On Some Geometric Optimal Path and Network Problems

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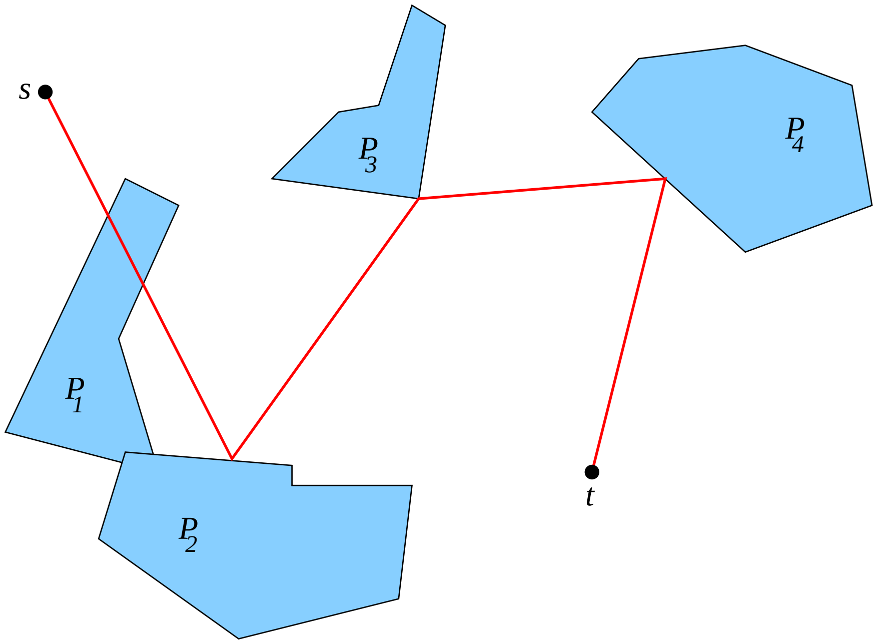
Overview of Talk:

- Some shortest path problems
  - Visiting a sequence of regions: Touring Polygons Problem (TPP)
  - Some "simple" 3D shortest path problems that are hard
  - Shortest paths in 3D, over a terrain
- A TSP variant: TSP with Neighborhoods
- Min-diameter bounded degree spanning trees: Freeze-Tag Problem

The Touring Polygons Problem (TPP) [Dror-Efrat-Lubiw-M]:

Given a sequence of k polygons in the plane, a start point s, and a target point, t, we seek a shortest path that starts at s, visits in order each of the polygons, and ends at t.



Related Problem: TSPN:

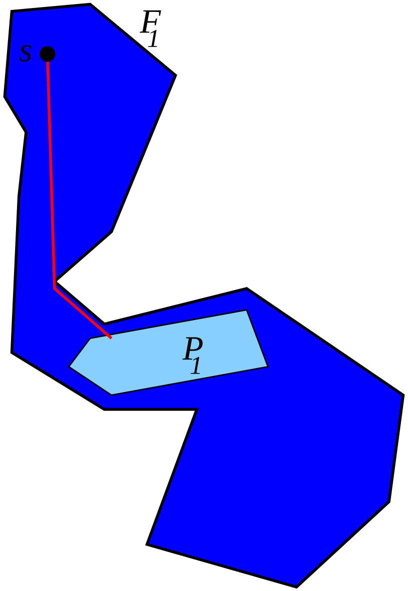
If the order to visit $\{P_1, P_2, \ldots, P_k\}$ is **not** specified, we get the NP-hard TSP with Neighborhoods problem.

TSPN: $O(\log n)$-approx in general

$O(1)$-approx, PTAS in special cases

The Fenced Problem:

Here that part of the path connecting P_i to P_{i+1} must lie inside a a simple polygon F_i, called the *fence*.



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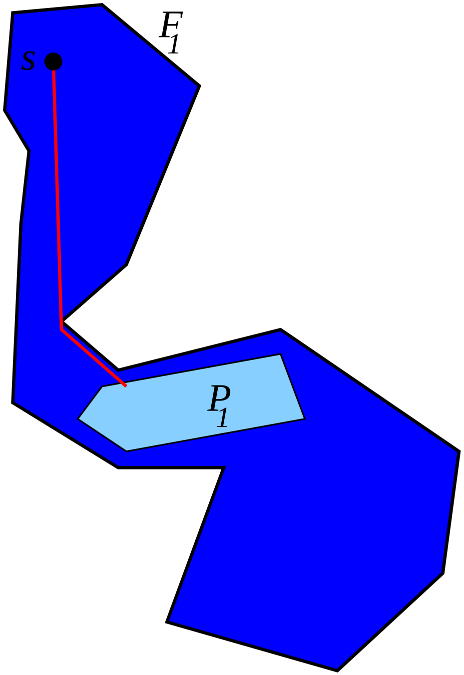

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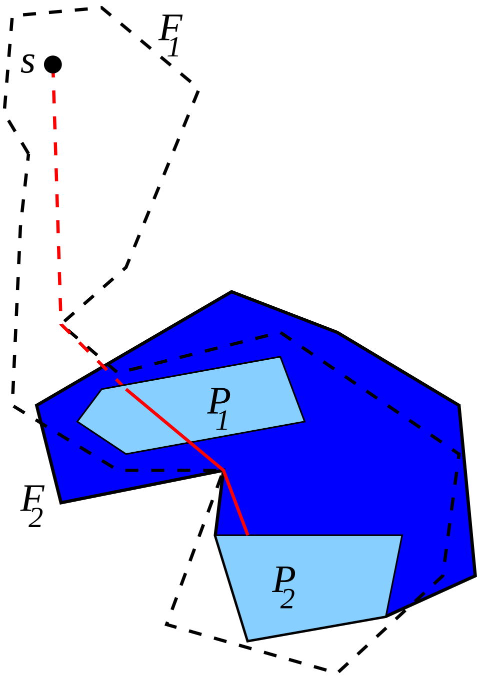

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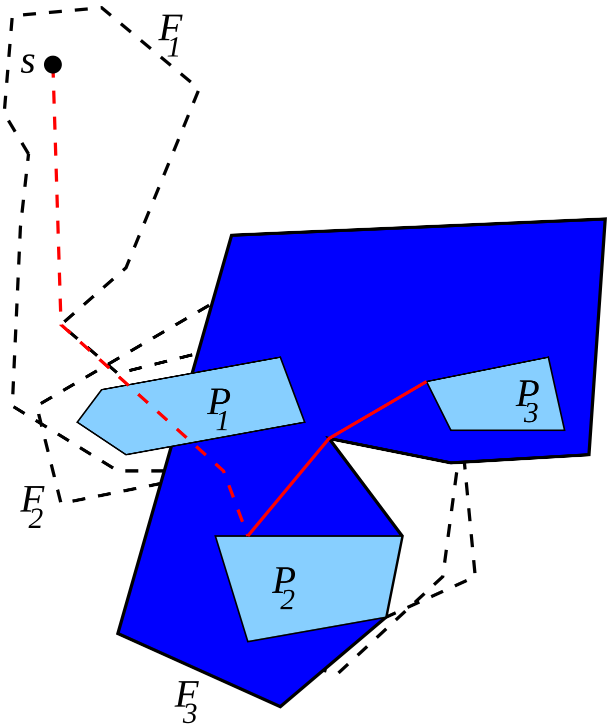

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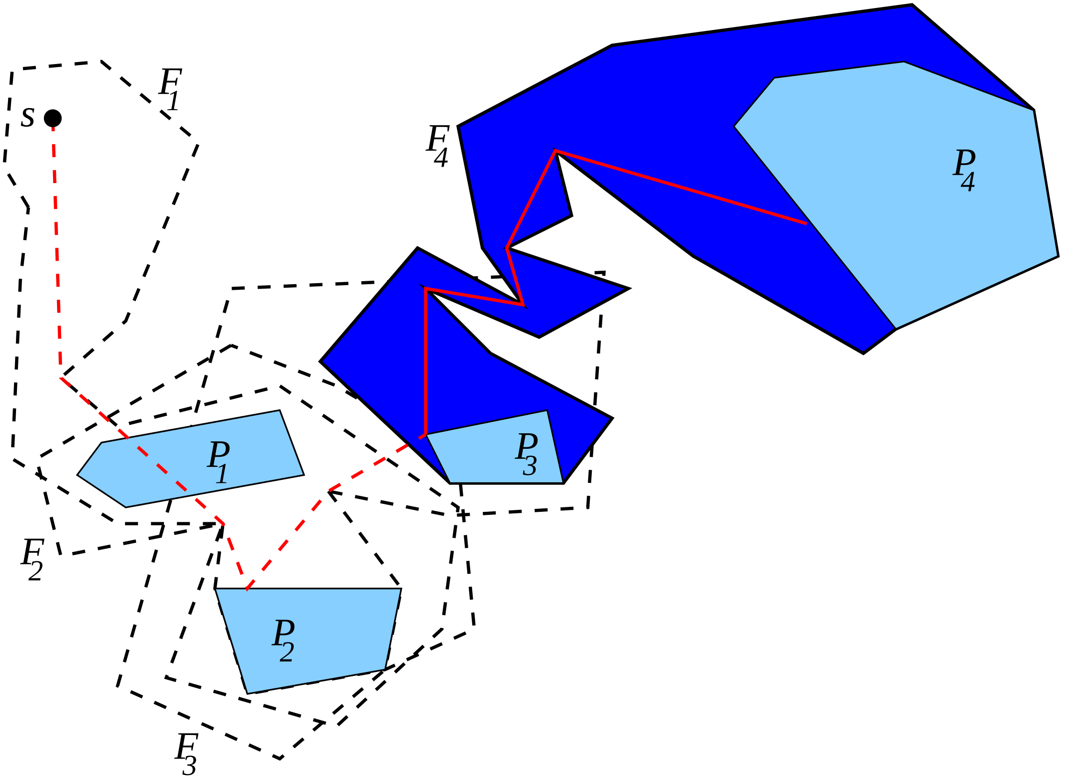

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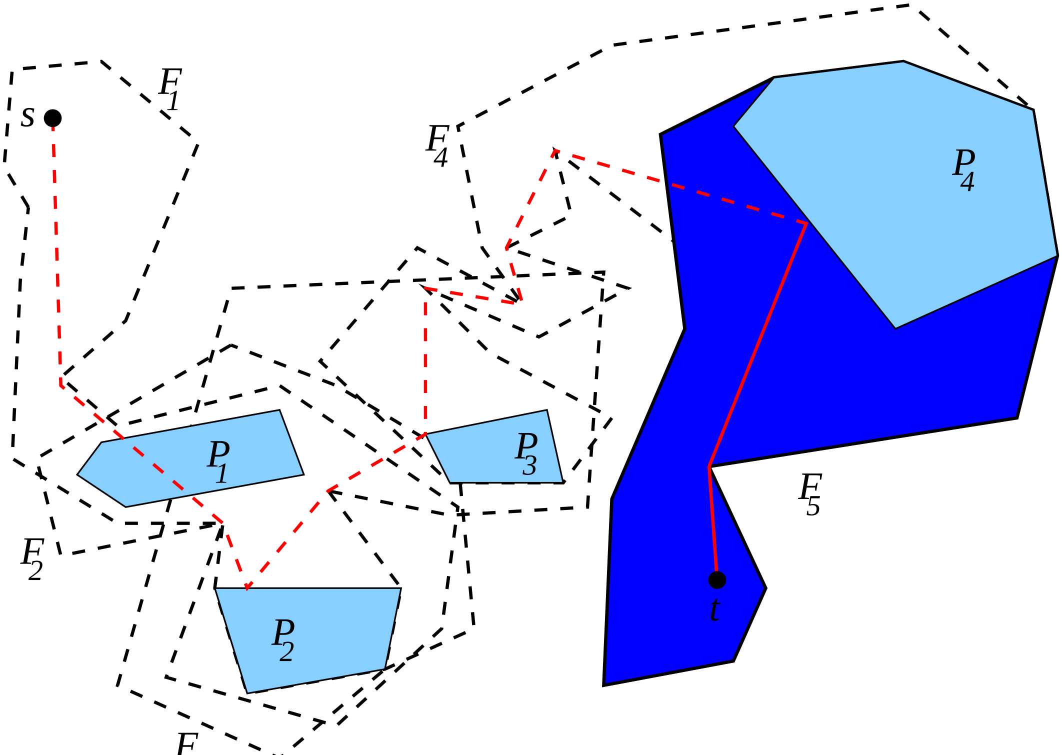
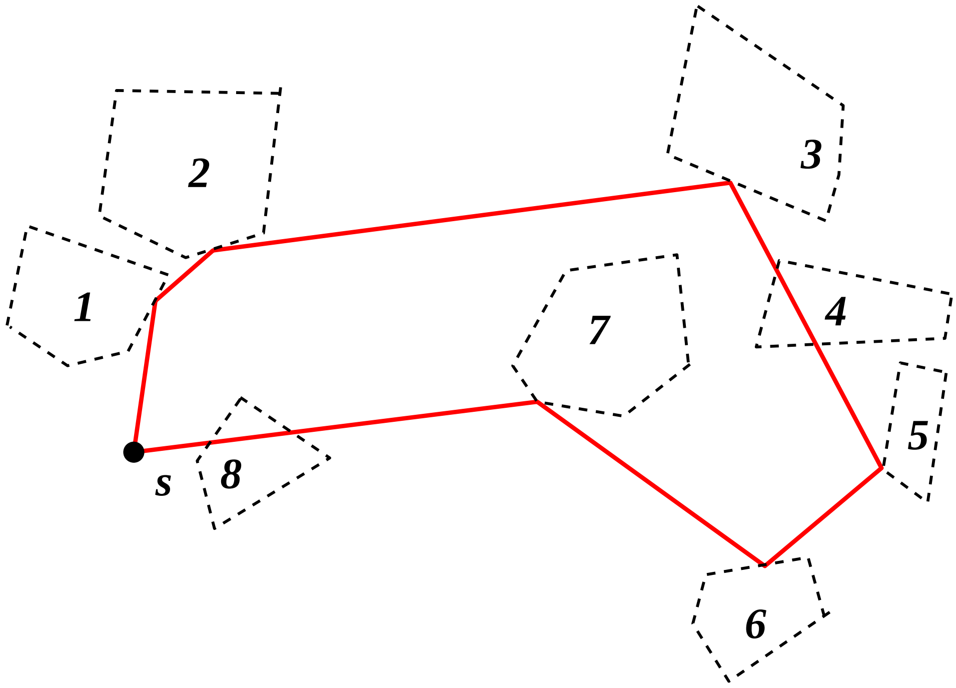
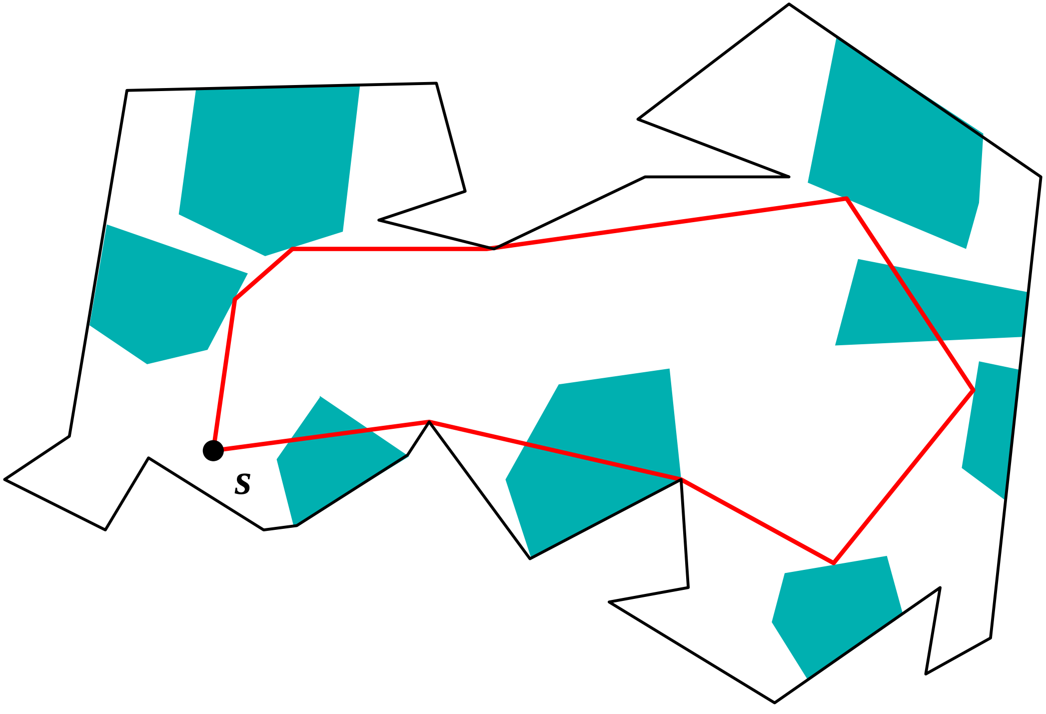


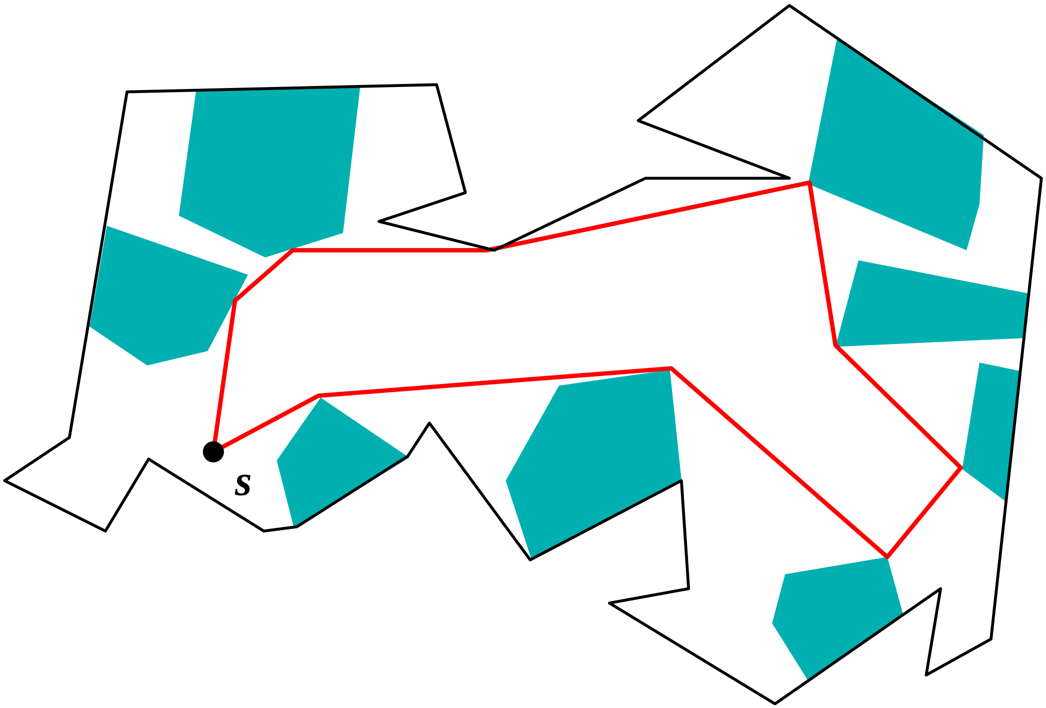
Applications: Parts Cutting Problem:



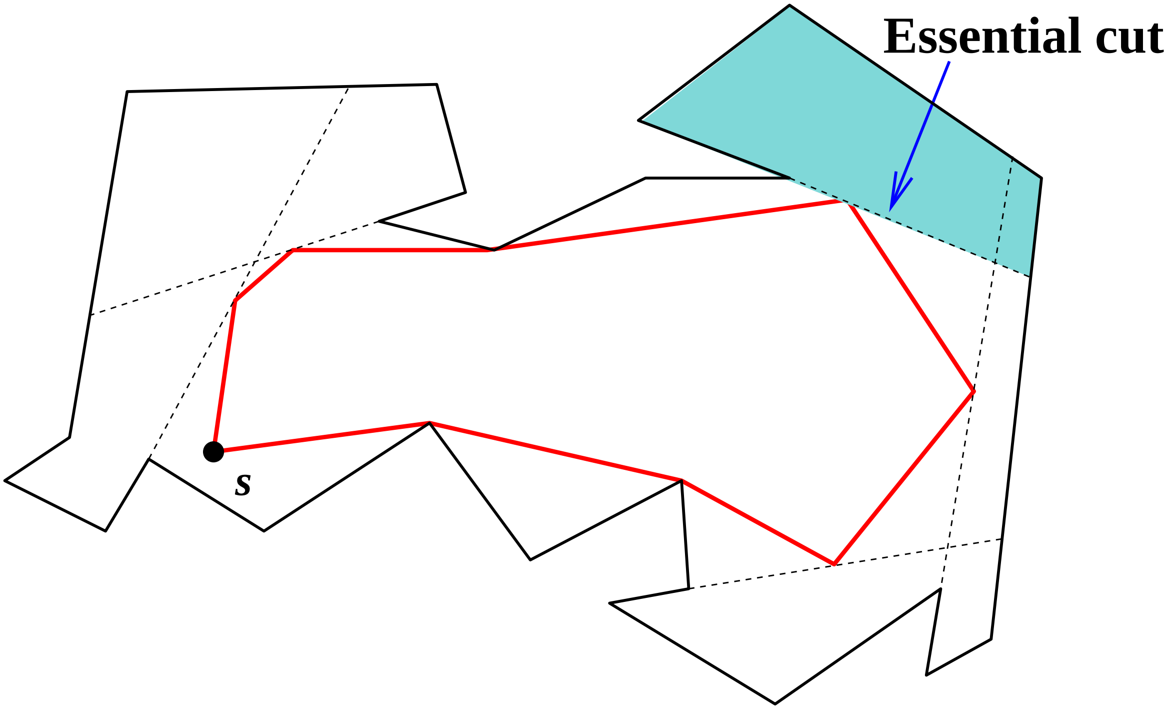
Applications: Safari Problem:



Applications: Zookeeper Problem:



Applications: Watchman Route Problem:



Fact: The optimal path visits the essential cuts in the order they appear along ∂P.

Summary of TPP Results:

- Disjoint convex polygons: $O(kn \log(n/k))$ time, $O(n)$ space

  (For fixed $s, \{P_1, P_2, \ldots, P_k\}$, $O(k \log(n/k))$ shortest path queries to t.)

- Arbitrary convex polygons: $O(nk^2 \log n)$ time, $O(nk)$ space

- Full combinatorial map: worst-case size $\Theta((n-k)2^k)$

  Output-sensitive algorithm; $O(k + \log n)$-time shortest path queries.

- TPP for nonconvex polygons: NP-hard

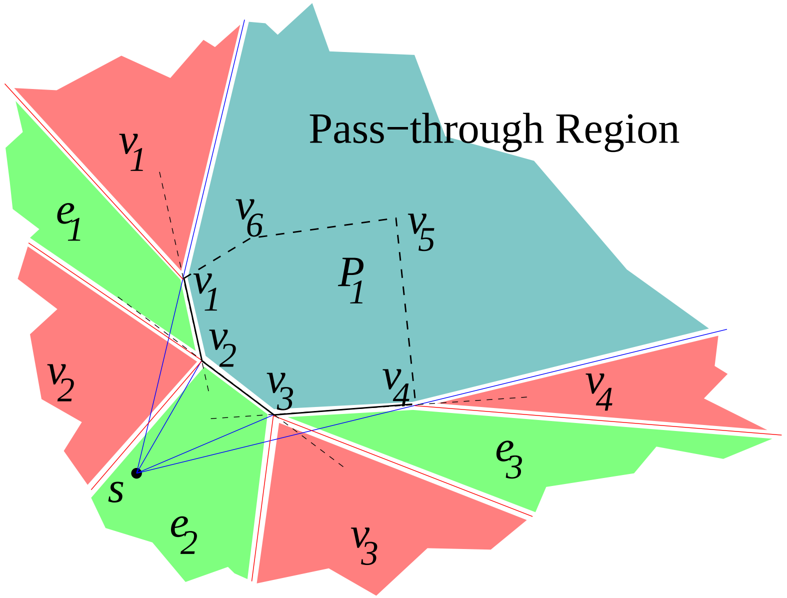
  FPTAS, as special case of 3D shortst paths

- Applications:
  - Safari: $O(n^2 \log n)$ vs. $O(n^3)$
  - Watchman: $O(n^3 \log n)$ vs. $O(n^4)$

    floating watchman: $O(n^4 \log n)$ vs. $O(n^5)$

    We avoid use of complicated path “adjustments” arguments, DP
  - Parts cutting: $O(kn \log(n/k))$

The Last Step Shortest Path Map:



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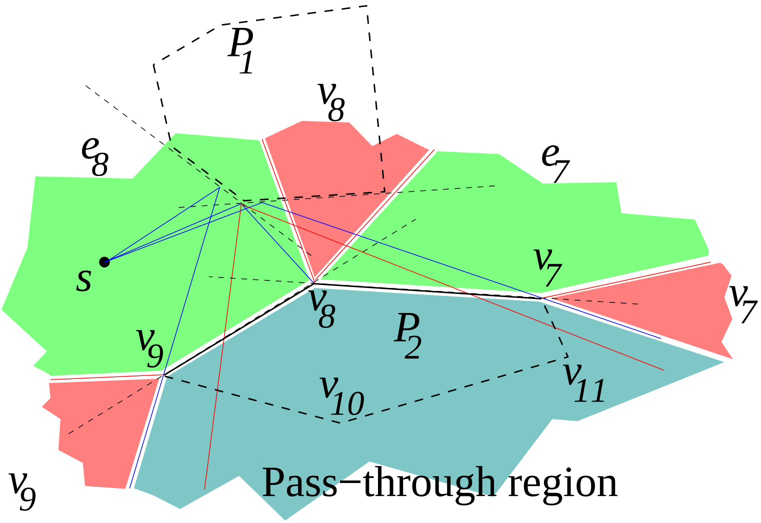
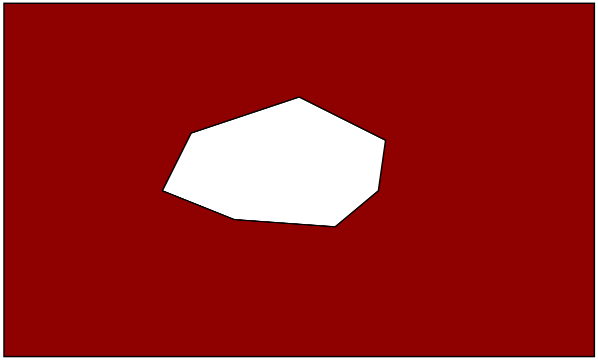


Relationship to 3D Shortest Paths:



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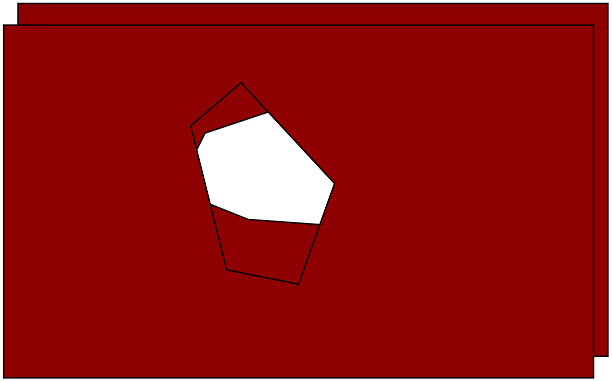

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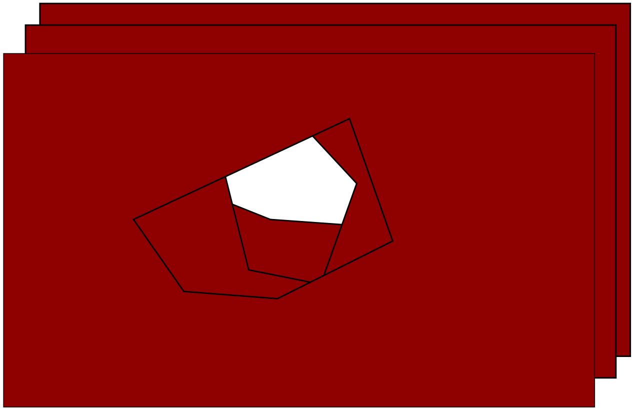

We show:

- Holes are convex: poly-time — last-step SPM
- Non-convex holes: NP-hard

3D Shortest Paths: Background:

- NP-hard in general **[CR]**
- FPTAS **[Pa],[Cl],[CSY],[H-P]**
- Special cases: surfaces, k convex polytopes, buildings of k heights

Shortest Paths Among Stacked (Flat) Obstacles:

If obstacles are **complements** of convex polygons: TPP solves

(case includes halfplanes)

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New result: Still NP-hard **[M-Sharir]**

Hardness Proof:

Theorem: The Euclidean shortest path problem is NP-hard for a stack of axis-parallel rectangles as obstacles.

Proof: from 3-SAT, based on modified Canny-Reif proof

- Use a cascade of path splitter gadgets to get 2^n combinatorially distinct path classes

  Paths encode an assignment of the n variables: path # i encodes assignment given by the (n-bit) binary representation of i.

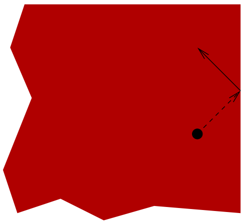
- Use path shuffle gadgets to rearrange paths within a class

- Use shuffle gadgets to construct a literal filter: the only path classes that pass through unobstructed are those having bit b_i set accordingly

- Assemble 3 literal filters per clause filter: output of clause filter will contain short path classes only for those assignments (if any) that satisfy the instance of 3SAT
- Collect paths back into one path class, using inverted path splitting gadgets.
- Final question: Is there a path from s to t of length L?

  Yes, iff the formula is satisfied.

Path Splitting Gadget:



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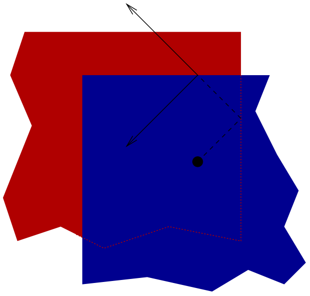

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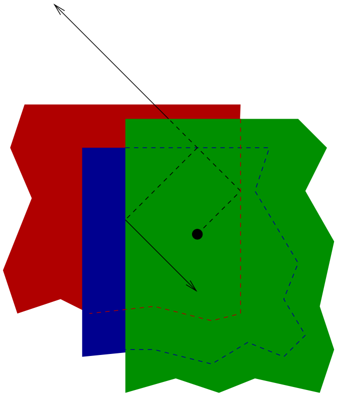

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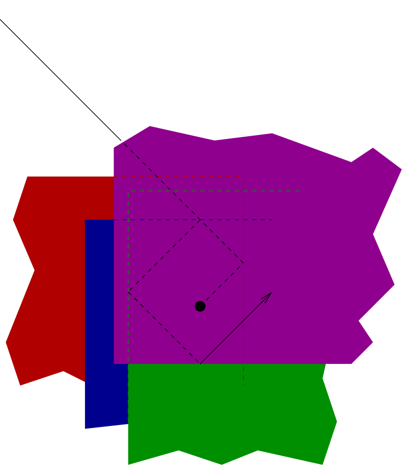

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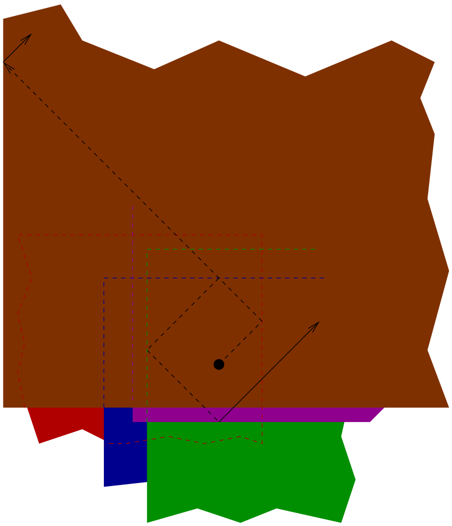
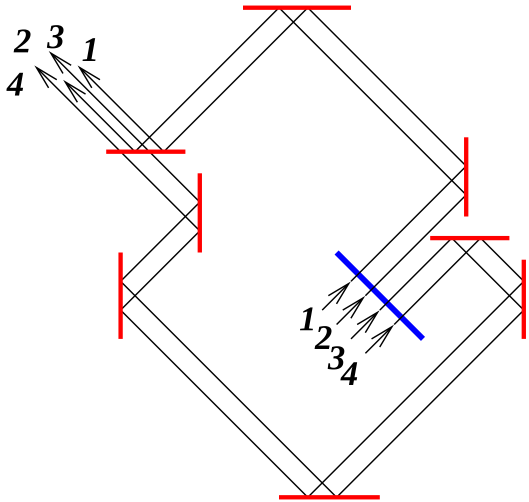


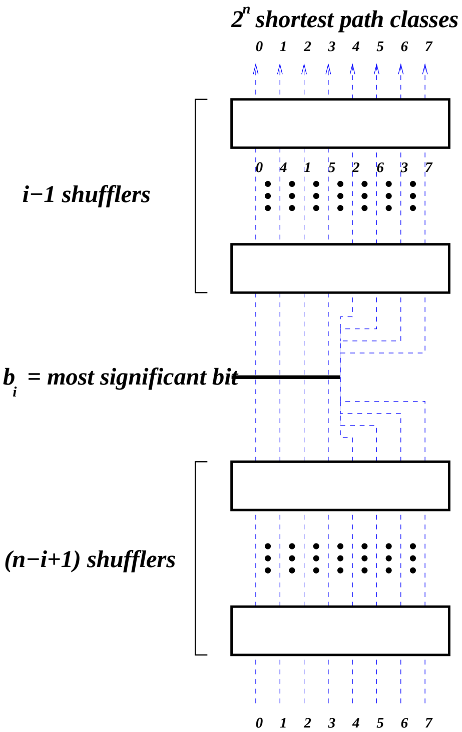
Path Shuffle Gadget:



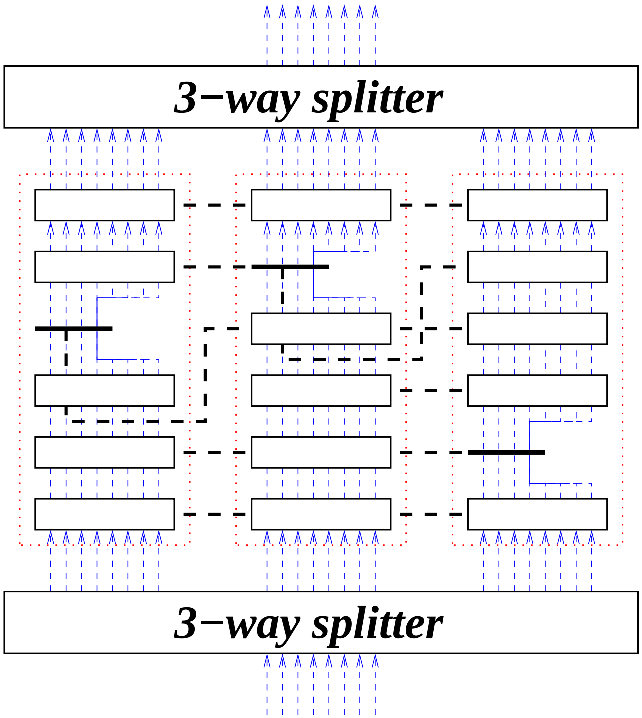
Input Order: 1,2,3,4

Output Order: 4,2,3,1

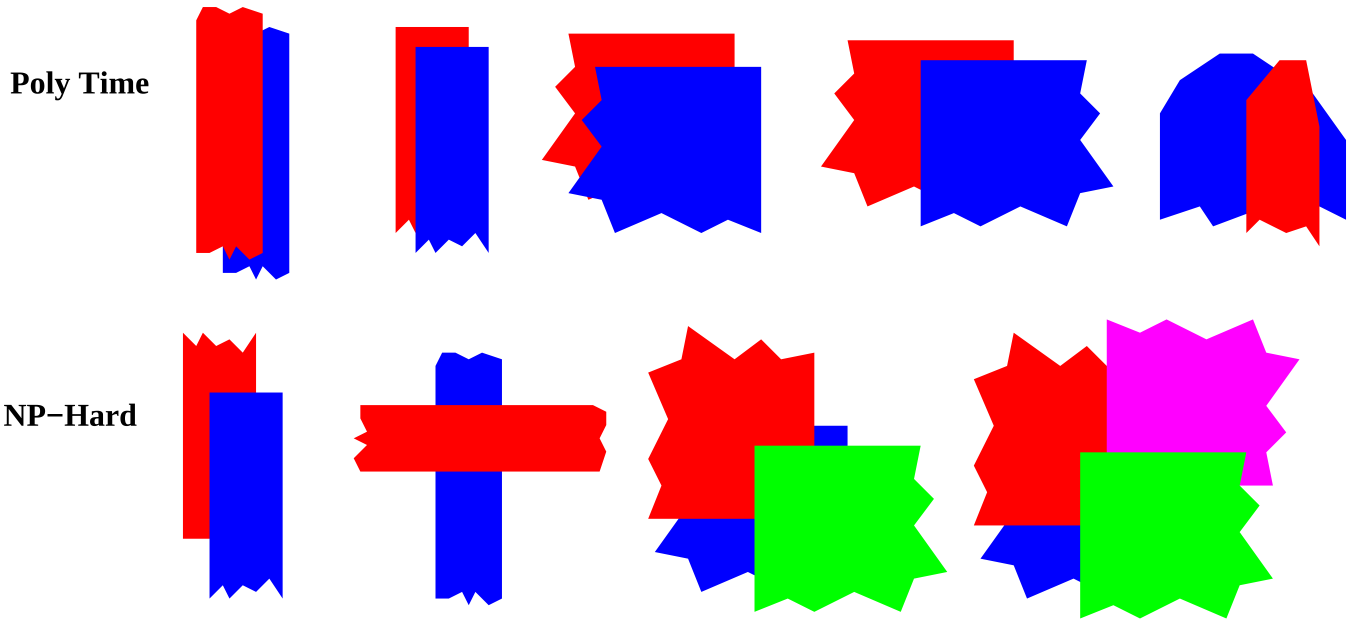
Literal Filter:



Clause Filter:



Instances of Stacked Obstacles:



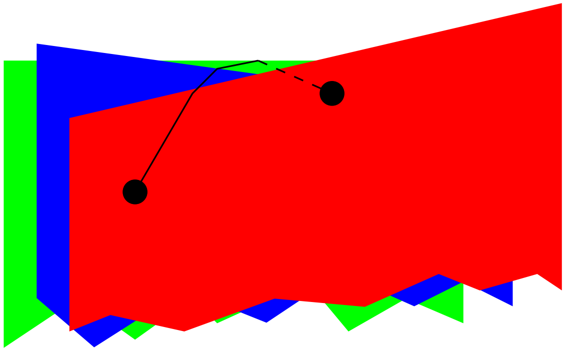
Shortest Paths Among Balls:

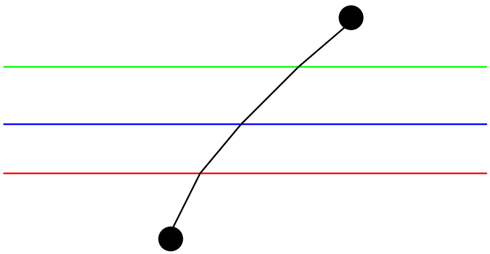
Also NP-Hard: L_1 shortest paths among balls in 3D

OPEN: Euclidean shortest paths among balls in 3D? Unit balls?

OPEN: Euclidean shortest paths among aligned cubes in 3D? Unit cubes?

Shortest Path Over Walls **[M-Sharir]:**





Top View

n lines in 3D: $e_1, \ldots, e_n$, each $\perp$ to y-axis

e_i: $y = a_i$, $z = b_i x + c_i$, with $a_1 < a_2 < \cdots < a_n$

Each e_i defines a (vertical halfplane) *wall* W_i

Goal: Find L_2-shortest path from s to t avoiding the interiors of walls

Some properties of $\pi(\cdot)$ and $L(\cdot)$:

$\pi(\zeta) =$ shortest path from s to $\zeta \in \Re^3$
$L(\zeta) =$ length of path $\pi(\zeta)$

(1) $\pi(\zeta)$ is y-monotone, polygonal, bending on some of the edges e_i

(2) $\pi(\zeta) = \pi_1 \| \pi_2$, with π_1 ascending (in z), π_2 descending

(3) The path $\pi(\zeta)$ is unique

Corollary: As ζ varies along a line ℓ,

(i) $L(\zeta)$ is a convex function of $\zeta \in \ell$

(ii) $\pi(\zeta)$ varies continuously (Hausdorff metric)

(iii) The combinatorial structure of $\pi(\zeta)$ changes only when it passes through 3 collinear (mutually visible) points on 3 distinct edges

(4) Solution by convex programming: LP-type problem

The Shortest Path Map:

Combinatorial Complexity of the Shortest-Path Map:

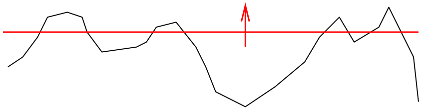
Lemma: For each $i < n$, the set

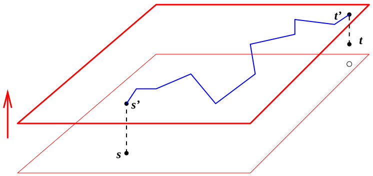
$$C_i = \{\zeta \in e_n \mid \pi(\zeta) \cap e_i \neq \emptyset\}$$

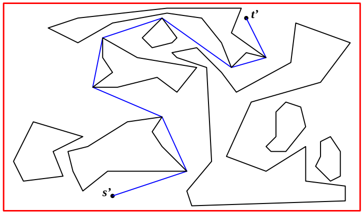
is connected.

Theorem: The number of combinatorial changes in the structure of $\pi(\zeta)$, as ζ moves along e_n, is $O(n)$.

L_1 Shortest Paths Over Terrains **[M-Sharir]:**







Structure: Can assume path goes up from s to s' (altitude h), then along a shortest path in the plane $z = h$ to t', then down to t

Atomic intervals: Partition heights h according to vertex heights (v_z), and critical heights at which $\exists$ edges e, e' of T for which the points $e(h)$ and $e'(h)$ have the same x- or y-coordinate.

$O(n^2)$ atomic intervals

L_1 Shortest Paths Over Terrains (cont):

Lemma: The length function, $L(h)$, is concave, piecewise-linear over each atomic interval

Algorithm: Compute shortest path for each of the $O(n^2)$ critical heights, and take best: $O(n^3 \log n)$.

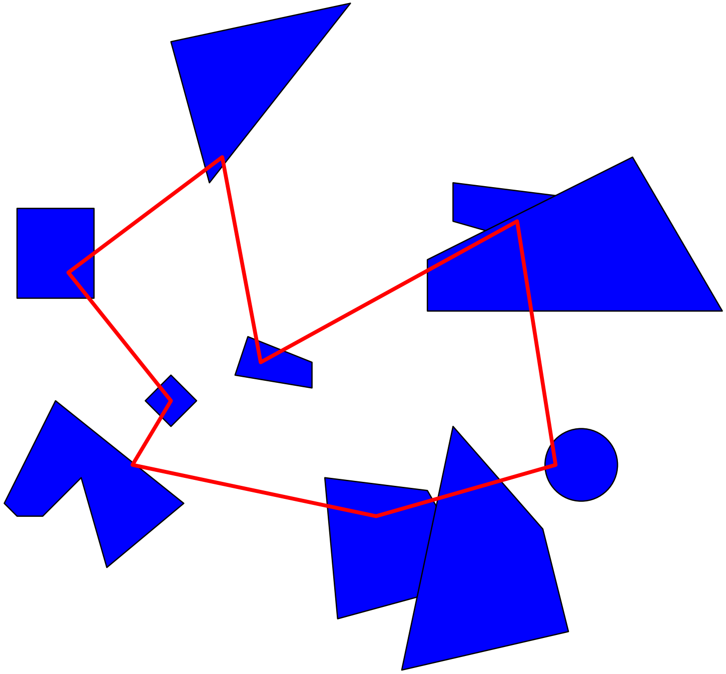
Work in progress: Analyzing the SPM for L_1 shortest paths over a terrain.

OPEN: Euclidean shortest paths over a terrain?

Note: SPM has worst-case exponential size

TSP with Neighborhoods (TSPN):

$S = \{X_1, X_2, \ldots, X_k\}$, a set of regions that must be visited



TSP with Neighborhoods (cont):

Problem introduced by Arkin-Hassin

- "obvious" heuristics do not work:

  TSP approx on centroids (as representative points)

  greedy algorithms (Prim- or Kruskal-like)

- $O(1)$-approx for "nice" regions:

  (a) parallel unit segments

  (b) unit disks

  (c) translates of a polygon P

- "Combination Lemma"

TSPN – More Approximation Results:

- General (connected) regions: $O(\log k)$-approx [MM95]

  use guillotine rectangular subdivisions (GRS), DP

  $O(n^5)$ time

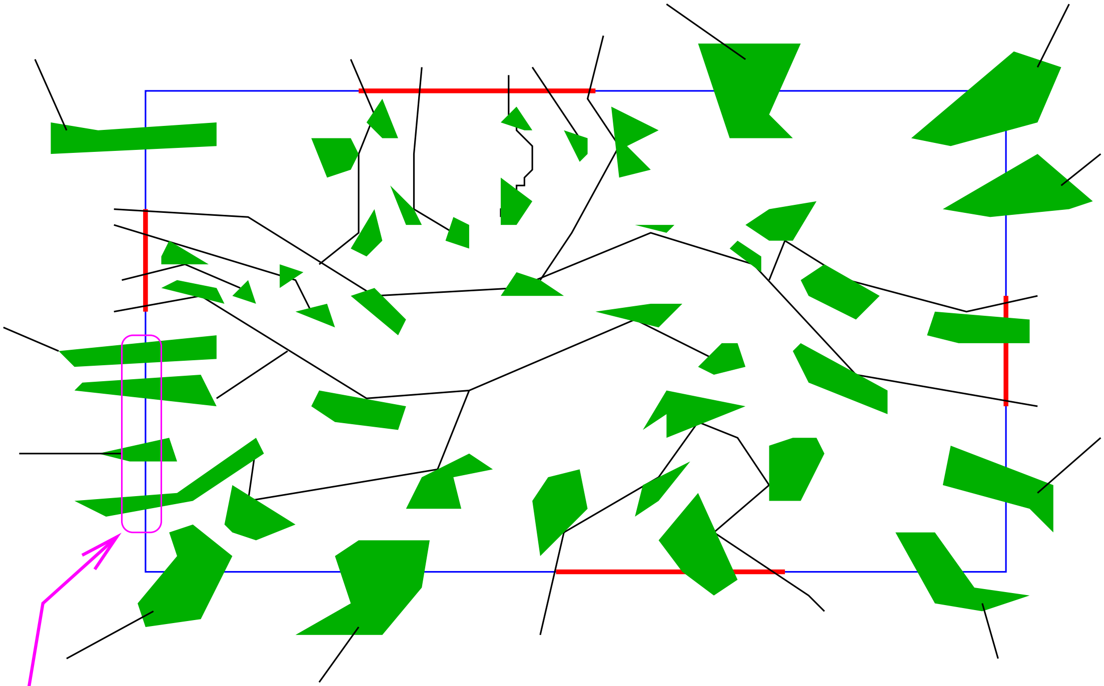
  (difficulty is in DP)

- Improved running time, using modified GRS [GL99]

  $O(n + k \log k)$

Difficulty in Applying TSP Methods to TSPN:

How to define a subproblem succinctly?



Which regions must be visited inside R?

Recent Progress on TSPN:

- $O(1)$-approx for regions of comparable size (diameter) [DM]
- PTAS for disjoint fat objects of comparable size [DM]

  (use a new charging scheme and m-guillotine subdivisions)
- $O(1)$-approx for disjoint fat objects of any size [BGK+]
- Hardness of approx:

  No c-approx with $c < 391/390$, unless P=NP [BGK+]

  No c-approx with $c < 2$, unless $NP \subseteq TIME(n^{O(\log\log n)})$ [SS]

Recent Progress on TSPN (cont):

- $O(1)$-approx, PTAS for planes in $\Re^3$ [ADM'03]
- Latest breakthrough:

  $O(1)$-approx for arbitrary length horizontal segments in $\Re^2$ [Mi03]

  (novel charging scheme, with m-guillotine method)

Work in progress: Extend to general disjoint regions? PTAS?