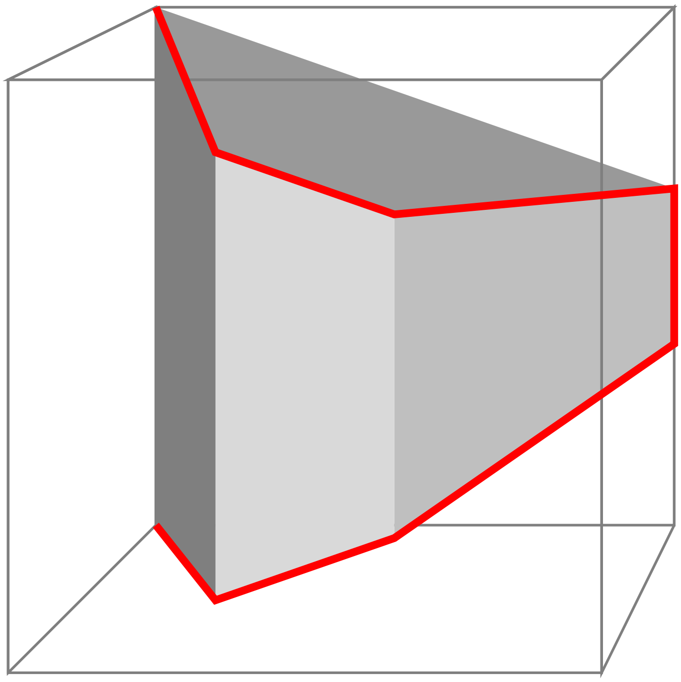
Linear Algebra
and
Cubes



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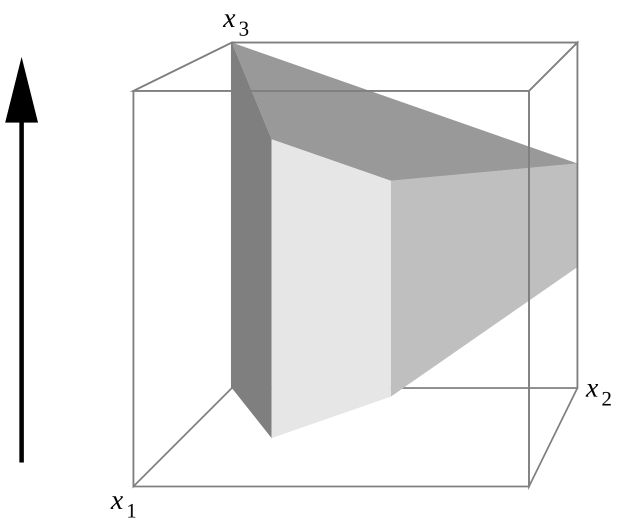
Cube Optimization (I)

Given an *acyclic unique sink orientation*:

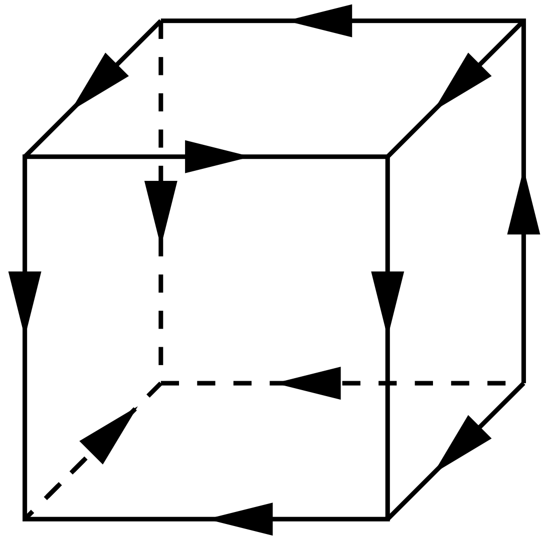
> *acyclic* orientation of the n-cube graph, such that *every nonempty face has a unique sink*.

Orientation might be defined...

in a geometric way



in an abstract way

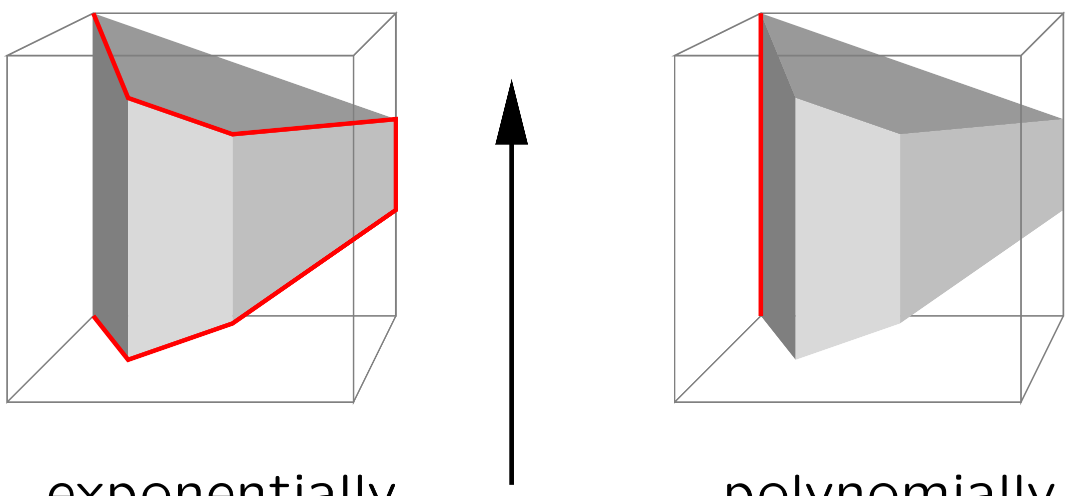


Cube Optimization (II)

Wanted:

Algorithm for finding the global sink as quickly as possible

We consider *simplex-type methods* (walks along directed paths); efficiency depends on *pivot rule* being used.



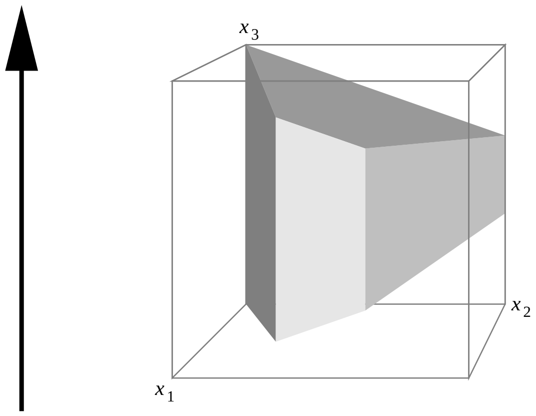
exponentially many steps

polynomially many steps

Linear Algebra. . .

. . . over the reals

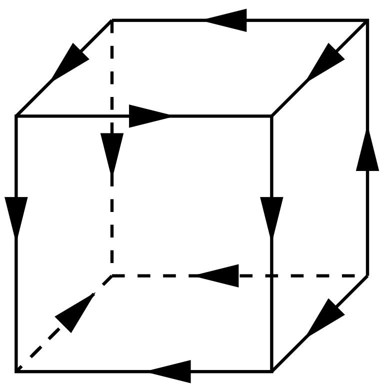
is used in constructing *geometric* cube orientations and analyzing *geometric* pivot rules



- Dantzig's rule
- Steepest Descent
- Largest Increase
- . . .

. . . over finite fields

is used in constructing *abstract* cube orientations and analyzing *combinatorial* pivot rules



- Bland's rule
- Random-Edge
- Random-Facet
- . . .

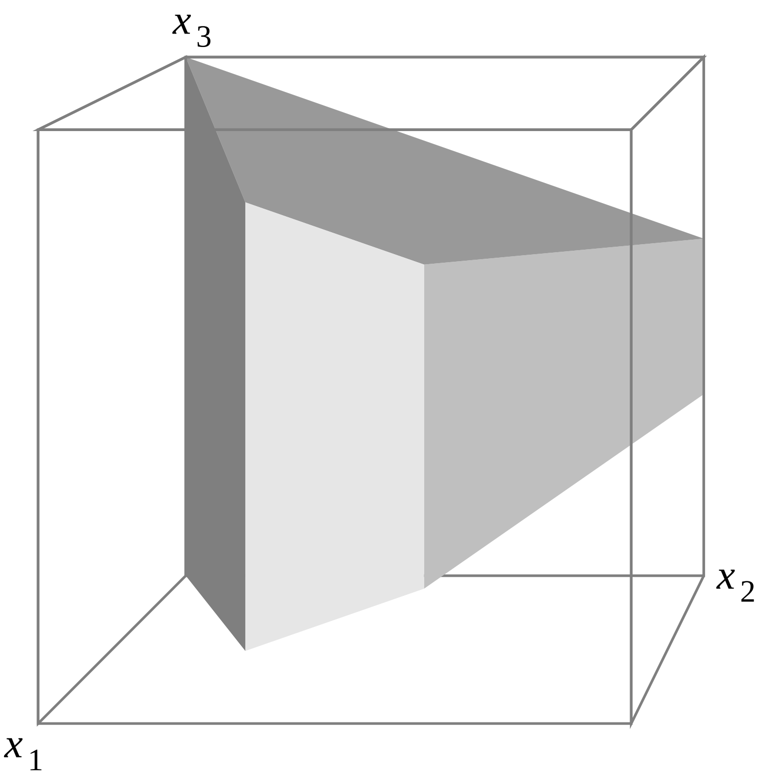
Part I:
Linear Algebra
Over the Reals

Klee & Minty's worst-case linear program, showing that *Dantzig's rule* may require exponentially many steps

Poor Man's Worst-Case LP

maximize x_n
subject to

$$\begin{array}{rcccl} 0 & \leq & x_1 & \leq & 1 \\ \varepsilon x_{i-1} & \leq & x_i & \leq & 1 - \varepsilon x_{i-1}, \quad i = 2, \ldots, n \end{array}$$



$\varepsilon = 1/3$

Observation: Simplex algorithm with "stupid" pivot rule may take $2^n - 1$ steps!

The Klee-Minty Cube (I)

Goal: Fool "smart" pivot rule!

Poor man's LP

maximize x_n
subject to

$$\begin{array}{rcccl} 0 & \leq & x_1 & \leq & 1 \\ \varepsilon x_{i-1} & \leq & x_i & \leq & 1 - \varepsilon x_{i-1}, \quad i = 2, \ldots, n \end{array}$$

$$y_1 := x_1$$
$$y_i := x_i - \varepsilon x_{i-1}, \quad i = 2, \ldots, n$$
$$\text{slack variables } s_i, \quad i = 1, \ldots, n$$

$$\Downarrow$$

Poor man's LP in standard equality form

maximize $\sum_{i=1}^{n} \varepsilon^{n-i} y_i$ subject to

$$\begin{array}{rcll} y_i + 2\sum_{j=1}^{i-1} \varepsilon^{i-j} y_j + s_i & = & 1, & i = 1, \ldots, n \\ y_i & \geq & 0, & i = 1, \ldots, n \\ s_i & \geq & 0, & i = 1, \ldots, n \end{array}$$

The Klee-Minty Cube (II)

$$\begin{array}{rcl} s_1 &=& 1 \qquad\qquad\qquad\qquad\qquad - \;\; y_1 \\ s_2 &=& 1 \qquad\qquad - \;\; y_2 \;\; - \;\; 2\varepsilon y_1 \\ s_3 &=& 1 \;-\; y_3 \;-\; 2\varepsilon y_2 \;-\; 2\varepsilon^2 y_1 \\ \hline z &=& \qquad y_3 \;+\; \varepsilon y_2 \;+\; \varepsilon^2 y_1 \end{array}$$

$\Downarrow$

Swap y_1, s_1 via $y_1 = 1 - s_1$

$\Downarrow$

$$\begin{array}{rcl} y_1 &=& 1 \qquad\qquad\qquad\qquad\qquad - \;\; s_1 \\ s_2 &=& 1 - 2\varepsilon \qquad\qquad - \;\; y_2 \;\; + \;\; 2\varepsilon s_1 \\ s_3 &=& 1 - 2\varepsilon^2 \;-\; y_3 \;-\; 2\varepsilon y_2 \;+\; 2\varepsilon^2 s_1 \\ \hline z &=& \varepsilon^2 \;+\; y_3 \;+\; \varepsilon y_2 \;-\; \varepsilon^2 s_1 \end{array}$$

$\Downarrow$

Swap y_2, s_2 via $y_2 = 1 - 2\varepsilon - s_2 + 2\varepsilon s_1$

$\Downarrow$

$$\begin{array}{rcl} \hline z &=& \varepsilon - \varepsilon^2 \;+\; y_3 \;-\; \varepsilon s_2 \;+\; \varepsilon^2 s_1 \end{array}$$

$\Downarrow$

$$\begin{array}{rcl} \hline z &=& \varepsilon \;+\; y_3 \;-\; \varepsilon s_2 \;-\; \varepsilon^2 y_1 \end{array}$$

$\Downarrow$

$$\begin{array}{rcl} \hline z &=& 1 - \varepsilon \;-\; s_3 \;+\; \varepsilon s_2 \;+\; \varepsilon^2 y_1 \end{array}$$

The Klee-Minty Cube (III)

Lemma (fooling the stupid): The "stupid" pivot rule of always choosing the variable with *smallest* positive coefficient in the z-row leads to $2^n - 1$ pivot steps on the poor man's worst-case LP...

... while one step suffices for the "smart" pivot rule of always choosing the variable with *largest* positive coefficient.

Lemma (fooling the smart): For $\varepsilon > 3$, the *Klee-Minty cube* is the tweaked poor man's LP

maximize $\sum_{i=1}^{n} \varepsilon^{n-i} y_i$ subject to

$$\begin{aligned} y_i + 2\sum_{j=1}^{i-1} \varepsilon^{i-j} y_j + s_i &= \varepsilon^{2i}, i = 1, \ldots, n \\ y_i &\geq 0, \quad i = 1, \ldots, n \\ s_i &\geq 0, \quad i = 1, \ldots, n \end{aligned}$$

It is a cube on which the "smart" pivot rule of always choosing the variable with *largest* positive coefficient in the z-row leads to $2^n - 1$ pivot steps. $\Leftarrow$ *Dantzig's rule*!

Part II: Linear Algebra Over $GF(2)$

- Ignorant (combinatorial) pivot rules
- Combinatorics of the Klee-Minty cube
- Random-Edge on the Klee-Minty cube
- Matoušek's abstract cubes
- Random-Facet on Matoušek cubes

Ignorant Pivot Rules (I)

Random-Edge: among the variables with positive coefficient in the z-row, *choose one uniformly at random*.

- Behavior only depends on *sign pattern* in the z-row, *not* on actual coefficients
- Expected number of steps is the same for the poor man's worst case LP and the Klee-Minty cube

Questions:

- What is this expected number of steps?
- Does the Klee-Minty cube fool the ignorant?

The Combinatorial z-row (I)

- $\left\{ \begin{array}{c} x_n \\ s_n \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}$

- $\left\{ \begin{array}{c} x_i \\ s_i \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}, \quad i = 1, \ldots, n-1$

- $\left\{ \begin{array}{c} + \\ - \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}$

z-row of optimal tableau:

z	$=$	1	$-$	s_3	$-$	εy_2	$-$	$\varepsilon^2 y_1$
vertex				0		0		0
value			0		0		0	

The Combinatorial z-row (II)

	z	$=$		$+$	y_3	$+$	εy_2	$+$	$\varepsilon^2 y_1$
vertex					1		0		0
value				1		1		1	

	z	$=$	ε^2	$+$	y_3	$+$	εy_2	$-$	$\varepsilon^2 s_1$
vertex					1		0		1
value				1		1		0	

	z	$=$	$\varepsilon - \varepsilon^2$	$+$	y_3	$-$	εs_2	$+$	$\varepsilon^2 s_1$
vertex					1		1		1
value				1		0		1	

	z	$=$	ε	$+$	y_3	$-$	εs_2	$-$	$\varepsilon^2 y_1$
vertex					1		1		0
value				1		0		0	

	z	$=$	$1 - \varepsilon$	$-$	s_3	$+$	εs_2	$+$	$\varepsilon^2 y_1$
vertex					0		1		0
value				0		1		1	

Combinatorial KM-cube (I)

- vertices $\equiv GF(2)^n$, values $\equiv GF(2)^n$

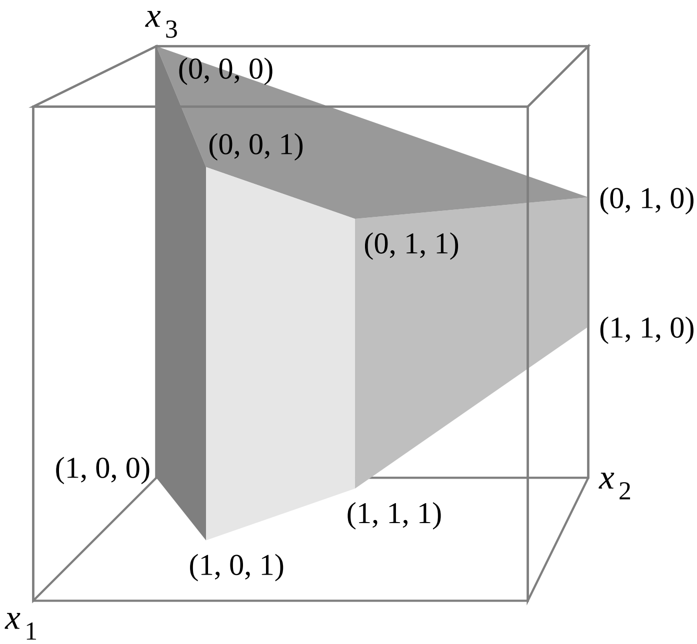
- adjacent vertices differ in exactly one coordinate

- vertex v has value Av, where
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in GF(2)^{n \times n}.$$

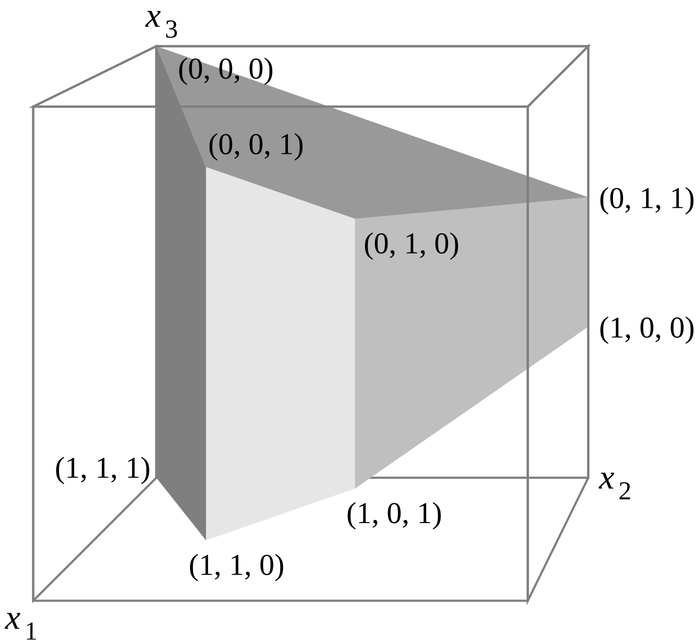
- v has better objective function value than v' iff $Av < Av'$ under lexicographic ordering of the values

- the step from v to $v + \mathbf{e}_i$ is a legal (improving) pivot iff $(Av)_i = 1$, equivalently if there is an odd number of 1-entries in $v_1 \ldots v_i$.

Combinatorial KM-cube (II)



Vertices



Values

Random-Edge Revisited

Fast Game (on *values*): among all 1-entries, choose one *uniformly at random* and flip it, along with all entries further to the right.

$$\begin{array}{ccccc}
1 & 1 & 0 & 1 & 1 \\
 & & & \downarrow & \\
1 & 1 & 0 & 0 & 0 \\
\downarrow & & & & \\
0 & 0 & 1 & 1 & 1 \\
 & & & \downarrow & \\
0 & 0 & 1 & 0 & 0 \\
 & & \downarrow & & \\
0 & 0 & 0 & 1 & 1 \\
 & & & \downarrow & \\
0 & 0 & 0 & 0 & 0
\end{array}$$

$\Rightarrow$ expected number of steps is $O(n^2)$

Lower bound???

The Slow Game

Slow Game: among *all* entries, choose one uniformly at random; if it is a 1-entry, flip it, along with all entries further to the right, otherwise, do nothing.

$$
\begin{array}{cccccl}
1 & 1 & 0 & 1 & 1 & \\
 & & & \downarrow & & \\
1 & 1 & 0 & 0 & 0 & \\
\downarrow & & & & & \\
0 & 0 & 1 & 1 & 1 & \\
 & \downarrow & & & & \text{void flip} \\
0 & 0 & 1 & 1 & 1 & \\
 & & & \downarrow & & \\
0 & 0 & 1 & 0 & 0 & \\
 & & & & \downarrow & \text{void flip} \\
0 & 0 & 1 & 0 & 0 & \\
 & & \downarrow & & & \\
0 & 0 & 0 & 1 & 1 & \\
 & & & \downarrow & & \\
0 & 0 & 0 & 0 & 0 &
\end{array}
$$

$\Rightarrow$ expected number of steps is $\Omega(n^2/\log n)$

Fast vs. Slow Game (I)

- flip of w at i (real or void): $w \to A^{(i)}w$, where

$$A^{(i)} = \begin{pmatrix} 1 & & & \downarrow \text{column } i & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & & \leftarrow \text{row } i & \\ & & & 1 & 1 & & \\ & & & \vdots & & \ddots & \\ & & & 1 & & & 1 \end{pmatrix}$$

- flip sequence: $s = (i_1, i_2, \ldots)$

- flip of w with (s,k): $w \to w^{(s,k)}$, where

$$\begin{aligned} w^{(s,k)} &:= A^{(s,k)}w, \\ A^{(s,k)} &:= A^{i_k} \cdots A^{i_2} A^{i_1} \end{aligned}$$

- $\mathcal{S} =$ set of flip sequences, $\quad \mathcal{V} = GF(2)^n$

Fast vs. Slow Game (II)

- $F(w), S(w)$: expected number of steps in fast and slow game, starting with w

- $F(n) := \frac{1}{2^n} \sum_{w \in \mathcal{V}} F(w)$

$$
\begin{aligned}
F(n) &= \sum_{k=1}^{\infty} \underset{\mathcal{S},\mathcal{V}}{\text{prob}}(w^{(s,k)} \neq w^{(s,k-1)}) \\
&= \sum_{k=1}^{\infty} \frac{1}{n} \sum_{i=1}^{n} \underset{\mathcal{S},\mathcal{V}}{\text{prob}}({w^{(s,k-1)}}_i = 1) \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \underset{\mathcal{S},\mathcal{V}}{\text{prob}}((A^{(s,k-1)} w)_i = 1) \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \underset{\mathcal{S},\mathcal{V}}{\text{prob}}(({\mathbf{e}_i}^T A^{(s,k-1)}) w = 1) \\
&= \frac{1}{2n} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \underset{\mathcal{S}}{\text{prob}}({\mathbf{e}_i}^T A^{(s,k-1)} \neq \mathbf{0}) \\
&= \frac{1}{2n} \sum_{i=1}^{n} S(A \mathbf{e}_{n-i+1})
\end{aligned}
$$

Random-Edge Performance

Theorem: There is a starting vertex of the n-dimensional Klee-Minty cube for which the simplex algorithm with the Random-Edge pivot rule requires an expected number of

$$\Omega(n^2/\log n)$$

steps.

$\Rightarrow$ Klee-Minty cube “mildly” fools Random-Edge

Ignorant Pivot Rules (II)

Random-Facet: among the variables with positive coefficient in the z-row, choose the one whose index comes first in an initially chosen *random permutation* π of the indices.

- $\pi = (1, 2, \ldots, n)$: "stupid" rule, $2^n - 1$ steps

- $\pi = (n, n-1, \ldots, 1)$: "smart" rule, 1 step

Theorem: For every starting vertex of the n-dimensional Klee-Minty cube, the simplex algorithm with the Random-Facet pivot rule requires $O(n^2)$ steps, and this bound is tight.

Beyond Klee-Minty Cubes

- Klee-Minty cube:

$$\text{vertex } v \to \text{value } w = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} v.$$

- Matoušek cube:

$$\text{vertex } v \to \text{value } w = Av,$$

  with $A \in GF(2)^{n \times n}$ being a lower-triangular, invertible matrix

Questions:

- Are Matoušek cubes combinatorial models of linear programs over (deformed) cubes?

- Does some Matoušek-cube fool Random-Facet? (It won't fool Random-Edge!)

Matoušek Cubes

Examples:

- The Matoušek cube $A = I_n$ is generated by the "unit cube" LP

  maximize $\sum_{i=1}^{n} i \cdot x_i$
  subject to

  $$0 \;\leq\; x_i \;\leq\; 1, \quad i = 1, \ldots, n$$

- The 3-dimensional Matoušek cubes

  $$A_1 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{pmatrix}$$

  do *not* come from any LP — they are generalized LPs

Matoušek Cubes — Results

Theorem: For a random starting vertex of a random n-dimensional Matoušek cube, the simplex algorithm with the Random-Facet pivot rule requires an expected number of $e^{\Omega(\sqrt{n})}$ steps, and this bound is tight.

$\Rightarrow$ Matoušek cubes fool Random-Facet

Theorem: For every starting vertex of every *LP-induced* n-dimensional Matoušek cube, the simplex algorithm with the Random-Facet pivot rule requires $O(n^2)$ steps, and this bound is tight.

$\Rightarrow$ still no LP known to fool Random-Facet

LP-induced Matoušek Cubes

Observation: If the n-dimensional Matoušek cube A is LP-induced, then A does not contain the "forbidden minors"

$$A_1 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{pmatrix}.$$

Lemma: Let $A \in GF(2)^{n \times n}$ be a matrix without forbidden minors. Then A^{-1} has *at most one* off-diagonal one-entry per row.

Corollary: Among the $2^{\binom{n}{2}}$ Matoušek cubes, at most $n! \approx 2^{n \log n}$ are LP-induced.

Examples:

Unit cube:

$$A = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Klee-Minty cube:

$$A = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Random-Facet: the LP-Case

- Random permutation π of the coordinates, start vertex v

- Among the coordinates i which are flippable ($(Av)_i = 1$), flip the first one in π; repeat until 0 is reached

k: last coordinate in π

v': first vertex where no coordinate different from k is flippable

v'': successor of v' (if existing)

Cheap Case: $v_k = 0$

v	1	0	0	1	1
π	1	5	3	4	2

$\Rightarrow \quad v' = 0 \quad 0 \quad 0 \quad 0 \quad 0$

$\Rightarrow$ done!

Expensive Case: $v_k = 1$

v	1	0	0	1	1
π	2	1	5	3	4

$\Rightarrow \quad Av' = 0 \quad 0 \quad 0 \quad 1 \quad 0$

- $Av'' = Av' + A_k = \mathbf{e}_k + A_k$
- $v'' = A_k^{-1} + \mathbf{e}_k$
- the possible v'' over all k with $v_k = 1$ are columns of $A^{-1} - I_n$
- in the LP-induced case, for any j, $v''_j = 1$ for *at most one* of the possible v''
- with j the second-to-last coordinate in π, v'' leads to the cheap case $v''_j = 0$ with high probability (over the random choice of k)

Beyond $GF(2)$?

- $\mathcal{V} = GF(q)^n$ can be interpreted as the vertex set of a polytope which is the product of n simplices of dimension $q-1$ each

- Any lower-triangular, invertible matrix $A \in GF(q)^{n \times n}$ induces an acyclic unique sink orientation on this polytope

- ... but relation between fast and slow game does not hold — flips are not linear functions anymore

- Performance of Random-Facet on such orientations? Lower bound of

$$e^{\Omega(\sqrt{n \log(nq)})}$$

  might hold for large q and would be significant