Computational Complexity of Stabbing, Visibility and Radii Problems

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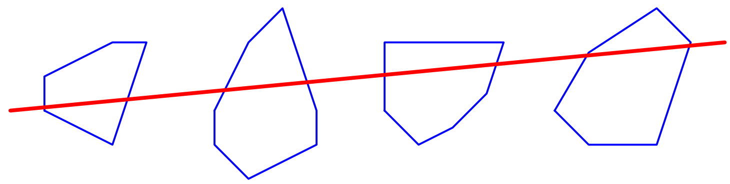
joint work with Peter Gritzmann

Interactions of lines with convex bodies in spaces of variable dimension:

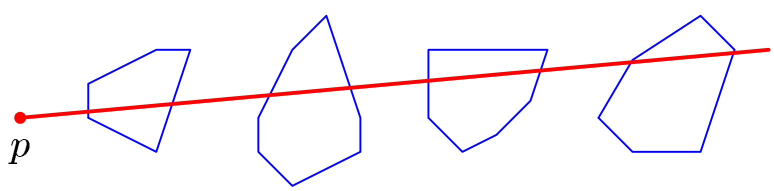
Fix a class $\mathcal{X}$ of convex bodies, e.g.,
$\mathcal{P}_{\mathcal{H}}$: rational $\mathcal{H}$-polytopes ($P = \{x \in \mathbb{R}^n \,:\, Ax \leq b\}$)
$\mathcal{P}_{\mathcal{V}}$: rational $\mathcal{V}$-polytopes ($P = \text{conv}\{v_1, \ldots, v_m\}$)
$\mathcal{B}$: 'rational' balls ($B = \{x \in \mathbb{R}^n \,:\, ||x - c|| \leq r\}$ with $c \in \mathbb{Q}^n$, $r^2 \in \mathbb{Q}$).

Input: $m, n \in \mathbb{N}$, bodies $C_1, \dots, C_m \subset \mathbb{R}^n$ from the class $\mathcal{X}$.

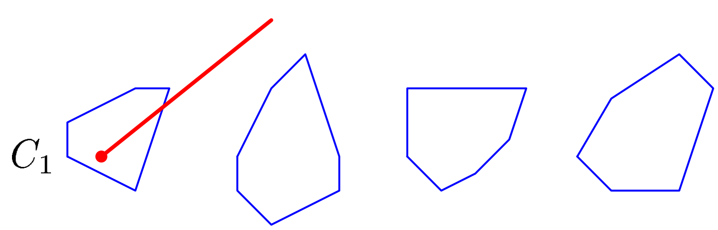
LINE STABBING$_{\mathcal{X}}$:
Does there exist a stabbing line for $C_1, \dots, C_m$?



ANCHORED RAY STABBING$_{\mathcal{X}}$:
(additionally part of the input: $p \in \mathbb{Q}^n$)
Question: Does there exist a stabbing ray issuing in p for $C_1, \dots, C_m$?

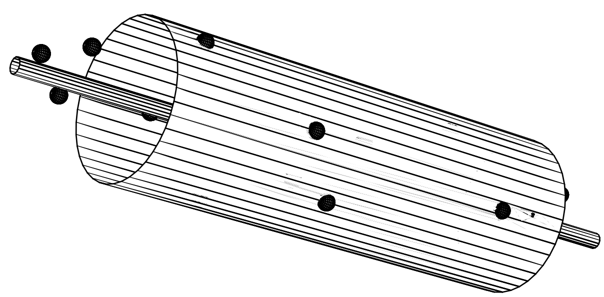


VISIBILITY$_{\mathcal{X}}$:
Question: Is C_1 visible with respect to $C_2, \dots, C_m$?



Radii problems

Outer j-radius $R_j(C)$ of a convex body C: radius of a smallest enclosing j-dimensional sphere of an optimal orthogonal projection of C onto a j-dimensional space ($j = n - 1$: smallest enclosing cylinder)



UPPER BOUNDING j-RADIUS$_{\mathcal{X}}$:
Instance: Given $j, n \in \mathbb{N}$, a body $C \subset \mathbb{R}^n$ from the class $\mathcal{X}$, $\rho^2 \in \mathbb{Q}^n$.
Question: Is $R_j(C)^2 \leq \rho^2$?

Radii problems $\longleftrightarrow$ stabbing problems:
There exists a stabbing line to the balls (c_1, r), $\ldots, (c_m, r)$
$\iff R_{n-1}(\text{conv}\{c_1, \ldots, c_m\}) \leq r$.

Motivation and previous results

- Much recent work on approximation of radii:
  - Chan (J. Comp. Geom. Appl. 2002)
  - Varadarajan, Venkatesh, Zhang (FOCS 2002)
  - Ye, Zhang (Approx. Algorithms for Comb. Opt. 2003)
  - Varadarajan, Har-Peled (SoCG 2003)

  – In some cases, hardness of R_j was not proven

- Few complexity results on stabbing and visibility for variable dimension

Main existing complexity results:

- Megiddo (1990):

  Stabbing for (non-)disjoint unit balls is $\mathbb{NP}$-hard

- Gritzmann and Klee (1993):
  - extensive complexity classification on $R_j(P)$ for various $\mathcal{X}$ and general ℓ_p-norms
  - do not cover cases where j comes close to n

Goal: Determine frontiers of hardness

Outline of Results

	$\mathcal{X} = \mathcal{P}_{\mathcal{H}}$	$\mathcal{X} = \mathcal{P}_{\mathcal{V}}$	$\mathcal{X} = \mathcal{B}$
LINE STABBING$_{\mathcal{X}}$	$\mathbb{NP}$-hard	$\mathbb{NP}$-hard	$\mathbb{NP}$-hard
ANCHORED LINE STABBING$_{\mathcal{X}}$	$\mathbb{NP}$-compl.	$\mathbb{NP}$-compl.	$\mathbb{NP}$-hard
ANCHORED RAY STABBING$_{\mathcal{X}}$	$\mathbb{P}$	$\mathbb{P}$	?
VISIBILITY$_{\mathcal{X}}$	$\mathbb{NP}$-hard	$\mathbb{NP}$-hard	$\mathbb{NP}$-hard
ANCHORED VISIBILITY$_{\mathcal{X}}$	$\mathbb{NP}$-compl.	$\mathbb{NP}$-compl.	$\mathbb{NP}$-hard

For fixed dimension: The problems can be solved in polynomial time for all $\mathcal{X} \in \{\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}, \mathcal{B}\}$.

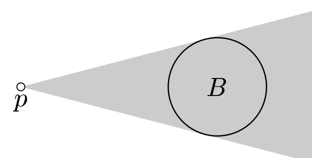
Radii: For each fixed $j \in \mathbb{N}$, upper bounding $R_{n-j}^2(P)$ for a $\mathcal{V}$-polytope:

ℓ_1	ℓ_2	ℓ_∞
$\mathbb{NP}$-hard	$\mathbb{NP}$-hard	?

Why is line stabbing hard at all?

Theorem. (a) For $\mathcal{X} \in \{\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}, \mathcal{B}\}$, Anchored Line Stabbing is $\mathbb{NP}$-hard.
(b) For $\mathcal{X} \in \{\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}\}$, Anchored Ray Stabbing can be solved in polynomial time.

Proof of (b): W.l.o.g. let the anchor $p = 0$.



The ray $[0, \infty)q$ ($q \in \mathbb{R}^n$) is a stabbing ray for $C_1, \dots, C_m$ if and only if the convex feasibility problem

$$q \quad \in \quad \text{pos}(C_i)\,, \quad 1 \leq i \leq m \quad \text{(positive hull)}$$

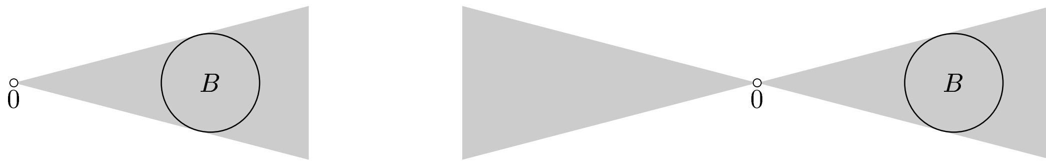
has a solution.

For $\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}} \rightsquigarrow$ linear programming.

Open problems

For balls: Convex quadratically constrained feasibility problem.
ANCHORED RAY STABBING$_{\mathcal{B}} \in \mathbb{P}$?

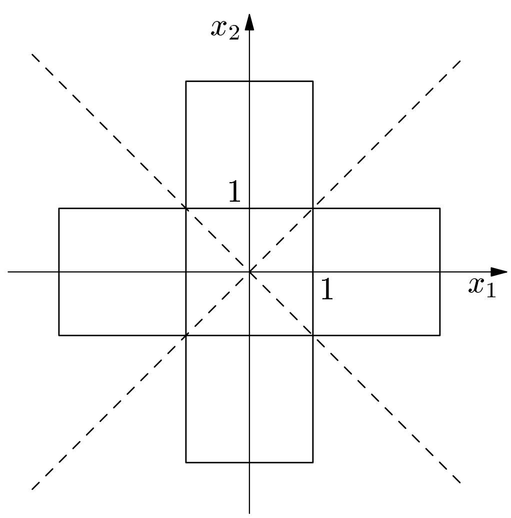
"Slightly" more general: SDFP (semidefinite programming feasibility problem) $\in \mathbb{P}$? Open.



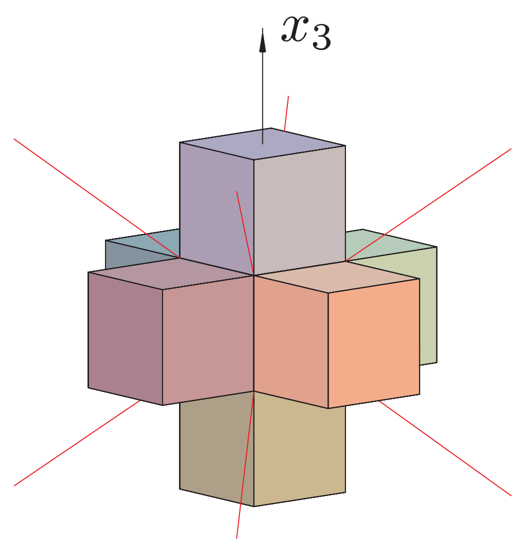
Theorem. ANCHORED LINE STABBING$_{\mathcal{P}_\mathcal{H}}$ is $\mathbb{NP}$-hard. The hardness persists if the polytopes are disjoint and restricted to be parallelotopes.

Reduction from 3-SAT: n variables, each clause:

$$C = y_i^{e_i} \vee y_j^{e_j} \vee y_k^{e_k}, \quad e_i, e_j, e_k \in \{-1, 1\}$$



$n = 1$

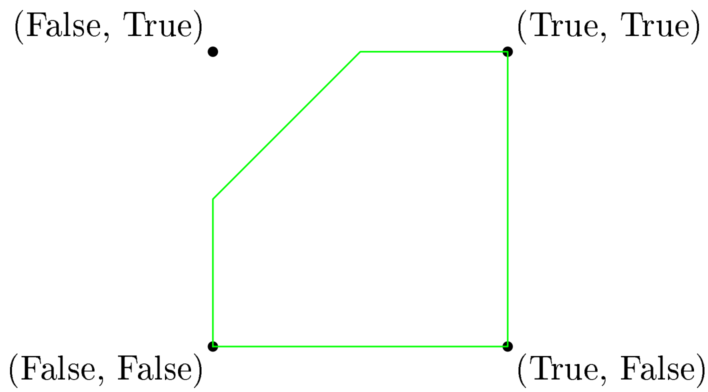


$n = 2$

With $t(\text{TRUE}) := 1$, $t(\text{FALSE}) := -1$:

$$\begin{aligned} &\text{assignment } (a_1, \ldots, a_n)^T \in \{\text{TRUE}, \text{FALSE}\}^n \\ \longleftrightarrow \quad &\text{line } \mathbb{R}(t(a_1), \ldots, t(a_n), 1)^T \subset \mathbb{R}^{n+1} \end{aligned}$$

Representing the clauses: Consider the 2-clause $y_1 \vee y_2^{-1}$ for $n = 2$. In the hyperplane $x_3 = 1$ in $\mathbb{R}^3$:



In general, cut of an $(n-3)$-dimensional face from the cube

$$2e_{n+1} + [-1, 1]^{n+1}.$$

For a clause $y_1^{-1} \vee y_2 \vee y_3^{-1}$:

$$(2e_{n+1} + [-1, 1]^{n+1}) \cap \{x \in \mathbb{R}^n \ : \ x_1 - x_2 + x_3 \leq 2\}.$$

Can also be achieved with parallelotopes. $\square$

Similarly for $\mathcal{V}$-polytopes. The hardness persists if the $\mathcal{V}$-polytopes are restricted to be cross polytopes.

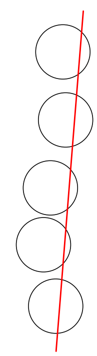
Open problems

For $\mathcal{X} \in \{\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}\}$:
Anchored Line Stabbing$_{\mathcal{X}} \in \mathbb{NP}$.

Line Stabbing$_{\mathcal{X}} \in \mathbb{NP}$ for $\mathcal{X} \in \{\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}, \mathcal{B}\}$?

For balls:

- In sufficiently generic situations, whenever there exists a stabbing line to the balls, then there exists a stabbing line tangent to $2n-2$ of them.



- Sottile, Th. (Trans. Am. Math. Soc., 2002):
  For $n \geq 3$, $2n-2$ general balls in $\mathbb{R}^n$ have at most $3 \cdot 2^{n-1}$ common tangent lines. This bound is tight.

$\rightsquigarrow$ algebraic problem of degree $3 \cdot 2^{n-1}$

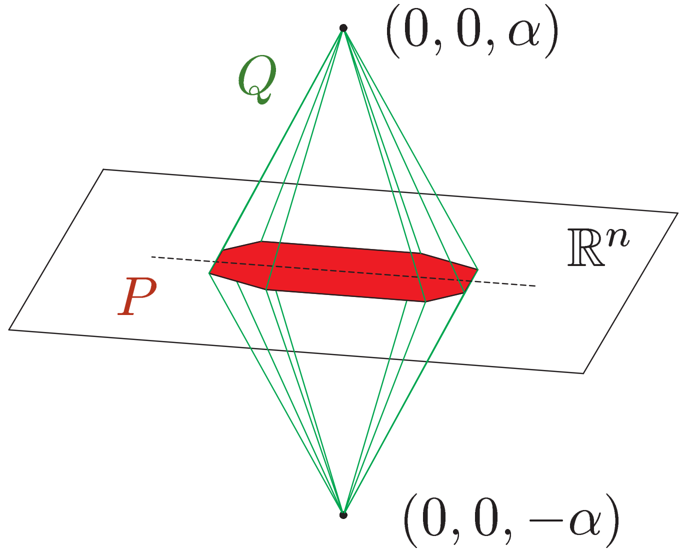
Can this be used to establish a witness of polynomial size that can be verified in polynomial time ?

Also open: Line Stabbing$_{\mathbb{B}}$ for disjoint unit balls $\mathbb{NP}$-hard?

Radii results

Theorem. For each fixed $j \in \mathbb{N}$, upper bounding $R_{n-j}(P)^2$ for a symmetric $\mathcal{V}$-polytope P is $\mathbb{NP}$-hard both in ℓ_1- and ℓ_2-space.

Proof: Transfer hardness from R_{n-1} to R_{n-j}.



For P symmetric, $\varepsilon > 0$ and $\alpha \geq \sqrt{n}2^{4L}(1 + 2^{4L}/\varepsilon)$ (L: input bit length):

$$R_{n-1}(Q) \leq R_{n-1}(P) \leq R_{n-1}(Q) + \varepsilon$$

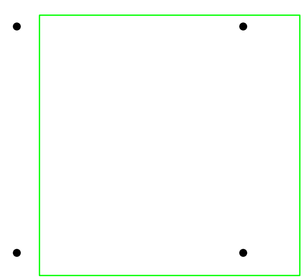
Can be generalized to R_{n-j} for any fixed j.

Open problems

- Containment in $\mathbb{NP}$
- ℓ_∞

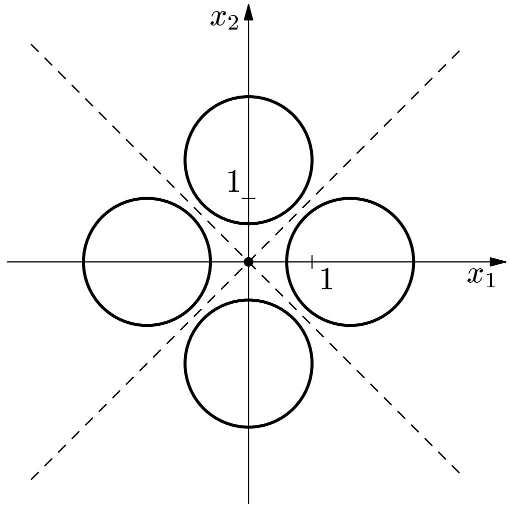
  $\leadsto$ Stabbing for axis-aligned unit cubes

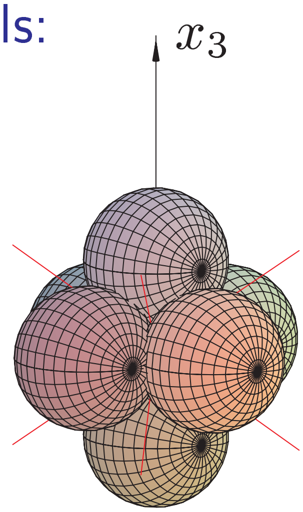
  Essential difficulty: For $n = 2$, not possible to cover exactly three points of $(\pm 1, \pm 1)$ by an axis-aligned square



Visibility computations

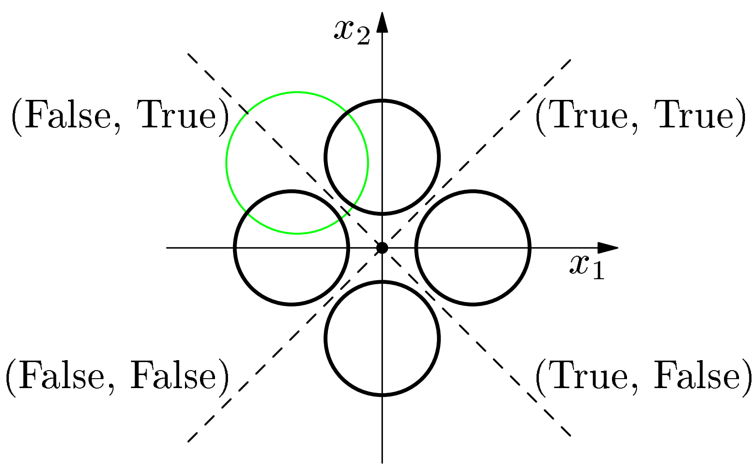
Consider anchored version for balls:





$n = 2$ $n = 3$

Represent the 2-clause $y_1 \vee y_2^{-1}$ for $n = 2$:



- $\mathbb{NP}$-hard also for axis-parallel unit cubes !
- open for disjoint unit balls.