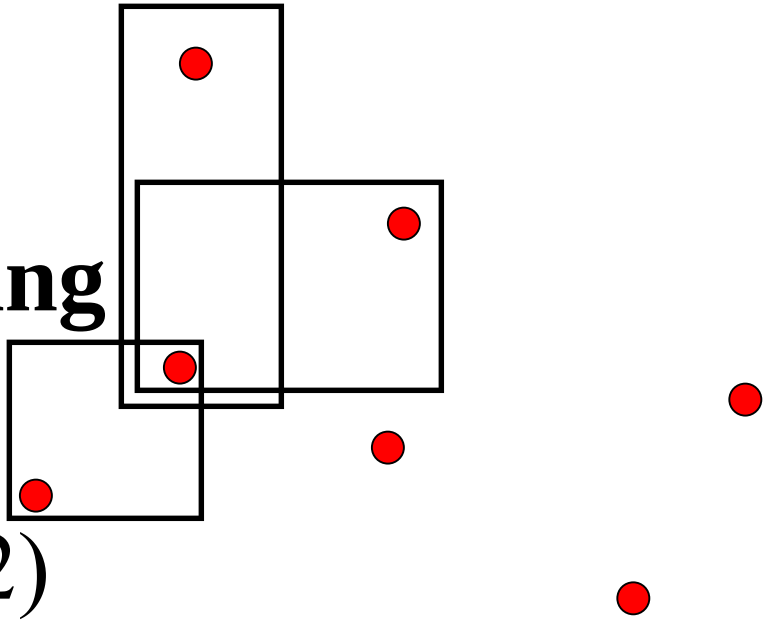
How about axis-parallel rectangles?



CF-coloring pts w.r.t axis-parallel rectangles

Thm [Har-Peled, S]: **O(sqrt (*n*))** colors suffice.

How <u>large</u> is an independent set in the “Delauney” graph ?



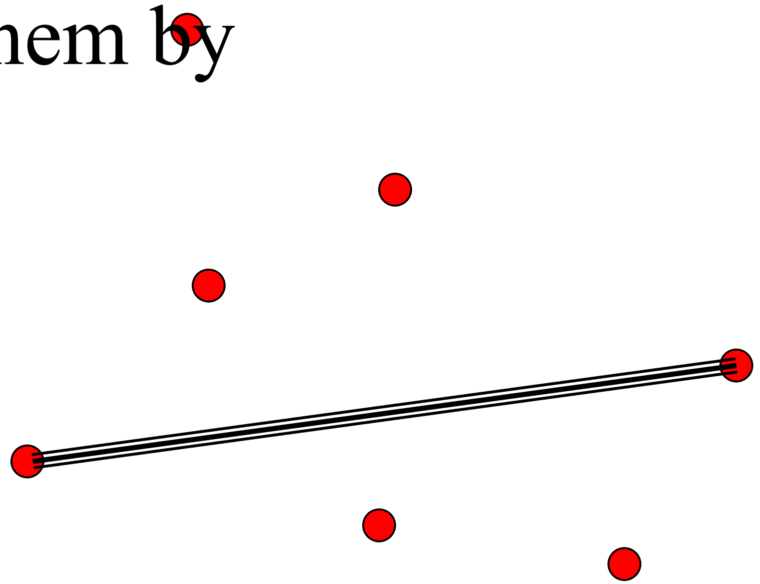
It’s a big and **long standing <u>open problem!</u>**

(dates back to …….. 2002)

CF-coloring pts w.r.t axis-parallel rectangles

Note: If rectangles are not axis parallel then …. not interesting

Any two points need distinct colors, as one can `isolate' them by a narrow rectangle



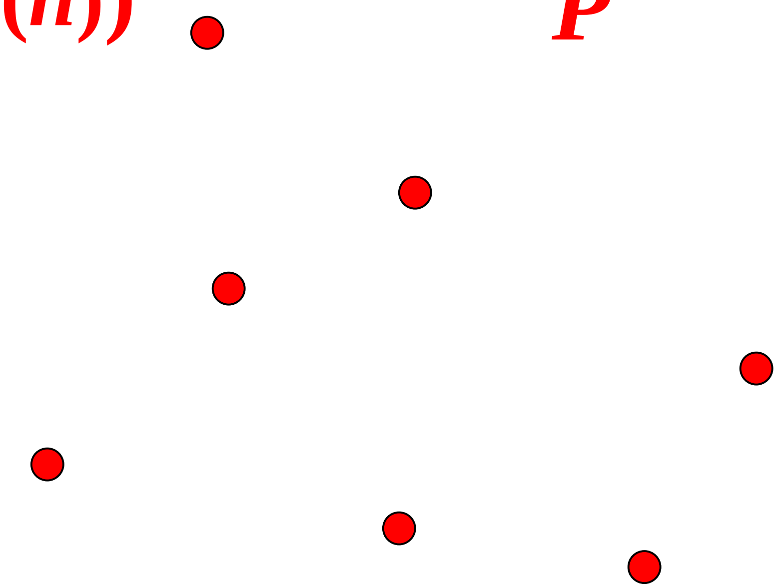
CF-coloring pts w.r.t axis-parallel rectangles

Thm: [Har-Peled – S]

f(n)= **O(sqrt (*n*))**

$\exists$ ***IS*** $\subset$ ***P*** ***s.t.*** **|*IS*| = Ω(sqrt (*n*))** ***P***

and ***IS*** is ***independent***

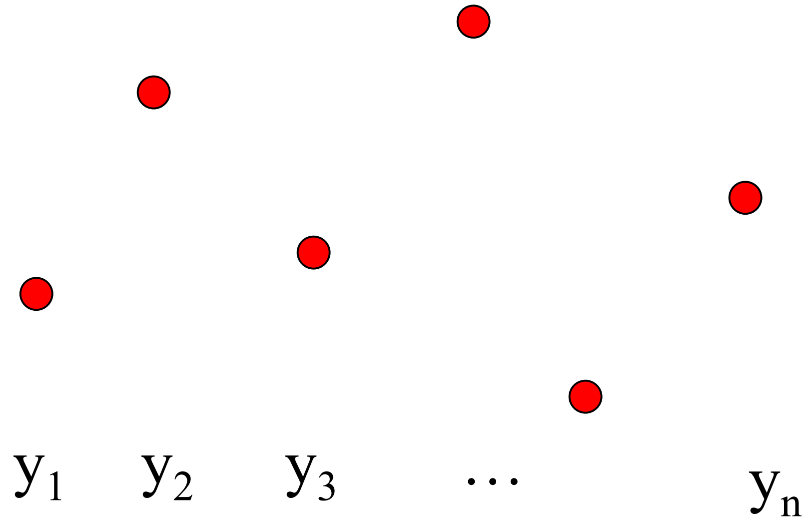


CF-coloring pts w.r.t axis-parallel rectangles (cont)

$\exists$ ***IS***$\subset$ ***P*** **|*IS*|** = **Ω(sqrt (*n*))** and

IS is ***independent***

Proof: Write the y-coordinates from left to right



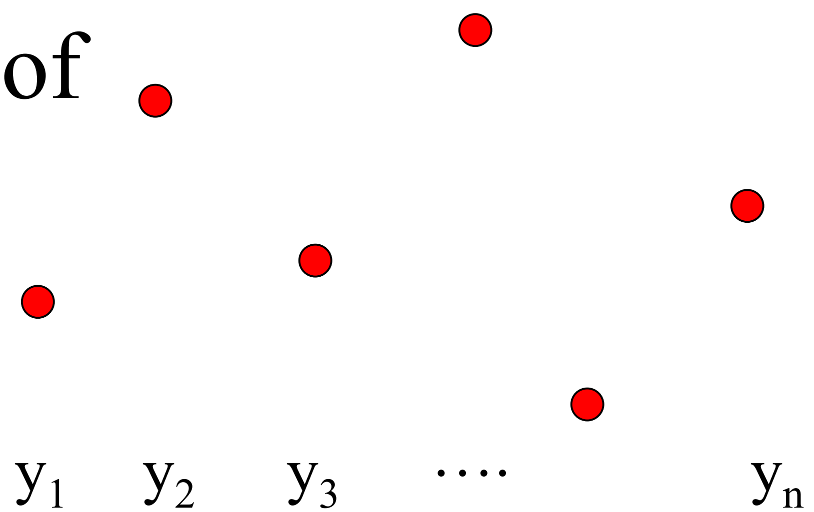
CF-coloring pts w.r.t axis-parallel rectangles (cont)

$\exists$ ***IS*** $\subset$ ***P*** **|*IS*|** = **Ω(sqrt (*n*))** and

IS is ***independent***

Thm: [Erdős-Szekeres]

Any sequence of ***n*** reals contains a ***monotone*** subsequence of length **sqrt (*n*)**

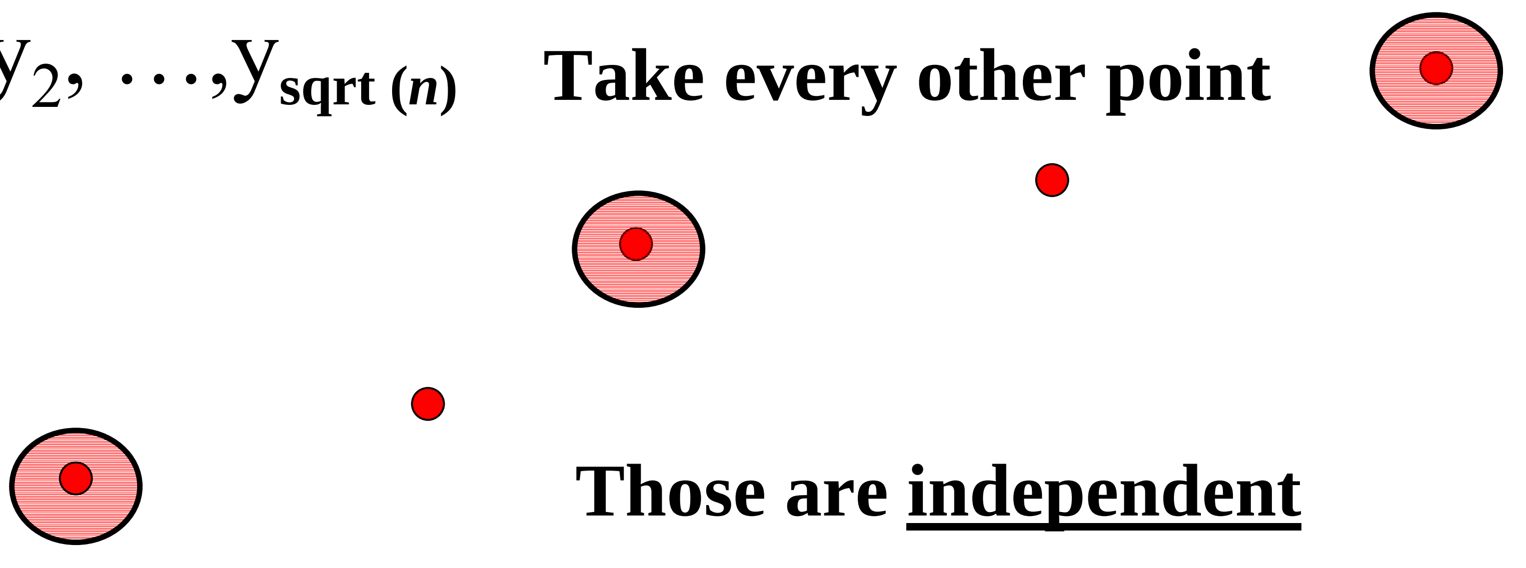


CF-coloring pts w.r.t axis-parallel rectangles (cont)

Thm: [Erdős-Szekeres]

Consider such a monotone (increasing) subsequence

$y_1, y_2, \ldots, y_{sqrt\,(n)}$ **Take every other point**



Those are independent

CF-coloring pts w.r.t axis-parallel rectangles (cont)

$\exists$ ***IS*** $\subset$ ***P*** $|$***IS***$|$ = Ω**(sqrt (*n*))** and

IS is ***independent***

Hence the above algorithm

iterates O**(sqrt (*n*))**.

Slight improvement:

Thm [Alon] [Chan] [Pach,Tóth 03]:

O**(sqrt (*n*)/sqrt(log n))**.

Back to CF-coloring <u>Regions</u>:

Problems Statement for <u>discs</u>

<u>Reminder:</u>

What is the minimum number ***f*(*n*)** s.t. any ***n*** **<u>discs</u>** can be CF-colored with only ***f*(*n*)** colors?

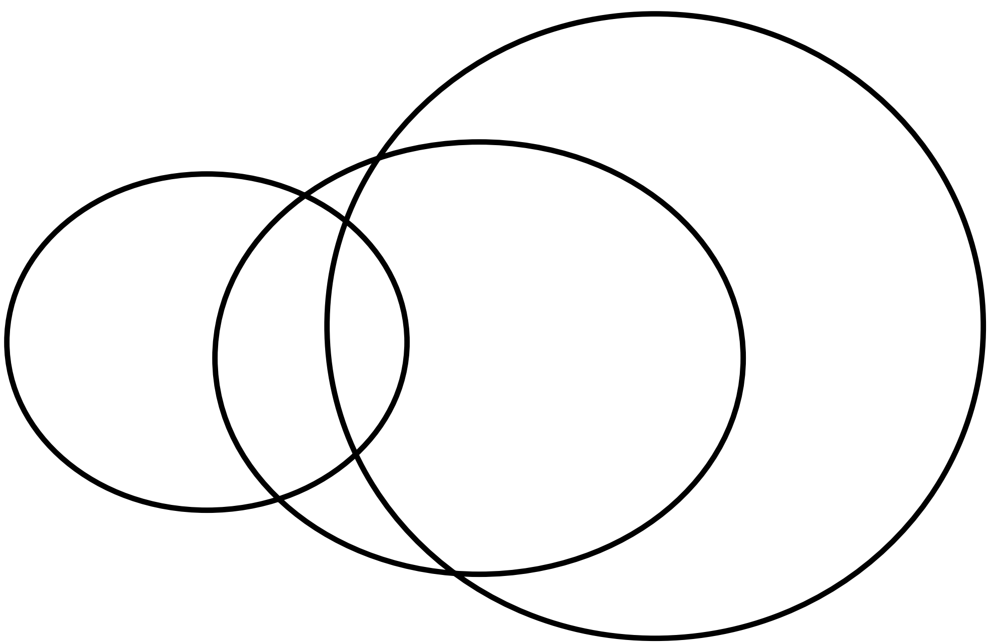
CF-coloring ***discs***

Thm [Even, Lotker, Ron, S]:

Any ***n*** ***discs*** admits a CF-coloring with **O(log *n*)** colors

↓

n pts in R^3 w.r.t halfspaces, using similar analysis



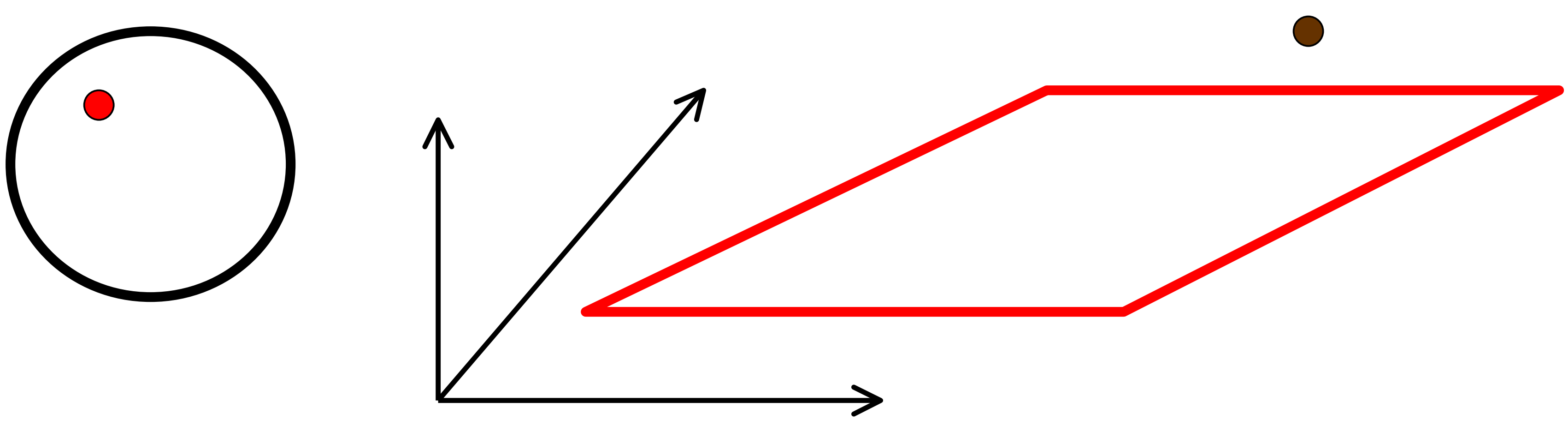
CF-coloring ***discs*** (cont)

We can transform the problem to that of

CF-coloring points in R^3

w.r.t. positive half-spaces.

$R^2 \Rightarrow R^3$

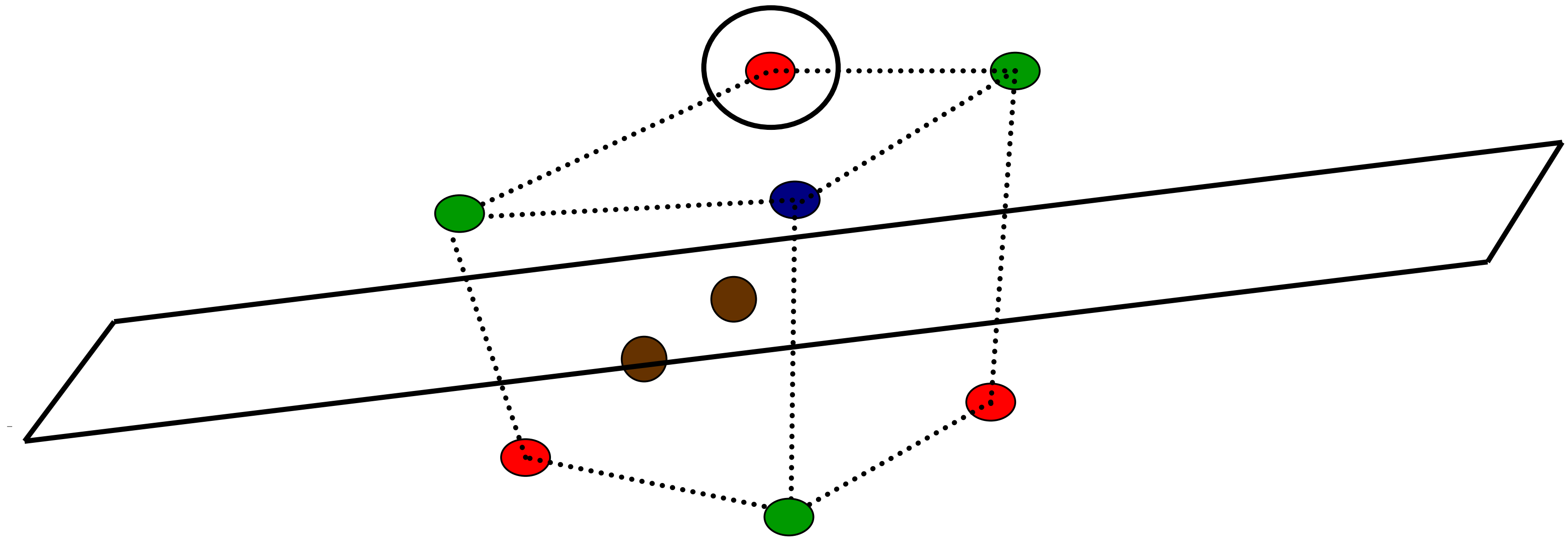


CF-coloring ***<u>discs</u>*** (cont)

***Observation*:**

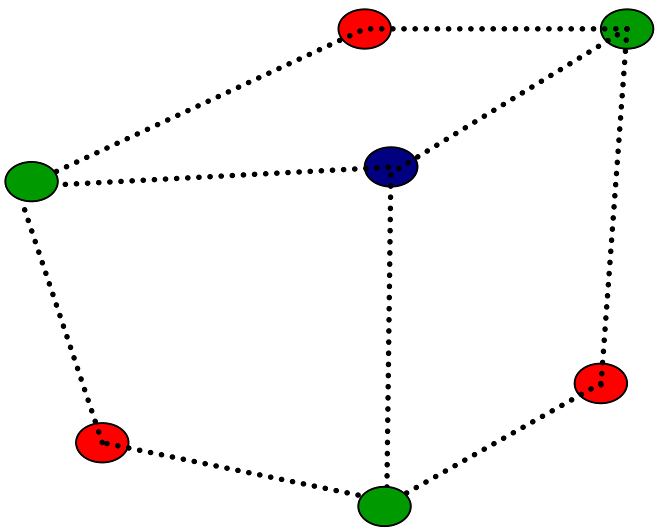
We can assume that all points are <u>extreme</u> i.e.,

in convex position!



CF-coloring ***discs*** (cont)

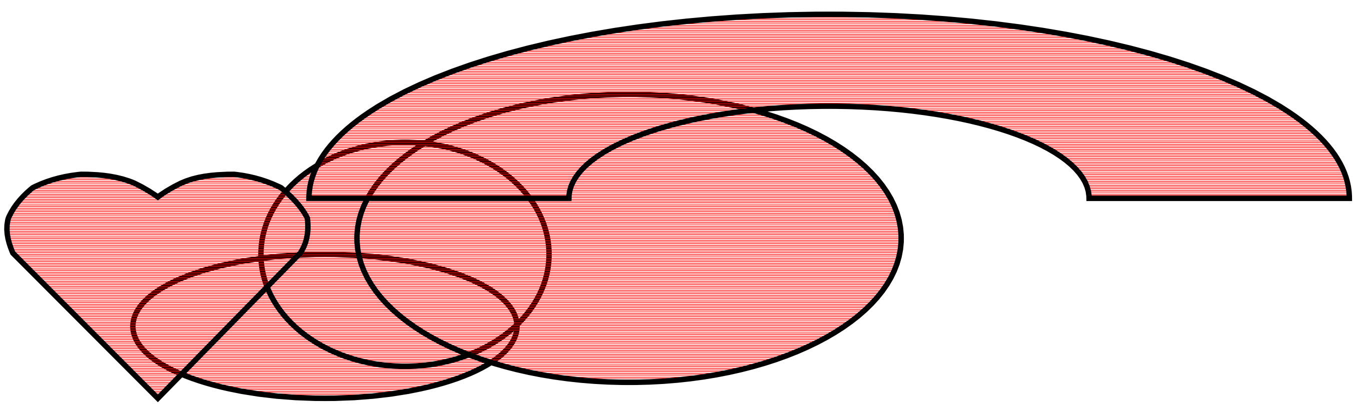
CF-Color the extreme points using the general framework



CF-coloring ***pseudo-discs***

Thm [Har-Peled, S]:

Let ***R*** be a collection of ***n*** ***pseudo-discs*****.** ***R*** admits a CF-coloring with **O(log *n*)** colors.

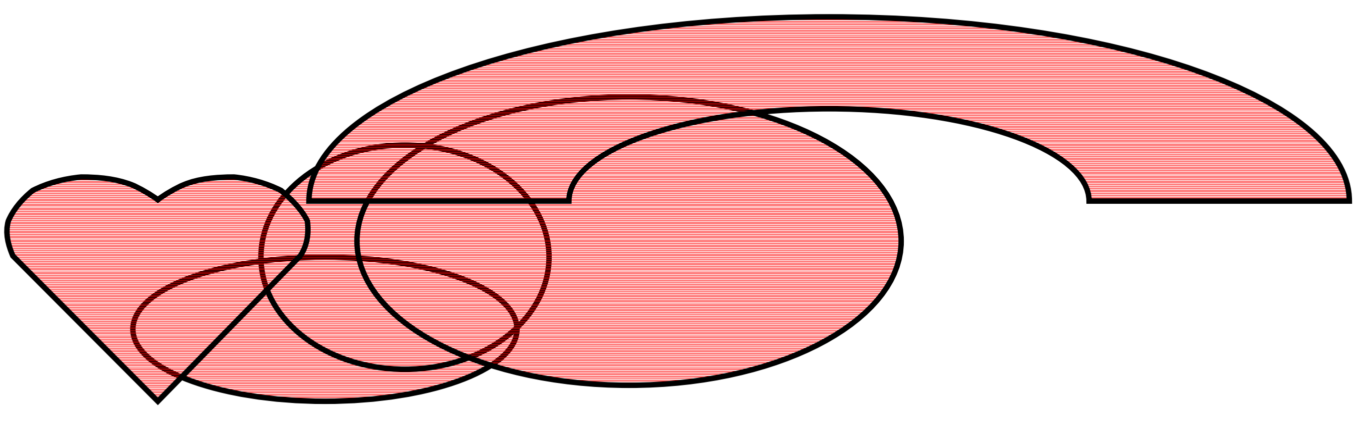


CF-coloring ***pseudo-discs***

What is so special about pseudo-discs ?

Thm **[Kedem, Livne, Pach, Sharir 86]:**

The **complexity** of the **union** of **any** ***n*** **pseudo-discs** is ***O*(*n*)**.



CF-coloring ***pseudo-discs***

More general Thm [Har-Peled, S]:

CF-coloring with 'small' # of colors for regions with 'low union complexity'

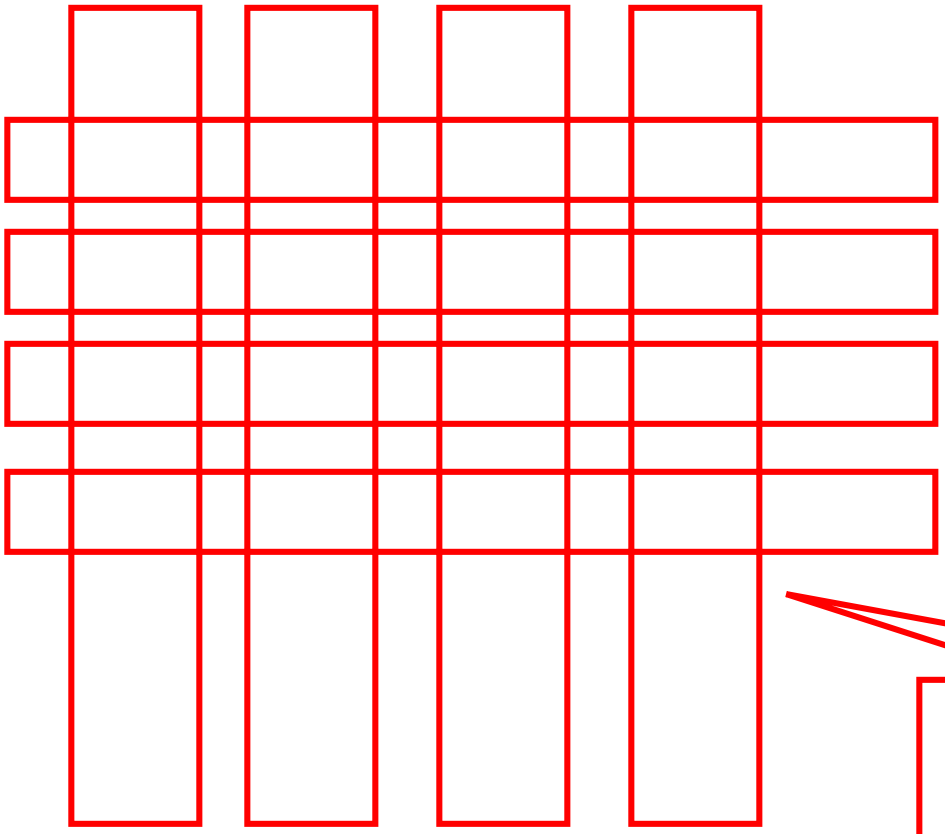
For example:

α-fat convex objects [Efrat, Sharir]

(α,β)-covered objects [Efrat 99]

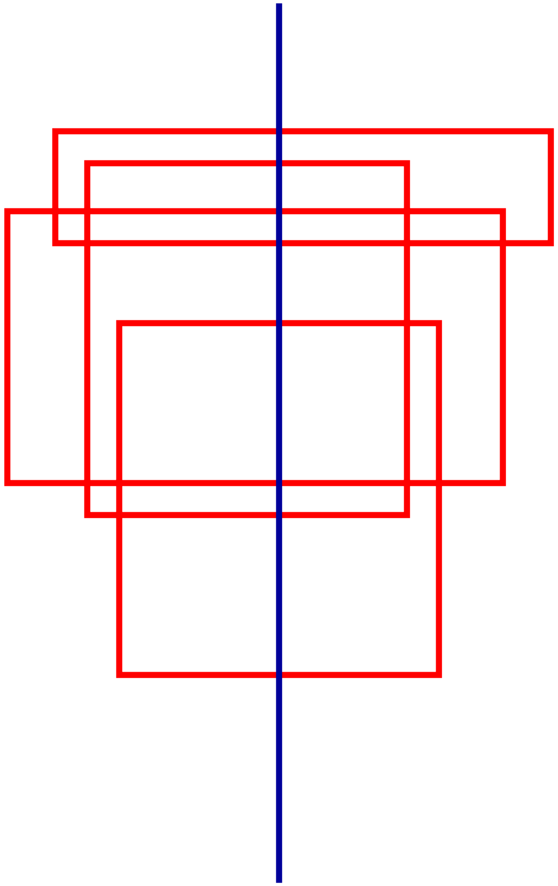
…

How about axis-parallel rectangles?



Union complexity could be quadratic !!!

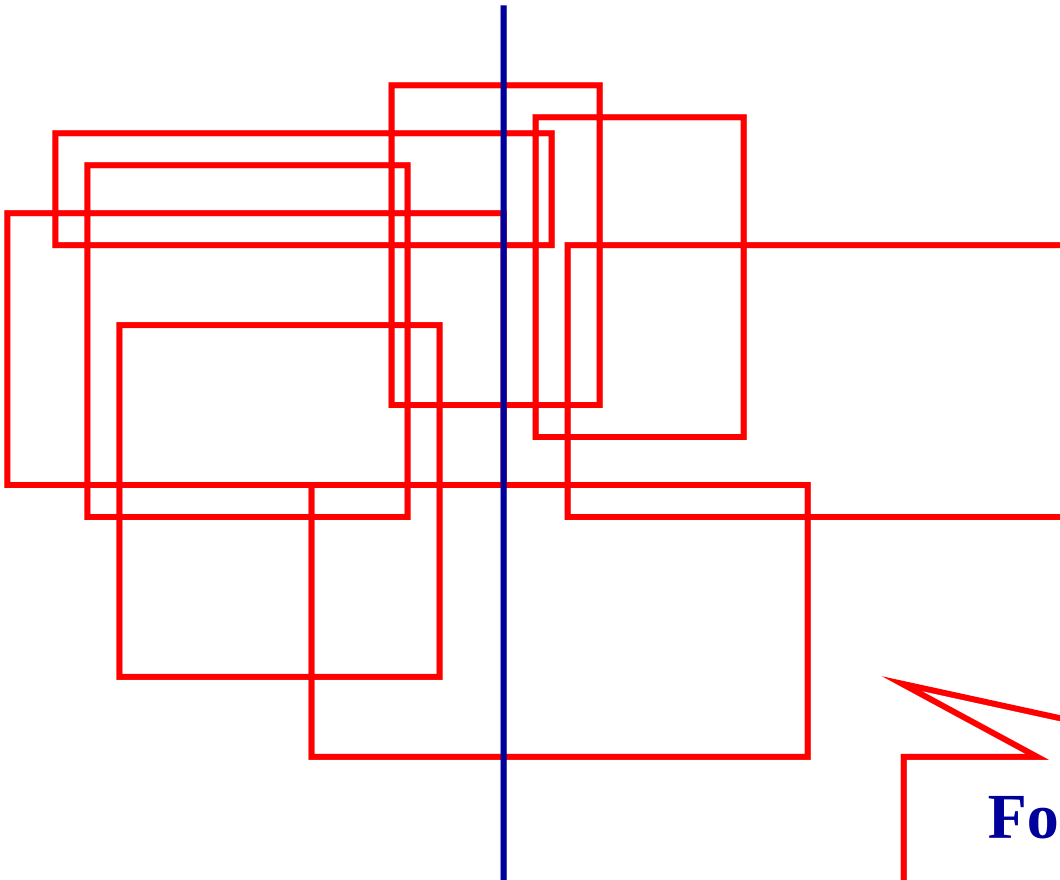
CF-coloring axis-parallel rectangles



If all intersected by a vertical line

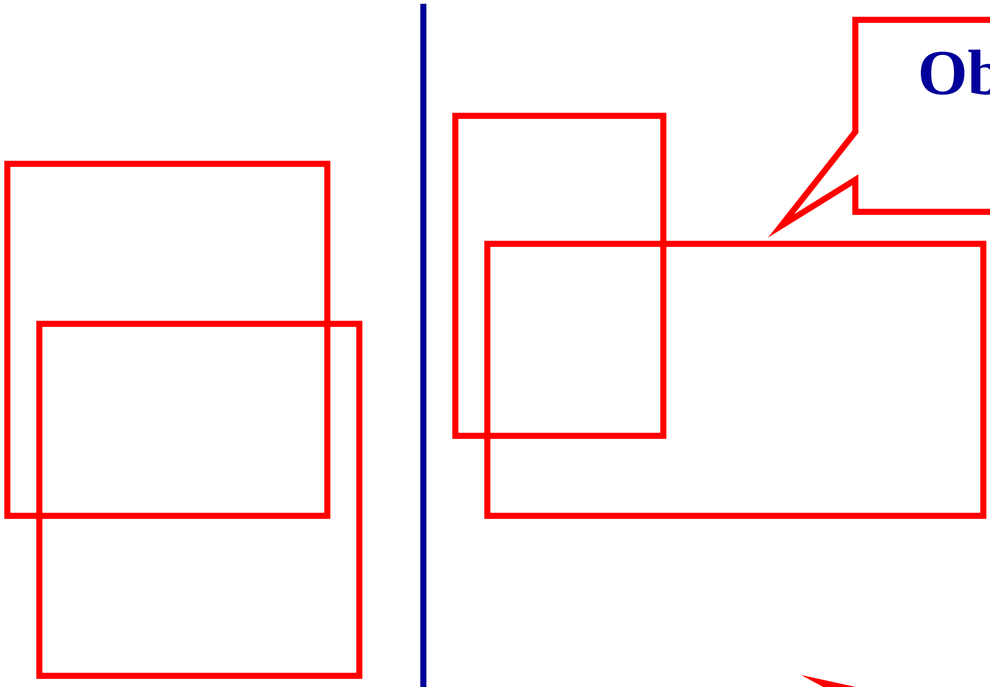
Union complexity is linear so *O*(*log n*) colors suffice

CF-coloring axis-parallel rectangles



For general case, apply divide and conquer

CF-coloring axis-parallel rectangles



Obtain CF-coloring with $O(\log^2 n)$ **colors**

For general case, apply divide and conquer

Open Problem:

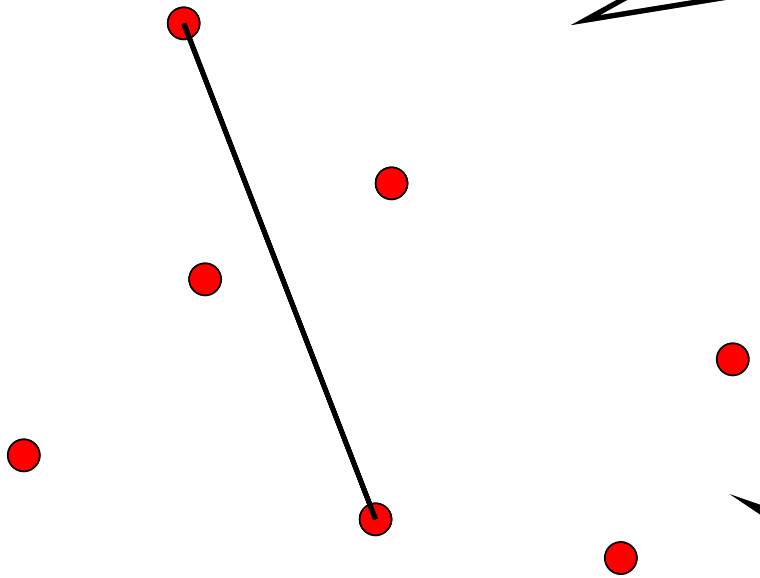
Improve the **bound**

How much time do I have?

OY VEY

CF-coloring a range space with a small VC-dimension

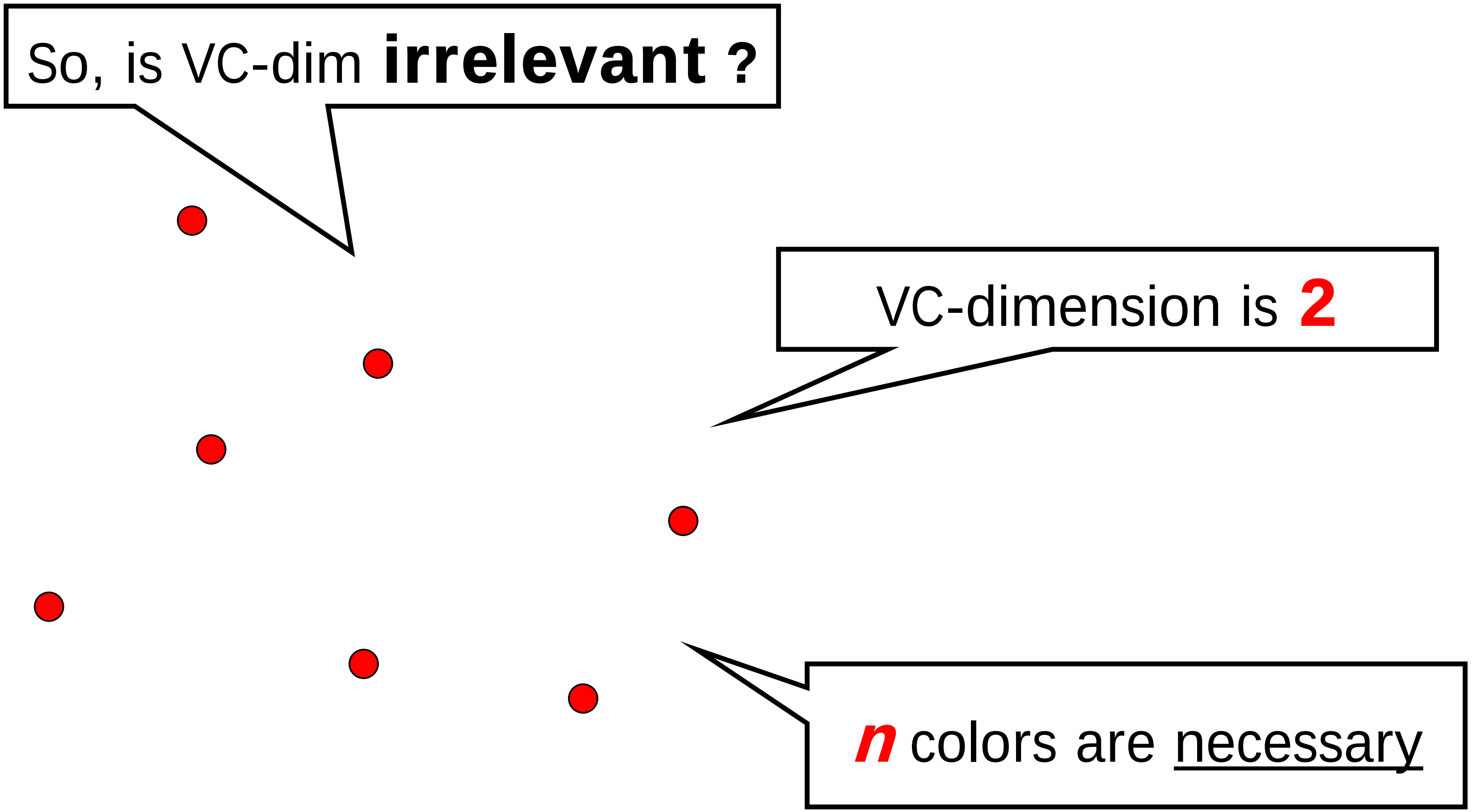
Consider a range space induced by *n* pts and straight line segments



VC-dimension is *2*

n colors are necessary

CF-coloring a range space with a small VC-dimension



CF-coloring a range space with a small VC-dimension

Well, … of course it is relevant!

We just need to change the problem!

Define *k*-CF-coloring similarly…

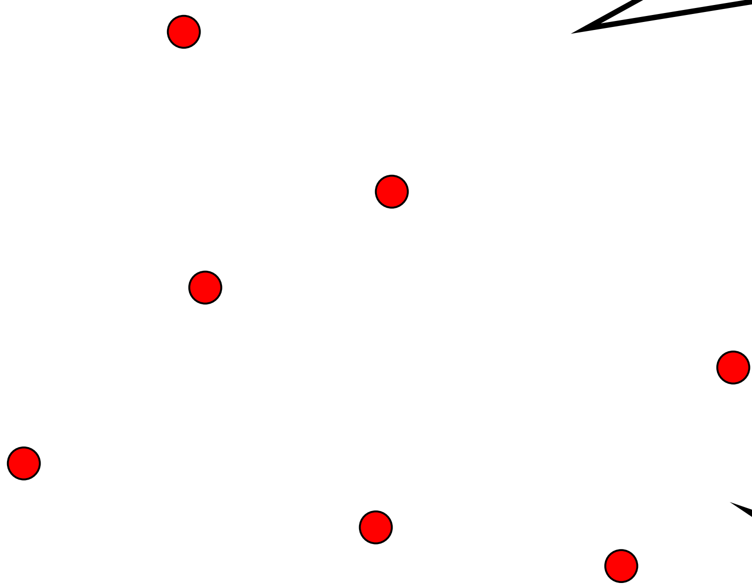
… require some color to appear at most *k* times.

CF-coloring a range space with a small VC-dimension

n pts and straight line segments

VC-dimension is **2**

one color suffice for **2**-CF-coloring



CF-coloring a range space with a small VC-dimension

How many colors we need for CF-coloring n pts in R^3 w.r.t balls?

Unfortunately, $\exists$ examples where n colors are necessary!

How about k-CF-coloring ($k > 1$)?

Thm [Har-Peled, S]:

$O(n^{1/k})$ colors suffice for k-CF-coloring.

CF-coloring a range space with a small VC-dimension

More generally

Thm [Har-Peled, S]:

For reasonably large k, $O(\log n)$ colors suffice for **for k-CF-coloring a range space with VC-dim d**.

CF-coloring ***pseudo-discs*** (cont)

Let S be a set of *n* pseudo-discs

Defnition: a subset $S' \subset S$ is *admissible* (w.r.t S) if for any point *p*

1. $p \in$ (at most one region of S').

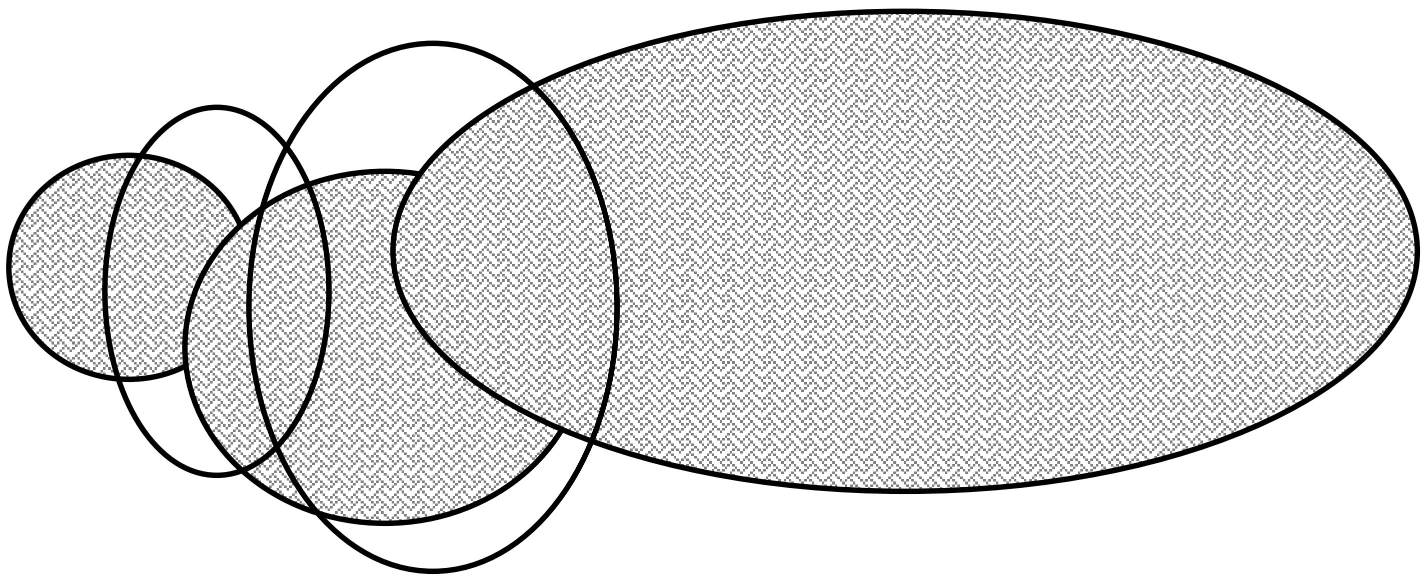
or

2. $\exists\, r \in S \setminus S'$ s.t $p \in r$.

CF-coloring *pseudo-discs* (cont)

Example of

S’⊂ S which is *admissible* (w.r.t S)



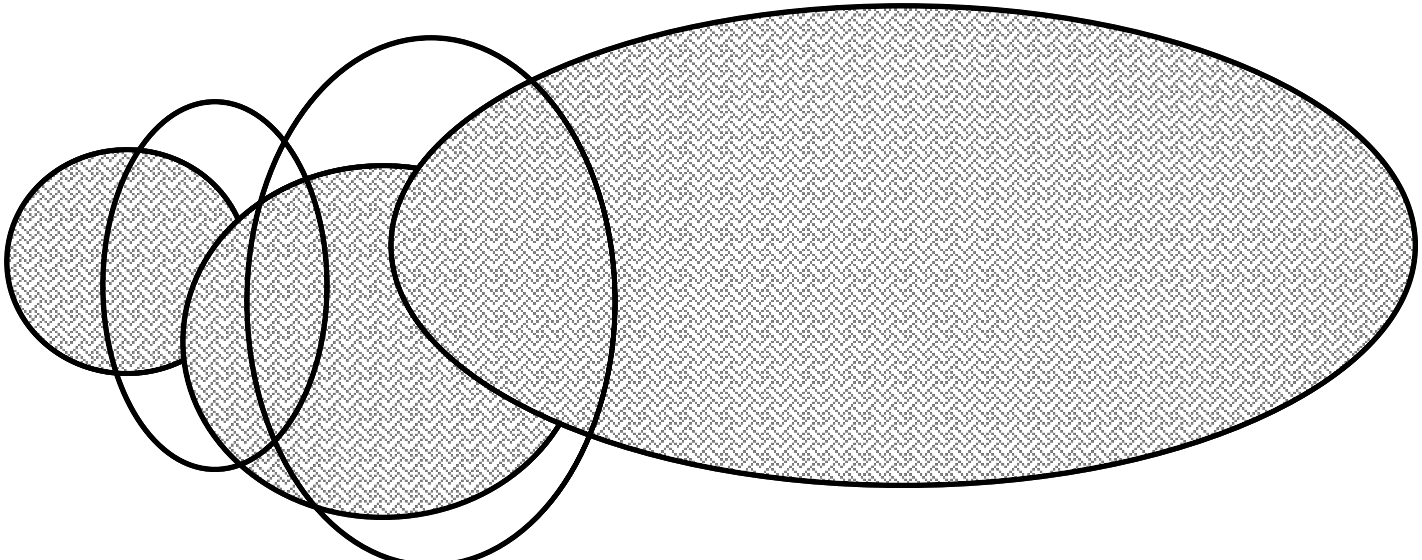
CF-coloring ***pseudo-discs*** (cont)

Algorithm for CF-coloring S:

1. find a "large" admissible set S'⊂ S

2. Color all regions of S' with *i*

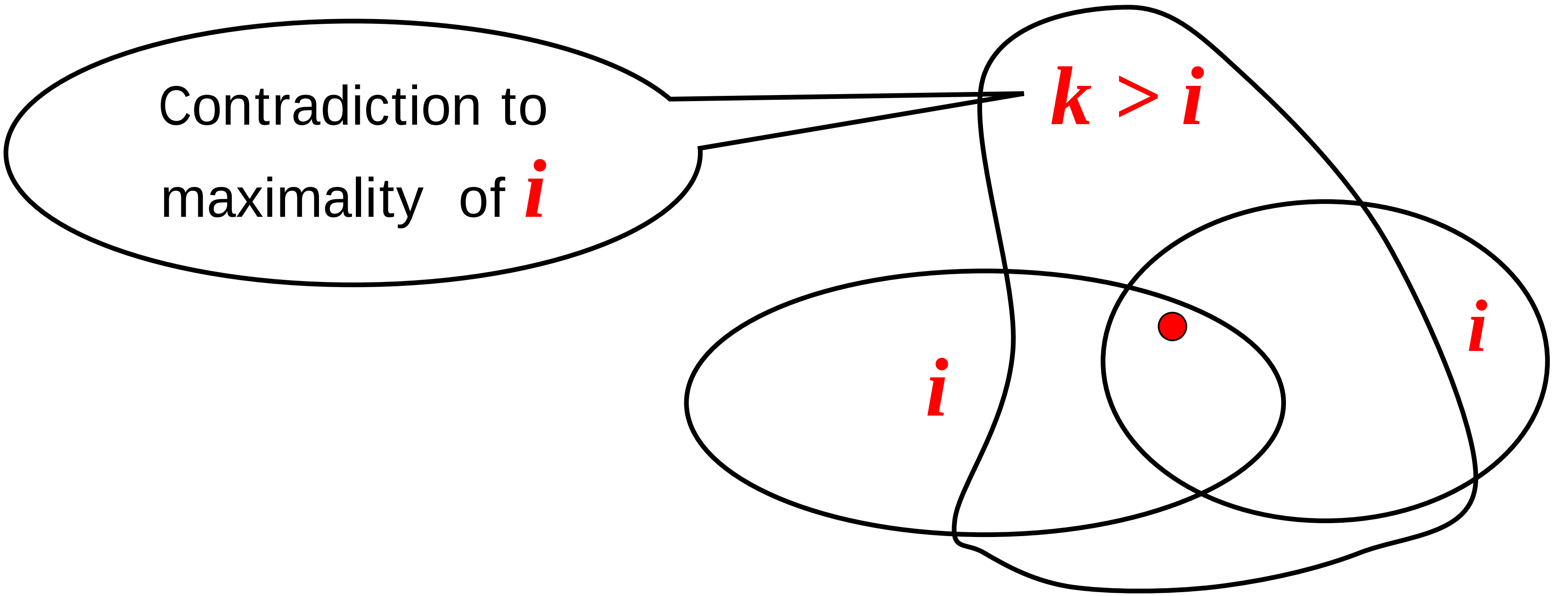
3. *i*:= *i*+1, recurse on S\S'



CF-coloring ***pseudo-discs*** (cont)

Proof: consider a point *p*.

i **= maximal color of regions containing *p*.**



CF-coloring ***pseudo-discs*** (cont)

Lemma [Har-Peled, S]

Let S be *n* pseudo-discs.

$\exists$ **admissible set S' s.t |S'| = Ω(*n*).**

Proof: Probabilistic plus some involved geometric observations…

Plugging to above algorithm gives CF-coloring with O(log *n*) colors!

Further extensions of this model

k-CF-coloring ….

non-highly overlapping discs …

Dynamic coloring …

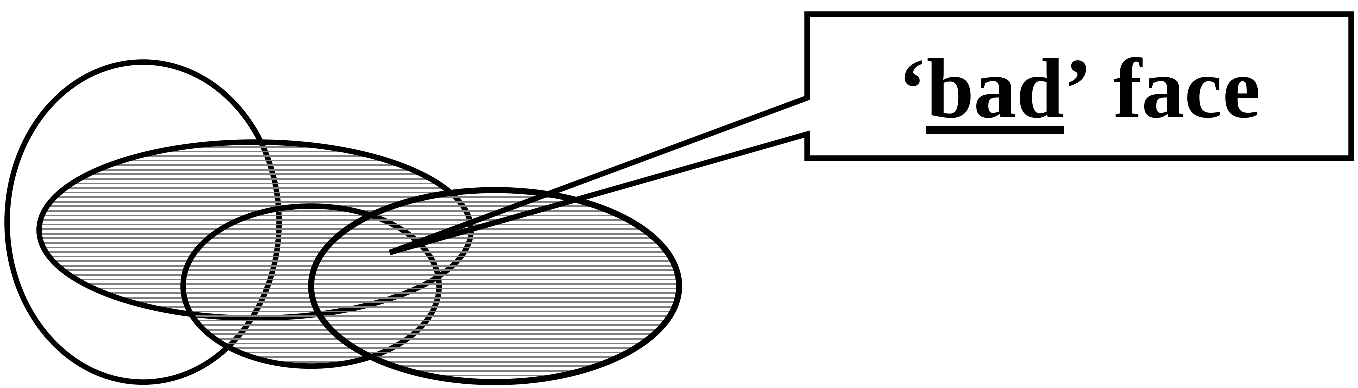

THANK YOU
WAKE UP!!!

$\exists$ admissible set S’ s.t $|S'| = \Omega(n)$. (cont)

Let S_b be the set of black regions.

***Def*: A face *f* of the arrangement of S is ‘bad’ if all regions containing *f* are ‘black’.**

Construct a graph *G* on S_b where two black regions (r_1, r_2) form an edge if $r_1 \cap r_2$ contains a bad face.



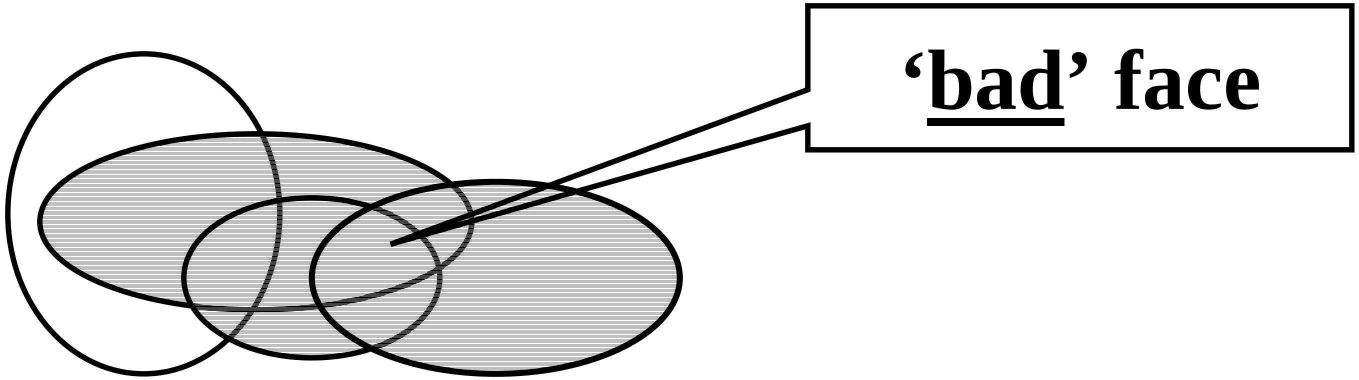
$\exists$ admissible set S' s.t $|S'| = \Omega(n)$. (cont)

Note: An independent set in G is admissible!

We will show that with constant prob:

1. G contains at least $n/3$ vertices.

2. G contains a 'large' independent set



$\exists$ admissible set S' s.t $|S'| = \Omega(n)$. (cont)

1. G contains at least $n/3$ vertices.

W.H.P by Chernoff.

2. Let X_f be the random variable having values: